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A new graph product and its spectrum

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A new graph product is introduced, and the characteristic polynomial of a graph so-formed is given as a function of the characteristic polynomials of the factor graphs. A class of trees produced using this product is shown to be characterized by spectral properties.

1. Notation and preliminaries

All graphs considered in this paper are finite, and without loops and multiple or directed edges. Any undefined graph-theoretical terms will have the meanings given to them in Behzad and Chartrand [1].

If G is a graph with adjacency matrix $A(G)$, then we denote the characteristic polynomial $\det(\lambda I-A(G))$ of $A(G)$ by $G(\lambda)$, and refer to it as the characteristic polynomial of G. If G is a rooted graph then we denote by G' the graph obtained from G when the root vertex is removed. The characteristic polynomial of the rooted graph G is just the characteristic polynomial of the unrooted graph with the same vertex and edge sets as G.

DEFINITION 1.1. Let H be a labelled graph on n vertices. Let G be a sequence of n rooted graphs $G_1, G_2, \ldots, G_n$. Then by $H(G)$ we denote the graph obtained by identifying the root of G_i with the ith vertex of H. We call $H(G)$ the *rooted product* of H by G.

Figure 1 illustrates this construction with H the path on three

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vertices and G consisting of three copies of the rooted path on two vertices

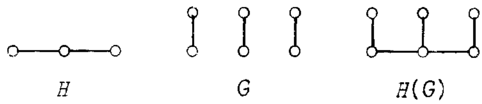

FIGURE 1

DEFINITION 1.2. Given a labelled graph H on n vertices and a sequence G of n rooted graphs, we define the matrix $A_\lambda(H, G)$ as follows:

$$A_\lambda(H, G) = (a_{ij})$$

where

$$a_{ij} = \begin{cases} G_i(\lambda) & , \quad i = j , \\ \\ -h_{ij} G_i'(\lambda) & , \quad i \neq j , \end{cases}$$

and $A(H) = (h_{ij})$ is the adjacency matrix of H.

If, for example, H and G are represented in Figure 1, then $A_\lambda(H, G)$ is the matrix

$$\begin{bmatrix} \lambda^2-1 & -\lambda & 0 \\ -\lambda & \lambda^2-1 & -\lambda \\ 0 & -\lambda & \lambda^2-1 \end{bmatrix} .$$

2. The polynomial of the rooted product

In this section we prove the following:

THEOREM 2.1. $H(G)(\lambda) = \det A_\lambda(H, G)$.

This result has already been proved by Schwenk [4] in the case where G consists of n isomorphic rooted graphs. The method we use to prove the result in general is quite different from his, however.

We will need the following lemma.

LEMMA 2.2. *Let K and L be rooted graphs, and let $K \cdot L$ denote the graph obtained by identifying the roots of K and L. Then*

$$K \cdot L(\lambda) = K(\lambda)L'(\lambda) + K'(\lambda)L(\lambda) - \lambda K'(\lambda)L'(\lambda) .$$

Proof. See Schwenk [4], or Godsil and McKay [2]. □

Proof of Theorem 2.1. We will use induction on the number of vertices of $H(G)$. Suppose this number is N, and that the theorem holds for all labelled graphs H and sequences G such that $H(G)$ has less than N vertices. For $n = 1$, the theorem follows from the definition of $A_\lambda(H, G)$, so we assume $n \geq 2$.

Let F denote the sequence of rooted graphs obtained from G by replacing the graph G_n by K_1, the graph with only one vertex. F' will be used to denote the subsequence of G consisting of the graphs $G_1, G_2, \ldots, G_{n-1}$. Let H' denote the graph obtained from H by deleting the vertex labelled n, and let $H(F)'$ denote the graph obtained from $H(F)$ be deleting the vertex which was labelled n in H. Clearly $H(F)' = H'(F')$. The situation is represented diagrammatically in Figure 2.

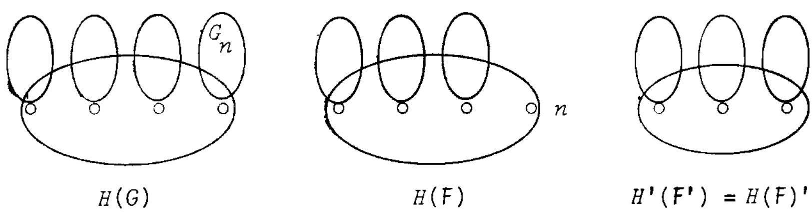

FIGURE 2

It follows at once from Lemma 2.2 that

$$H(G)(\lambda) = G_n(\lambda)H(F)'(\lambda) + G_n'(\lambda)H(F)(\lambda) - \lambda G_n'(\lambda)H(F)'(\lambda) . \tag{1}$$

Now

(2) $$\det A_\lambda(H, G) = \det \left[\begin{array}{c|c} & -h_{1n}G_1'(\lambda) \\ & -h_{2n}G_2'(\lambda) \\ A_\lambda(H', F') & \vdots \\ & -h_{n-1,n}G_{n-1}'(\lambda) \\ \hline -G_n'(\lambda)\cdot \mathbf{h}_n & G_n(\lambda) \end{array}\right],$$

where $\mathbf{h}_n$ denotes the row vector $(h_{n1}, h_{n2}, \ldots, h_{n,n-1})$. Since the determinant of a matrix is a linear function of any row, the right side of (2) can be expressed as

$$\det \left[\begin{array}{c|c} & -h_{1n}G_1'(\lambda) \\ A_\lambda(H', F') & \vdots \\ & -h_{n-1,n}G_{n-1}'(\lambda) \\ \hline -G_n'(\lambda)\cdot \mathbf{h}_n & \lambda G_n'(\lambda) \end{array}\right] + \det \left[\begin{array}{c|c} & -h_{1n}G_1'(\lambda) \\ A_\lambda(H', F') & \vdots \\ & -h_{n-1,n}G_{n-1}'(\lambda) \\ \hline 0 & G_n(\lambda)-\lambda G_n'(\lambda) \end{array}\right],$$

which equals

(3) $$G_n'(\lambda) \det A_\lambda(H, F) + \big(G_n(\lambda)-\lambda G_n'(\lambda)\big) \det A_\lambda(H', F') .$$

By our induction hypothesis $\det A_\lambda(H', F') = H'(F')(\lambda)$, and $\det A_\lambda(H, F) = H(F)(\lambda)$. Hence (3) may be rewritten as

(4) $$G_n'(\lambda)H(F)(\lambda) + G_n(\lambda)H'(F')(\lambda) - \lambda G_n'(\lambda)H'(F')(\lambda) .$$

Since $H'(F') = H(F)'$, a comparison of (4) with (1) shows that we have established the theorem.

We note that on dividing the ith row of $A_\lambda(H, G)$ by $G_i'(\lambda)$ for $i = 1, 2, \ldots, n$, one obtains a matrix of the form $\Lambda - A(H)$, where

$$\Lambda = \operatorname{diag}\left\{\frac{G_1(\lambda)}{G_1'(\lambda)}, \frac{G_2(\lambda)}{G_2'(\lambda)}, \ldots, \frac{G_n(\lambda)}{G_n'(\lambda)}\right\} .$$

Hence

(5) $$\det A_\lambda(H, G) = \det\big(\Lambda-A(H)\big) \cdot \prod_{i=1}^{n} G_i'(\lambda) .$$

In the special case where the G_i are all isomorphic, $\Lambda = \left(G_1(\lambda)/G_1'(\lambda)\right)I$ and so

(6) $$H(G)(\lambda) = G_1'(\lambda)^n H\left(\frac{G_1(\lambda)}{G_1'(\lambda)}\right) .$$

This is the formula given in [4].

Finally if G consists of n copies of P_2, the path on two vertices, one obtains, from (6),

(7) $$H(G)(\lambda) = \lambda^n H\left(\lambda - \frac{1}{\lambda}\right) ,$$

since $P_2(\lambda) = \lambda^2 - 1$, and $P_2'(\lambda) = \lambda$. We will use (7) in the next section.

3. A spectral characterization of a class of trees

NOTATION 3.1. A *matching* of a graph T is a set of mutually non-adjacent edges. An *m-matching* consists of m such edges. A matching M such that every vertex of T is an end vertex of some edge in M is called a *1-factor*.

We recall, from [3] for example, that if T is a tree on n vertices, then

(8) $$T(\lambda) = \sum_{m=0}^{[n/2]} (-1)^m a_{2m} \lambda^{n-2m} ,$$

where a_{2m} is the number of m-matchings of T.

We will use $T(P_2)$ to denote the rooted product of T by the collection consisting of one copy of P_2 for each vertex of T. It follows from (8) that, if T is a tree on n vertices, then $T(-\lambda) = (-1)^n T(\lambda)$, and so from (7) above we find

(9) $$T(P_2)\left(\frac{1}{\lambda}\right) = \left(\frac{-1}{\lambda}\right)^n T\left(\lambda - \frac{1}{\lambda}\right)$$
$$= (-1)^n \lambda^{-2n} T(P_2)(\lambda) .$$

We will call a polynomial of degree $2n$ satisfying (9) *symmetric*.

THEOREM 3.2. *Let T be a tree on $2n$ vertices. Then $T(\lambda)$ is symmetric if and only if $T = S(P_2)$ for some tree S.*

Proof. The sufficiency follows from the remarks above. We give the proof of the necessity in a number of steps.

We assume $n \geq 2$. Let a_{2m} denote the number of m-matchings of T.

(a) T has 1 n-matching and $2n - 1$ $(n-1)$-matchings.

Since $T(\lambda)$ is symmetric we have $a_0 = a_{2n}$ and $a_2 = a_{2n-2}$. But $a_0 = 1$ and a_2 is just the number of edges of T and so the claim follows.

(b) T has n end vertices.

Let M be the n-matching of T. By counting $(n-1)$-matchings we will show that every edge in M contains an end vertex of T.

Say that an $(n-1)$-matching N is of type I if it is a subset of M. Clearly there are n such matchings.

Let v_2v_3 be an edge of T not in M. Then there are vertices v_1 and v_4 of T such that both v_1v_2 and v_3v_4 lie in M. Let N be the $(n-1)$-matching obtained from M by replacing v_1v_2 and v_3v_4 by the edge v_2v_3. We will call N a type II $(n-1)$-matching. The number of type II $(n-1)$-matchings is just the number of edges of T not in M. This equals $n - 1$.

Since a type II $(n-1)$-matching is not a subset of M we have already found $2n - 1$ distinct $(n-1)$-matchings.

Let v_3v_4 be an edge in M such that neither v_3 nor v_4 is an end-vertex. Let v_2 and v_5 be vertices of T adjacent to v_3 and v_4 respectively. Then there exist vertices v_1 and v_6 in T such that v_1v_2 and v_5v_6 lie in M. Replacing the edges v_1v_2, v_3v_4, and v_5v_6 of M by the edges v_2v_3 and v_4v_5 we obtain an $(n-1)$-matching N.

Since $|N \cap M| \leq n - 3$, N is not of type I or II.

Thus the existence of an edge v_3v_4 in M such that neither v_3 nor v_4 is an end vertex of T implies that T has at least $2n$ $(n-1)$-matchings. Hence every edge in M contains at least one end vertex of T . If some edge in M consisted of two adjacent end-vertices, then T would be disconnected. Therefore T must have exactly $|M| = n$ end vertices.

(c) $T = S(P_2)$ for some tree S .

Let S be the tree obtained by removing the n end vertices from T . As T has a 1-factor, it cannot have a vertex adjacent to two end vertices. Hence $T = S(P_2)$. □

We remark that the proof of the theorem actually shows that a tree on $2n$ vertices with an n-matching, and $2n - 1$ $(n-1)$-matchings is a rooted product.

Note that Theorem 3.2 does not hold when the assumption that T is a tree is dropped. For example the graph shown in Figure 3 is obviously not a rooted product, although its characteristic polynomial is
$\lambda^8 - 9\lambda^6 + 16\lambda^4 - 9\lambda^2 + 1$, which is symmetric.

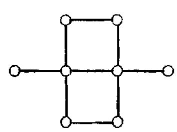

FIGURE 3

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