A new graph product and its spectrum
C.D. Godsil and B.D. McKay

A new graph product is introduced, and the characteristic polynomial of a graph so-formed is given as a function of the characteristic polynomials of the factor graphs. A class of trees produced using this product is shown to be characterized by spectral properties.

1. Notation and preliminaries

All graphs considered in this paper are finite, and without loops and multiple or directed edges. Any undefined graph-theoretical terms will have the meanings given to them in Behzad and Chartrand [1].

If $G$ is a graph with adjacency matrix $A(G)$, then we denote the characteristic polynomial $\det(\lambda I - A(G))$ of $A(G)$ by $G(\lambda)$, and refer to it as the characteristic polynomial of $G$. If $G$ is a rooted graph then we denote by $G'$ the graph obtained from $G$ when the root vertex is removed. The characteristic polynomial of the rooted graph $G$ is just the characteristic polynomial of the unrooted graph with the same vertex and edge sets as $G$.

DEFINITION 1.1. Let $H$ be a labelled graph on $n$ vertices. Let $G$ be a sequence of $n$ rooted graphs $G_1, G_2, \ldots, G_n$. Then by $H(G)$ we denote the graph obtained by identifying the root of $G_i$ with the $i$th vertex of $H$. We call $H(G)$ the rooted product of $H$ by $G$.

Figure 1 illustrates this construction with $H$ the path on three

Received 14 September 1977.
vertices and \( G \) consisting of three copies of the rooted path on two vertices

![Diagram of vertices and graphs](attachment:image)

**FIGURE 1**

**DEFINITION 1.2.** Given a labelled graph \( H \) on \( n \) vertices and a sequence \( G \) of \( n \) rooted graphs, we define the matrix \( A_\lambda(H, G) \) as follows:

\[
A_\lambda(H, G) = \begin{pmatrix}
A_{i,j}
\end{pmatrix}
\]

where

\[
A_{i,j} = \begin{cases} 
G_i(\lambda) , & i = j , \\
-\kappa_{i,j}G'_i(\lambda) , & i \neq j ,
\end{cases}
\]

and \( A(H) = \begin{pmatrix} h_{i,j} \end{pmatrix} \) is the adjacency matrix of \( H \).

If, for example, \( H \) and \( G \) are represented in Figure 1, then \( A_\lambda(H, G) \) is the matrix

\[
\begin{bmatrix}
\lambda^2 - \lambda & -\lambda & 0 \\
-\lambda & \lambda^2 - \lambda & -\lambda \\
0 & -\lambda & \lambda^2 - 1
\end{bmatrix}
\]

**2. The polynomial of the rooted product**

In this section we prove the following:

**THEOREM 2.1.** \( H(G)(\lambda) = \det A_\lambda(H, G) \).

This result has already been proved by Schwenk [4] in the case where \( G \) consists of \( n \) isomorphic rooted graphs. The method we use to prove the result in general is quite different from his, however.

We will need the following lemma.
Lemma 2.2. Let $K$ and $L$ be rooted graphs, and let $K \cdot L$ denote the graph obtained by identifying the roots of $K$ and $L$. Then

$$K \cdot L(\lambda) = K(\lambda)L'(\lambda) + K'(\lambda)L(\lambda) - \lambda K'(\lambda)L'(\lambda).$$

Proof. See Schwenk [4], or Godsil and McKay [2]. □

Proof of Theorem 2.1. We will use induction on the number of vertices of $H(G)$. Suppose this number is $N$, and that the theorem holds for all labelled graphs $H$ and sequences $G$ such that $H(G)$ has less than $N$ vertices. For $n = 1$, the theorem follows from the definition of $A_\lambda (H, G)$, so we assume $n \geq 2$.

Let $F$ denote the sequence of rooted graphs obtained from $G$ by replacing the graph $G_n$ by $K_1$, the graph with only one vertex. $F'$ will be used to denote the subsequence of $G$ consisting of the graphs $G_1, G_2, \ldots, G_{n-1}$. Let $H'$ denote the graph obtained from $H$ by deleting the vertex labelled $n$, and let $H(F)'$ denote the graph obtained from $H(F)$ by deleting the vertex which was labelled $n$ in $H$. Clearly $H(F)' = H'(F')$. The situation is represented diagrammatically in Figure 2.

![Diagram](image)

FIGURE 2

It follows at once from Lemma 2.2 that

$$H(G)(\lambda) = G_n(\lambda)H(F)'(\lambda) + G_n'(\lambda)H(F)(\lambda) - \lambda G_n'(\lambda)H(F)'(\lambda).$$

Now
where $h_n$ denotes the row vector $(h_{n1}, h_{n2}, \ldots, h_{nn-1})$. Since the determinant of a matrix is a linear function of any row, the right side of (2) can be expressed as

$$\det A^\lambda(H', F') = \det A^\lambda(H', F') + \det A^\lambda(H', F')$$

which equals

$$G_n(\lambda) \det A^\lambda(H, F) + \{G_n(\lambda) - \lambda G_n'(\lambda)\} \det A^\lambda(H', F')$$

By our induction hypothesis $\det A^\lambda(H', F') = H'(F')(\lambda)$, and $\det A^\lambda(H, F) = H(F)(\lambda)$. Hence (3) may be rewritten as

$$G_n(\lambda)H(F)(\lambda) + G_n(\lambda)H'(F')(\lambda) - \lambda G_n'(\lambda)H'(F')(\lambda)$$

Since $H'(F') = H(F)'$, a comparison of (4) with (1) shows that we have established the theorem.

We note that on dividing the $i$th row of $A^\lambda(H, G)$ by $G_i(\lambda)$ for $i = 1, 2, \ldots, n$, one obtains a matrix of the form $\Lambda - A(H)$, where

$$\Lambda = \text{diag}\left(\frac{G_1(\lambda)}{G_1'(\lambda)}, \frac{G_2(\lambda)}{G_2'(\lambda)}, \ldots, \frac{G_n(\lambda)}{G_n'(\lambda)}\right)$$

Hence

$$\det A^\lambda(H, G) = \det(\Lambda - A(H)) \cdot \prod_{i=1}^{n} G_i'(\lambda)$$
In the special case where the $G_i$ are all isomorphic, \( \Lambda = \{G_1(\lambda)/G_1'(\lambda)\}I \) and so

\[
H(G)(\lambda) = G'_1(\lambda)\frac{H(G_1(\lambda))}{G_1'(\lambda)}.
\]

This is the formula given in [4].

Finally if $G$ consists of $n$ copies of $P_2$, the path on two vertices, one obtains, from (6),

\[
H(G)(\lambda) = \lambda^n H \left( \lambda - \frac{1}{\lambda} \right),
\]

since $P_2(\lambda) = \lambda^2 - 1$, and $P_2'(\lambda) = \lambda$. We will use (7) in the next section.

3. A spectral characterization of a class of trees

**NOTATION 3.1.** A matching of a graph $T$ is a set of mutually non-adjacent edges. An $m$-matching consists of $m$ such edges. A matching $M$ such that every vertex of $T$ is an end vertex of some edge in $M$ is called a 1-factor.

We recall, from [3] for example, that if $T$ is a tree on $n$ vertices, then

\[
T(\lambda) = \sum_{m=0}^{[n/2]} (-1)^m a_{2m} \lambda^{n-2m},
\]

where $a_{2m}$ is the number of $m$-matchings of $T$.

We will use $T(P_2)$ to denote the rooted product of $T$ by the collection consisting of one copy of $P_2$ for each vertex of $T$. It follows from (8) that, if $T$ is a tree on $n$ vertices, then

\[
\hat{T}(\lambda) = (-1)^n T(\lambda),
\]

and so from (7) above we find

\[
T(P_2)\left(\frac{1}{\lambda}\right) = \left(\frac{-1}{\lambda}\right)^n T\left(\lambda - \frac{1}{\lambda}\right)
= (-1)^n \lambda^{-2n} T(P_2)(\lambda).
\]
We will call a polynomial of degree $2n$ satisfying (9) symmetric.

**Theorem 3.2.** Let $T$ be a tree on $2n$ vertices. Then $T(\lambda)$ is symmetric if and only if $T = S[P_2]$ for some tree $S$.

**Proof.** The sufficiency follows from the remarks above. We give the proof of the necessity in a number of steps.

We assume $n \geq 2$. Let $a_{2m}$ denote the number of $m$-matchings of $T$.

(a) $T$ has 1 $(n-1)$-matching and $2n - 1$ $(n-1)$-matchings.

Since $T(\lambda)$ is symmetric we have $a_0 = a_{2m}$ and $a_2 = a_{2n-2}$. But $a_0 = 1$ and $a_2$ is just the number of edges of $T$ and so the claim follows.

(b) $T$ has $n$ end vertices.

Let $M$ be the $n$-matching of $T$. By counting $(n-1)$-matchings we will show that every edge in $M$ contains an end vertex of $T$.

Say that an $(n-1)$-matching $N$ is of type I if it is a subset of $M$. Clearly there are $n$ such matchings.

Let $v_2v_3$ be an edge of $T$ not in $M$. Then there are vertices $v_1$ and $v_4$ of $T$ such that both $v_1v_2$ and $v_3v_4$ lie in $M$. Let $N$ be the $(n-1)$-matching obtained from $M$ by replacing $v_1v_2$ and $v_3v_4$ by the edge $v_2v_3$. We will call $N$ a type II $(n-1)$-matching. The number of type II $(n-1)$-matchings is just the number of edges of $T$ not in $M$. This equals $n-1$.

Since a type II $(n-1)$-matching is not a subset of $M$ we have already found $2n - 1$ distinct $(n-1)$-matchings.

Let $v_3v_4$ be an edge in $M$ such that neither $v_3$ nor $v_4$ is an end-vertex. Let $v_2$ and $v_5$ be vertices of $T$ adjacent to $v_3$ and $v_4$ respectively. Then there exist vertices $v_1$ and $v_6$ in $T$ such that $v_1v_2$ and $v_5v_6$ lie in $M$. Replacing the edges $v_1v_2$, $v_3v_4$, and $v_5v_6$ of $M$ by the edges $v_2v_3$ and $v_4v_5$ we obtain an $(n-1)$-matching $N$. 
Since \(|N \cap M| \leq n - 3\), \(N\) is not of type I or II.

Thus the existence of an edge \(v_3v_4\) in \(M\) such that neither \(v_3\) nor \(v_4\) is an end vertex of \(T\) implies that \(T\) has at least \(2n\) \((n-1)\)-matchings. Hence every edge in \(M\) contains at least one end vertex of \(T\). If some edge in \(M\) consisted of two adjacent end-vertices, then \(T\) would be disconnected. Therefore \(T\) must have exactly \(|M| = n\) end vertices.

(c) \(T = S(P_2)\) for some tree \(S\).

Let \(S\) be the tree obtained by removing the \(n\) end vertices from \(T\). As \(T\) has a 1-factor, it cannot have a vertex adjacent to two end vertices. Hence \(T = S(P_2)\) .

We remark that the proof of the theorem actually shows that a tree on \(\geq n\) vertices with an \(n\)-matching, and \(\geq n - 1\) \((n-1)\)-matchings is a rooted product.

Note that Theorem 3.2 does not hold when the assumption that \(T\) is a tree is dropped. For example the graph shown in Figure 3 is obviously not a rooted product, although its characteristic polynomial is \(\lambda^8 - 9\lambda^6 + 16\lambda^4 - 9\lambda^2 + 1\), which is symmetric.

\[\text{FIGURE 3}\]

References


Department of Mathematics,
Syracuse University,
Syracuse,
New York,
USA;

Department of Mathematics,
University of Melbourne,
Parkville,
Victoria.