Asymptotics for Symmetric 0-1 Matrices with Prescribed Row Sums

by

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Abstract

Let $M(n,k)$ be the number of labelled regular simple graphs with order n and degree k. Let $0 < \epsilon < 2/9$. Then

$$M(n,k) = \frac{(nk)!}{(nk/2)!\,2^{nk/2}(k!)^n} \exp\bigl(-(k^2-1)/4 + O(k^3/n)\bigr)$$

uniformly as $n \to \infty$ with $1 \le k \le \epsilon n$. This is generalised to labelled graphs which have an arbitrary degree sequence and avoid a specified set of edges.

1. Introduction

Let $\mathbf{g} = (g_1, g_2, \ldots, g_n)$ be a sequence of non-negative integers, and define $\mathcal{M}(\mathbf{g})$ to be the set of symmetric 0-1 matrices of order n with zero diagonal elements and row sums $g_1, g_2, \ldots, g_n$, respectively. For obvious reasons, we will always assume that $\sum_{i=1}^n g_i$ is even. We will be concerned with the asymptotic properties of $\mathcal{M}(\mathbf{g})$ as $n \to \infty$, in particular with its cardinality $N(\mathbf{g})$.

We start by summarising the previous results. Define $g_{\max} = \max_{i=1}^n g_i$, $e(G) = \frac{1}{2}\sum_{i=1}^n g_i$ and $\lambda = \sum_{i=1}^n g_i(g_i-1)/(4e(G))$. Define $P(\mathbf{g})$ by

$$N(\mathbf{g}) = \frac{(2e(G))!}{e(G)!\,2^{e(G)}\prod_{i=1}^n g_i!} P(\mathbf{g}).$$

The first result of interest to us was that of Read [9], who proved that $P(\mathbf{g}) = e^{-\lambda-\lambda^2} + o(1)$ if $g_i = 3$ for all i. The same result with arbitrary but bounded row sums was established by Bender and Canfield [1] and by Wormald [10]. The first attempt to allow the degrees to increase with n was made by Bollobás [2], who obtained $P(\mathbf{g}) = e^{-\lambda-\lambda^2}(1+o(1))$ for $g_{\max} \le \sqrt{2\log n} - 1$. Most recently, Bollobás and McKay [4] have shown that $P(\mathbf{g}) = e^{-\lambda-\lambda^2}(1 + o(e^{(\log n)^{4/5}}/n))$ for $g_{\max} \le (\log n)^{1/3}$.

Actually, some of the results quoted above have more generality. Bender and Canfield [1], for example, optionally allow non-zero diagonal entries and integer entries greater than one. Also, [1], [4] and [10] prescribe a restricted set of matrix entries which must be zero. This extension enables one to investigate such things as the expected number of submatrices of specified form, a matter which has been studied by other means in [5].

Leaving aside differences of terminology, all the papers cited above except [5] and possibly [9] use essentially the same model; only the method and accuracy of the analysis varies. This paper is no exception. Our contribution is a new method of analysis which enables us to considerably extend and strengthen all previous results for the 0-1 case with zero diagonal.

A parallel study of rectangular 0-1 matrices (not necessarily symmetric) has appeared in [6].

2. The Model

Consider a collection of disjoint sets $v_1, v_2, \ldots, v_n$, where v_i has cardinality g_i for each i. These will henceforth be called *cells*. Define $V = \{v_1, v_2, \ldots, v_n\}$. A *pairing* is a set of unordered pairs (called the *edges* of the pairing) such that

(i) each edge has the form $\{x, x'\}$ where $x, x' \in \bigcup_{i=1}^n v_i$, and
(ii) each element of $\bigcup_{i=1}^n v_i$ is in exactly one edge.

Given a pairing P, we can obtain a multigraph $G(P)$. The vertices of $G(P)$ are $v_1, v_2, \ldots, v_n$. The number of graph edges joining v_i to v_j is the number of edges $\{x, x'\}$ of P such that $x \in v_i$ and $x' \in v_j$. Equivalently, P yields an $n \times n$ symmetric integer matrix whose (i, j)-th entry equals the number of edges in $G(P)$ from v_i to v_j. For the remainder of the paper we will use the graph terminology rather than the matrix terminology.

Given $\mathbf{g}$, the number of possible pairings is exactly $(2e(G))!/(e(G)!\,2^{e(G)})$. Furthermore, each element of $\mathcal{M}(\mathbf{g})$ corresponds to exactly $\prod_{i=1}^n g_i!$ pairings. $P(\mathbf{g})$ can thus be interpreted as the probability that a randomly chosen pairing P produces a multigraph $G(P)$ which has no multiple edges or loops. Previous estimates of $P(\mathbf{g})$ were made using either inclusion-exclusion or the method of moments. In the next section we present a method which is more complex yet substantially more accurate.

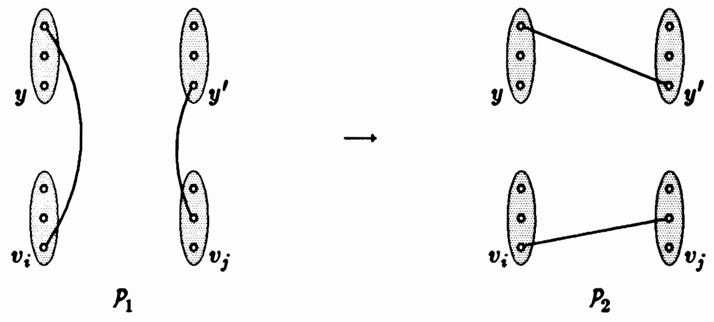
3. Basic Analysis

For notational convenience, we make no distinction between graphs and their edge sets. We will always assume that $g_{\max} \geq 1$.

Let L be a graph with vertex set V which is simple except that it has a loop on every vertex. Let $l_{\max}$ be the maximum degree of L, and let $e(L)$ be the number of edges. Note that a loop contributes 2 to the degree of its vertex. Let H be a multigraph, also with vertex set V, with the restriction that if xx' is an edge of non-zero multiplicity of H, then xx' is an edge of L. Let h_i be the degree of vertex v_i in H and let $e(H) = \frac{1}{2}\sum_{i=1}^n h_i$ be the total number of edges. If $x, x' \in V$, then $\mu_H(xx')$ denotes the multiplicity of edge xx' in H. Also, $H + xx'$ is the multigraph obtained from H by adding an extra edge from x to x'. Let $C(L, H)$ be the set of all pairings P such that, for $x, x' \in V$, if xx' is an edge of L then $\mu_{G(P)}(xx') = \mu_H(xx')$

and if xx' is not an edge of L then $\mu_{G(P)}(xx') \le 1$. More informally, $G(P)$ is simple outside L and matches H inside L.

Let $P_1 \in C(L, H)$ and $P_2 \in C(L, H + v_iv_j)$, where $v_iv_j \in L$. Then P_1 and P_2 are said to be ***closely related*** if P_2 can be obtained from P_1 by an operation of the following type.



In the diagram, only the relevant parts of P_1 and P_2 are drawn; the other parts are unchanged by the operation. In detail, the requirements for the legality of the operation are

(i) v_i, v_j, y, $y' \in V$ (distinct except that $v_i = v_j$ is allowed),

(ii) $v_iv_j \in L$, and

(iii) yv_i, v_jy', $yy' \notin L$.

Define $N_1 = |C(L, H)|$ and $N_2 = |C(L, H + v_iv_j)|$. Furthermore, let M be the number of closely related pairs (P_1, P_2) such that $P_1 \in C(L, H)$ and $P_2 \in C(L, H+v_iv_j)$. Our aim for the remainder of this section will be to bound the ratio of N_1 to N_2. We will do this via four separate bounds on M.

Lemma 3.1.

$$M \le \begin{cases} (g_i - h_i)(g_j - h_j)N_1, & \textit{if } i \ne j, \\ \binom{g_i - h_i}{2} N_1, & \textit{if } i = j. \end{cases}$$

Proof. Choose an arbitrary $P_1 \in C(L, H)$. If $i \ne j$, y can be chosen in at most $g_i - h_i$, and y' in at most $g_j - h_j$ ways. If $i = j$, y and y' can be chosen in at most $\binom{g_i - h_i}{2}$ ways. ∎

Lemma 3.2. *Let* $\Delta = g_{\max}(g_{\max} + l_{\max} - 3)$. *Then*

$$M \ge 2(\mu_H(v_iv_j) + 1)(e(G) - e(H) - \Delta)N_2.$$

Proof. Choose an arbitrary $P_2 \in C(L, H + v_iv_j)$. We wish to bound the number of pairings $P_1 \in C(L, H)$ which are closely related to P_2. The edge from v_i to v_j can be chosen in $\mu(v_i v_j) + 1$ ways. For the ordered pair (y, y') we have the $e(G) - e(H) - 1$ edges of P_2 which lie outside $H + v_i v_j$. Each of these gives two possibilities, except that some cases are excluded. The latter are listed below, together with an upper bound on each.

	$i \neq j$	$i = j$
$y = v_i,\ y = v_j,\ y' = v_i,\ y' = v_j$	$4(g_{\max} - 1)$	$2(g_{\max} - 2)$
$yv_i \in G,\ y'v_j \in G$	$2(g_{\max} - 1)^2$	$2(g_{\max} - 1)(g_{\max} - 2)$
$yv_i \in L,\ y'v_j \in L$	$2g_{\max}(l_{\max} - 3)$	$2g_{\max}(l_{\max} - 2)$

Since $g_{\max} \geq 1$, the largest total is for the left column. The Lemma now follows when this total is subtracted from $2(e(G) - e(H) - 1)$. ∎

Lemmas 3.1 and 3.2 can be combined to give an upper bound on the ratio N_2/N_1. By applying this upper bound repeatedly, the following Theorem is obtained.

Let J be a multigraph which satisfies the same requirements as H, and define $e(J)$ and $\{j_i\}$ consistently with $e(H)$ and $\{h_i\}$. Let $H + J$ be the multigraph with $\mu_{H+J}(xx') = \mu_H(xx') + \mu_J(xx')$ for all $x,\ x' \in V$. Let $\lambda(J)$ be the number of loops in J, that is $\sum_{x \in V} \mu_J(xx)$. For integers a, b define $a^{[b]} = a(a-1)\cdots(a-b+1)$.

Theorem 3.3. *If $e(G) - e(H) - \Delta \geq e(J)$ and $C(L, H) \neq \emptyset$, then*

$$\frac{|C(L, H+J)|}{|C(L,H)|} \leq \frac{\prod_{i=1}^{n}(g_i - h_i)^{[j_i]}}{2^{\lambda(J)+e(J)}(e(G) - e(H) - \Delta)^{[e(J)]}\prod \mu_{H+J}(xx')^{[\mu_J(xx')]}}$$

where the product in the denominator is over all $x,\ x' \in V$. ∎

Our next task is to find a bound complementary to Theorem 3.3. The first prerequisite follows easily from the proof of Lemma 3.2.

Lemma 3.4. $M \leq 2(\mu_H(v_i v_j) + 1)(e(G) - e(H) - 1)N_2$. ∎

The other prerequisite is not so easy to come by.

Lemma 3.5. *Let $\Delta' = g_{\max}(g_{\max} + l_{\max} - 1) + 2$, and suppose $e(G) - e(H) - \Delta' > 0$. Then*

$$M \geq \begin{cases} (g_i - h_i)(g_j - h_j)(1 - E)N_1, & \text{if } i \neq j, \\ \binom{g_i - h_i}{2}(1 - E)N_1, & \text{if } i = j, \end{cases}$$

where

$$E = \frac{\Delta}{2(e(G) - e(H) - \Delta')}.$$

Proof. There is no useful lower bound on the number of pairings $P_2 \in C(L, H + v_i v_j)$ which are closely related to an arbitrary $P_1 \in C(L, H)$. Instead, we will choose a *random* $P_1 \in C(L, H)$ and bound the *expected* number of closely related pairings in $C(L, H + v_i v_j)$.

The upper bound given in Lemma 3.1 is high because of the possibilities that $yy' \in G(P_1)$, $yy' \in L \setminus G(P_1)$ and $y = y'$.

As a typical case, consider the possibility that $yy' \le G(P_1)$ when $i \neq j$. Choose a random $P_1 \in C(L, H)$. There are at most $g_i - h_i$ choices of y, then at most $g_{\max} - 1$ choices of y' such that $yy' \in G(P_1)$. Given these choices, the probability that $y'v_j \in G(P_1)$ is at most

$$\frac{|C(L \cup \{v_iy, yy', y'v_j\}, H + v_iy + yy' + y'v_j)|}{|C(L \cup \{v_iy, yy', y'v_j\}, H + v_iy + yy')|} \le \frac{(g_{\max} - 1)(g_j - h_j)}{2(e(G) - e(H) - \Delta')},$$

by Theorem 3.3. The other cases can be handled in the same way. ∎

Lemmas 3.4 and 3.5 can now be combined to give a bound complementary to Theorem 3.3.

Theorem 3.6. *If $e(G) - e(H) - \Delta' > e(J)$ and $C(L, H) \neq \emptyset$, then*

$$\frac{|C(L, H + J)|}{|C(L, H)|} \ge \frac{\prod_{i=1}^{n} (g_i - h_i)^{[j_i]}}{2^{\lambda(J)+e(J)}(e(G) - e(H) - 1)^{[e(J)]} \prod \mu_{H+J}(xx')^{[\mu_J(xx')]}} \times \left(1 - \frac{\Delta}{2(e(G) - e(H) - e(J) - \Delta')}\right)^{e(J)},$$

where the product in the denominator is over all $x, x' \in V$. ∎

For later convenience, we combine Theorems 3.3 and 3.6 for the special case where H has no edges.

Theorem 3.7. *If $e(G) - \Delta' > e(J)$ and $C(L, \emptyset) \neq \emptyset$, then*

$$\frac{|C(L, J)|}{|C(L, \emptyset)|} = \frac{\prod_{i=1}^{n} g_i^{[j_i]}}{2^{\lambda(J)+e(J)}(e(G) - 1)^{[e(J)]} \prod_{x,x' \in V} \mu_J(xx')!} D(\mathbf{g}, L, J),$$

where

$$\left(1 - \frac{\Delta}{2(e(G) - e(J) - \Delta')}\right)^{e(J)} \le D(\mathbf{g}, L, J) \le \frac{(e(G) - 1)^{[e(J)]}}{(e(G) - \Delta)^{[e(J)]}}.$$ ∎

4. Synthesis

The results of the previous section can be used to find an estimate of the probability $P(\mathbf{g})$. In fact we will do this with a little more generality. Let X be a simple graph with vertex set V and maximum degree $x_{\max}$. Define $P(\mathbf{g}, X)$ to be the probability that a random pairing P produces a simple graph $G(P)$ with no edges in common with X. In the matrix formulation, X specifies a symmetric set of matrix entries which must be zero.

Choose a pairing $\mathcal{P}$ and form the multigraph $G(\mathcal{P})$. A ***naughty*** edge of $G(\mathcal{P})$ is one that is either parallel to another edge (i.e. is part of a multiple edge), is a loop, or coincides with an edge of X. The naughty edges of $G(\mathcal{P})$ together form a multigraph called the ***naughty graph*** of $G(\mathcal{P})$ (and of $\mathcal{P}$). Our problem is to estimate the probability $P = P(\mathbf{g}, X)$ such that a random $\mathcal{P}$ has a naughty graph with no edges. By definition, this is equal to the number of pairings with empty naughty graph divided by the total number. Thus we have

$$\frac{1}{P} = \sum_K \frac{\nu(K)}{\nu(\emptyset)}$$

if $P \neq 0$, where $\nu(K)$ is the number of pairings with naughty graph K, and the sum is over all possible K. The ratio $\nu(K)/\nu(\emptyset)$ can be written in terms of the ratio bounded in Theorem 3.7. To do this, separate each possible naughty graph K into edge-disjoint multigraphs $A(K)$ and $B(K)$. The edges of $A(K)$ are those edges of K which are loops or which coincide with edges of X, and the edges of $B(K)$ are the others. Then clearly

$$\frac{\nu(K)}{\nu(\emptyset)} = \frac{|C(L,K)|/|C(L,\emptyset)|}{\sum_S |C(L,S)|/|C(L,\emptyset)|},$$

where L includes every edge of $B(K)$, every edge of X and a loop on every vertex, and the sum in the denominator is over all simple subgraphs S of $B(K)$.

The value of $1/P$ can now be estimated by comparing it to a more tractible expression. For $k > 0$ and $1 \leq i, j \leq n$, define

$$\alpha_k(ij) = \begin{cases} g_i^{[k]} g_j^{[k]}/k!, & \text{if } i \neq j, \\ g_i^{[2k]}/(2^k k!), & \text{if } i = j, \end{cases}$$

$$\beta_k(ij) = \alpha_k(ij)/(2e(G))^k$$

and

$$\Psi = \Psi(\mathbf{g}, X) = \prod_{i \leq j} (1 + \beta_1(ij) + \beta_2(ij)) \prod_{i<j} (1 + \beta_2(ij)),$$

where the first product is over all i, j such that $i = j$ or $v_i v_j \in X$, while the second is over all i, j such that $i \neq j$ and $v_i v_j \notin X$. When Ψ is completely expanded, it yields a summation some terms of which can be identified with possible naughty graphs K. Precisely, K can be identified with the term

$$\psi(K) = \prod_{i \leq j} \beta_{\mu_K(v_i v_j)}(ij).$$

Such a term is present if K has no edges of multiplicity greater than two.

We now present a series of lemmas which will enable us to compare the values of Ψ and $1/P$. Define the function $m(x) = (2x)!/(2^x x!)$.

Lemma 4.1. *Let K be a possible naughty graph. Then if $\nu(\emptyset) \neq 0$,*

$$\sum_{J \supseteq K} \frac{\nu(J)}{\nu(\emptyset)} \leq \frac{m(e(G)-e(K))}{Pm(e(G))} \prod_{i \leq j} \alpha_{\mu_K(v_i v_j)}(v_i v_j),$$

where the sum of J is over all possible naughty graphs which contain K.

Proof. Since the total number of pairings is $m(e(G))$, we have $\nu(\emptyset) = Pm(e(G))$. The product in the Lemma bounds the number of ways of choosing the pairing edges corresponding to K. The remaining pairing edges can be chosen in $m(e(G)-e(K))$ ways. ∎

Lemma 4.2. *Let K be a multigraph with vertex set V. Then*

$$\sum_{J \supseteq K} \psi(J) \leq \psi(K)\Psi,$$

where the sum is over all multigraphs which contain K.

Proof. This is immediate from the definition of Ψ. ∎

For a possible naughty graph K, define

$$r(K) = \frac{\prod_{i<j} g_i^{[\mu_K(v_i v_j)]} g_j^{[\mu_K(v_i v_j)]} \prod_i g_i^{[2\mu_K(v_i v_i)]}}{\prod_i g_i^{[k_i]}}.$$

Note that $r(K) \geq 1$. Also, define $\hat{\Delta} = 2 + g_{\max}(\frac{3}{2}g_{\max} + x_{\max} + 1)$.

Lemma 4.3. *Let K be a possible naughty graph with no edges of multiplicity greater than two. Assume that $\hat{\Delta} \leq \epsilon_1 e(G)$ and $e(K) \leq \epsilon_2 e(G)$, where ϵ_1 and ϵ_2 are fixed positive constants with $\frac{3}{2}\epsilon_1 + \epsilon_2 < 1$. Then, if $\nu(\emptyset) \neq 0$,*

$$1 + \left| \frac{\nu(K)/\nu(\emptyset)}{\psi(K)} - 1 \right| \leq r(K) \exp\bigl(O(\hat{\Delta} e(K)/e(G))\bigr) \frac{e(G)^{e(K)}}{e(G)^{[e(K)]}}.$$

Proof. By Theorem 3.7,

$$1 + \left| \frac{\nu(K)/\nu(\emptyset)}{\psi(K)} - 1 \right| \leq r(K) \frac{e(G)^{e(K)}}{(e(G)-\hat{\Delta})^{[e(K)]}} \left(1 - \frac{\hat{\Delta}/2}{e(G)-e(K)-\hat{\Delta}}\right)^{-e(K)} \times \sum_{S \subseteq B(K)} \frac{|C(L,S)|}{|C(L,\emptyset)|},$$

where the sum is over all simple subgraphs S of $B(K)$ and L is defined near the start of this section. The value of $\hat{\Delta}$ is valid since $l_{\max} \leq 2 + x_{\max} + g_{\max}/2$.

The assumed bounds on $\hat{\Delta}$ and $e(K)$ ensure that

$$\frac{e(G)^{e(K)}}{(e(G)-\hat{\Delta})^{[e(K)]}} = \exp\bigl(O(\hat{\Delta} e(K)/e(G))\bigr) \frac{e(G)^{e(K)}}{e(G)^{[e(K)]}}$$

and

$$\left(1-\frac{\hat{\Delta}/2}{e(G)-e(K)-\hat{\Delta}}\right)^{-e(K)}=\exp\bigl(O(\hat{\Delta}e(K)/e(G))\bigr).$$

To bound the sum, let m be the number of edges of $B(K)$, *not* counting multiplicities. The number of simple subgraphs S of $B(K)$ with exactly r edges is at most $\binom{m}{r}$. Thus, using Theorem 3.7, the sum is bounded by

$$\sum_{r=0}^{m}\binom{m}{r}\frac{g_{\max}^{2r}}{(e(G)-\hat{\Delta})^{[r]}2^r}\le\left(1+\frac{eg_{\max}^2}{2(e(G)-\hat{\Delta})}\right)^m$$
$$=\exp\bigl(O(\hat{\Delta}e(K)/e(G))\bigr).\quad\blacksquare$$

Lemma 4.4. *Suppose that $\hat{\Delta}\le\epsilon_1 e(G)$, where $\epsilon_1<2/3$. Then*

$$\frac{1}{P}=\Psi\exp\bigl(O(\hat{\Delta}^2/e(G))\bigr).$$

Proof. We proceed by breaking the problem into a number of pieces. Firstly, choose a constant ϵ_2 such that $0<\epsilon_2<2/3-\epsilon_1$. We can dispose of those K with $e(K)>\epsilon_2 e(G)$ by arguing as follows. The function

$$f(x)=\frac{1}{\Psi}\prod_{i\le j}(1+\beta_1(ij)x+\beta_2(ij)x^2)\prod_{i<j}(1+\beta_2(i)x^2),$$

where the products are restricted as in the definition of Ψ, can be interpreted as the probability generating function of a non-negative random variable which is the sum of $n(n+1)/2$ simpler independent random variables. The mean of the random variable is bounded by

$$2\sum_{i\le j}\beta_2(ij)+\sum_i\beta_1(ii)+\sum_{v_iv_j\in X}\beta_1(ij)<\hat{\Delta}/2.$$

Therefore, the contribution to Ψ of those K with $e(K)>\epsilon_2 e(G)$ is at most $\hat{\Delta}\Psi/(2\epsilon_2 e(G))=\Psi O(\hat{\Delta}/e(G))$. A parallel argument (working directly with pairings) shows that the contribution to $1/P$ of the same K is at most $O(\hat{\Delta}/e(G))/P$.

At this point, it is convenient to note that the displayed bound above guarantees the existence of at least one actually occuring naughty graph K for which $e(K)\le\hat{\Delta}/2$. Because $\epsilon_1<2/3$, this gives us a starting point from which we can produce at least one pairing with empty naughty graph, using the switching operation employed in Section 3. Thus $\nu(\emptyset)\ne 0$, justifying our use of Theorem 3.9, Lemma 4.1 and Lemma 4.3.

Next we consider a few of the less common possibilities for K. The contribution to Ψ of all K with an edge of multiplicity greater than two is zero, by the definition of Ψ. The contribution of the same naughty graphs to $1/P$ is, by Lemma 4.1, at most

$$\frac{1}{P}\frac{m(e(G)-3)}{m(e(G))}\left(\sum_{i<j}\frac{g_i^{[3]}g_j^{[3]}}{6}+\sum_i\frac{g_i^{[6]}}{48}\right)=\frac{1}{P}O(g_{\max}^4/e(G)).$$

In a similar manner, we can dispose of those K which have a vertex of degree greater than, 2 not counting multiplicities. The contributions to Ψ and $1/P$ are respectively $\exp(O(\hat{\Delta}^2/e(G))\Psi$ and $\exp(O(\hat{\Delta}^2/e(G)))/P$. Next, consider those terms of Ψ which correspond to impossible naughty graphs, because of excessive degrees. These have been counted already, except for a few cases where $1 \le g_i \le 3$. The contribution to Ψ here is easily seen (with the help of Lemma 4.2) to be $O(\hat{\Delta}^2/e(G))\Psi$.

We are left with the naughty graphs K for which the conditions of Lemma 4.3 hold, and for which the maximum degree, not counting multiplicities, is at most two. We must consider the maximum possible relative errors associated with the factors $e(G)^{e(K)}/e(G)^{[e(K)]}$, $\exp(O(\hat{\Delta}e(K)/e(G)))$ and $r(K)$.

To handle the first two factors, compare the term $1+\beta_1(ij)x+\beta_2(ij)x^2$ of $f(x)$ with $\exp(\beta_1(ij)x+\beta_2(ij)x^2)$ and $1+\beta_2(ij)x^2$ with $\exp(\beta_2(ij)x^2)$. We find that the coefficient of $x^{e(k)}$ in $\Psi f(x)$ is at most that of $x^{e(K)}$ in $\exp(x\sum_{i\le j}\beta_1(ij)+x^2\sum_{i\le j}\beta_2(ij)+x^2\sum_{i<j}\beta_2(ij))$, where the sums are restricted as before. This is clearly at most $\sum_{k=\lceil e(K)/2\rceil}^{e(K)}(\hat{\Delta}/2)^k/k!$. A straightforward argument now shows that the maximum effect of $\exp(O(\hat{\Delta}e(K)/e(G)))e(G)^{e(K)}/e(G)^{[e(K)]}$ is a factor of $\exp(O(\hat{\Delta}^2/e(G)))$.

Finally, we must investigate the factor $r(K)$. Ignoring those K we have eliminated, non-trivial contributions to $r(K)$ come from those vertices of K which have no loops and two other edges, not counting multiplicities. There are three possibilities (0, 1 or 2 double edges) which together provide a contribution of at most $g_i^2\hat{\Delta}^2\Psi/e(G)^2$ to Ψ for vertex v_i (by Lemma 4.2). The effect on $r(K)$ of this event is a factor of $1+O(1/g_i)$, for $g_i \ne 0$. After some routine calculations, we find that the overall effect of $r(K)$ on Ψ is a factor of $\prod_{g_i\ne0}(1+O(1/g_i))^{g_i^2\hat{\Delta}^2/e(G)^2}=\exp(O(\hat{\Delta}^2/e(G)))$. ∎

In order to estimate P we now only need to estimate Ψ. Define

$$\lambda=\frac{1}{4e(G)}\sum_{i=1}^{n}g_i^{[2]} \quad\text{and}\quad \mu=\frac{1}{2e(G)}\sum_{v_iv_j\in X}g_ig_j.$$

Lemma 4.5. *If $g_{\max}^2=O(e(G))$, then $\Psi=\exp(\lambda+\lambda^2+\mu+O(\hat{\Delta}^2/e(G)))$.*

Proof. Since $g_{\max}^2=O(e(G))$, we can uniformly write $1+\beta_2(ij)=\exp(\beta_2(ij)+O(\beta_2(ij)^2))$ and $1+\beta_1(ij)+\beta_2(ij)$ similarly. The expression for Ψ then becomes a product of exponentials and thus the exponential of a sum. The rest is easy. ∎

Theorem 4.6. *Suppose that $g_{\max} \ge 1$ and $\hat{\Delta} \le \epsilon_1 e(G)$, where $\epsilon_1 < 2/3$ and $\hat{\Delta} = 2 + g_{\max}(\frac{3}{2}g_{\max} + x_{\max} + 1)$. Then the number of simple graphs with degree sequence* **g** *and no edge in common with X is uniformly*

$$\frac{(2e(G))!}{e(G)!\,2^{e(G)}\prod_{i=1}^{n} g_i!}\exp\left(-\lambda - \lambda^2 - \mu + O(\hat{\Delta}^2/e(G))\right)$$

as $n \to \infty$, where λ and μ are as defined above. ∎

Let $RG(n,k)$ be the number of regular simple graphs of order n and degree k.

Corollary 4.7. *If $1 \le k \le \epsilon_3 n$, where $\epsilon_3 < 2/9$ is fixed. Then uniformly*

$$RG(n,k) = \frac{(nk)!}{(nk/2)!\,2^{nk/2}(k!)^n}\exp\left(-(k^2-1)/4 + O(k^3/n)\right). \quad ∎$$

Corollary 4.8. *If $1 \le k = o(n)$, then*

$$\log RG(n,k) \sim \log\left(\frac{(nk)!}{(nk/2)!\,2^{nk/2}(k!)^n}\right). \quad ∎$$

5. Potpourri

The results of the previous section show that the known asymptotic estimate of $N(\mathbf{g})$ for $g_{\max} \le \sqrt{2\log n} - 1$ is in fact accurate for $g_{\max} = o(e(G)^{1/4})$. It would be of considerable interest to know if this is the limit of its validity. It is possible that the methods of this paper could be improved enough to settle this question. To begin with, Theorem 3.7 as we have it could be used to sharpen Lemmas 3.1, 3.2, 3.4 and 3.5. These in turn would imply a more accurate version of Theorem 3.7. This process could in principle be repeated, but even the first iteration would be quite complicated.

In the case $\mathbf{g} = (k, k, \ldots, k)$, we have a little experimental evidence of the accuracy of Corollary 4.7. In [7] we presented exact values of $N(\mathbf{g})$ for $1 \le n \le 21$ and $0 \le k < n$. A careful numerical analysis of them suggested the following possibility.

Conjecture 5.1. [7] *If $k \ge 1$ is fixed and $\mathbf{g} = (k, k, \ldots, k)$, then*

$$N(\mathbf{g}) = \frac{(nk)!}{(nk/2)!\,2^{nk/2}(k!)^n}\exp\left(-\frac{(k-1)(k+1)}{4} - \frac{(k-1)(k+2)(k^2-k+1)}{12kn} - \frac{(k-1)^4(k^2+4k+6)}{24k^2n^2} + O(k^5/n^3)\right).$$

If the conjecture is true, it would undoubtedly be true if k increased not too quickly with n.

In [8], McKay and Wormald prove that, if $3 \le g_i \le o(n^{1/2-\epsilon})$ for all i, almost no isomorphism class of $\mathcal{M}(\mathbf{g})$ has members with non-trivial automorphisms. The same result for fixed constant g_i has been obtained by Bollobás [3]. It follows, under these conditions, that the number of equivalence classes in $\mathcal{M}(\mathbf{g})$ is asymptotically $N(\mathbf{g})/n!$. In fact, the estimates in [8] lead quickly to results of the following kind.

Theorem 5.2. *Let $URG(n,k)$ be the number of unlabelled regular simple graphs of order n and degree k. If $\epsilon > 0$ is fixed, and $3 \le k = O(n^{1/2-\epsilon})$, then uniformly*

$$URG(n,k) = \frac{(nk)!}{(nk/2)!\,2^{nk/2}(k!)^n n!}\exp\bigl(-(k^2-1)/4 + O(k^3/n)\bigr).$$

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