PRODUCTS OF GRAPHS AND THEIR SPECTRA

C. GODSIL AND B. MCKAY

In this paper some new methods of constructing infinite families of cospectral graphs are presented. As an example of their application it is shown that given any graph G on n vertices one can construct at least $\binom{2n-2}{n-2}$ *non-isomorphic pairs of connected cospectral graphs on 3n vertices such that each member of each of the pairs contains three disjoint subgraphs isomorphic to G.*

The same procedure can be used to construct pairs of non-isomorphic and non-cospectral graphs with the same spectral radius containing any two given graphs as disjoint induced subgraphs.

1. INTRODUCTION

Throughout this paper all graphs considered have a finite number of vertices and no loops or multiple edges. All undefined graph theoretic terms will have the meanings given in Harary [2]. Similarly, undefined matrix theoretic terms will be found in Lancaster [5]. All matrices will be assumed to have non-negative integral entries though, with the exception of the statements concerning the spectral radii of graphs, the results are valid without this restriction. When writing down partitioned matrices we use 0 to denote a block with all entries zero.

We take the definition of a *multigraph* to permit both loops and multiple directed edges. The adjacency matrix $A = (a_{ij})$ of a labelled multigraph on n vertices is the $n \times n$ matrix with a_{ij} equal to the number of directed edges going from vertex i to vertex j. This definition is consistent with the definition of the adjacency matrix of a graph. Note that we have a 1-1 correspondence between labelled multigraphs and matrices with non-negative integral entries.

Two graphs (multigraphs) will be called *cospectral* if the characteristic polynomials of their adjacency matrices are the same. We also apply this term to the matrices themselves. The *spectral radius* of a graph (or matrix) is just the largest eigenvalue of its characteristic polynomial. We will refer to two matrices as isomorphic if the labelled graphs (multigraphs) they represent are *isomorphic* (i.e., relabellings of each other). In other words two matrices A and B are isomorphic if there exists a permutation matrix P such that $P^{-1}AP = B$. We write $G(\lambda)$ and $A(\lambda)$ to denote the characteristic polynomial of the graph (or multigraph) G and the matrix A, respectively.

The rest of this paper falls into two parts; the first of which (Section 2) consists of the statements of the main results, and a discussion of some of their consequences. The second part (Section 3) contains the main proofs. These have been presented separately, as they are entirely matrix theoretical and somewhat technical.

2. THE PARTITIONED TENSOR PRODUCT

By Newton's relations (see [6]), the roots of a polynomial are determined, up to ordering, by the sequences of sums of the r^{th} powers of its roots for $r = 0,1,2,\ldots$ Now for any matrix A, $\mathrm{tr}\, A^r$ (the trace of the r^{th} power of A) is just the sum of the r^{th} powers of the characteristic roots of A. Taking $A^0 = I$ (where I is the identity matrix of the same size as A) we have the following result:

Lemma 2.1. *If* A *and* B *are any two matrices, then they are cospectral iff* $\mathrm{tr}\, A^r = \mathrm{tr}\, B^r$ *for* $r = 0,1,2,\ldots$.

Definitions 2.2. We define the *tensor product* of the matrices A and B, where $A = (a_{ij})$ is $m \times n$, to be the matrix consisting of m rows of n blocks where the j^{th} block in the i^{th} row is the matrix $a_{ij}B$. We denote it $A \times B$. The main properties of the tensor product may be found in [5] (where it is called the direct product). Clearly we can define the tensor product of two graphs (or multigraphs) as the graph represented by the tensor product of their adjacency matrices. The properties of the tensor product of graphs are outlined in [3] (where it is called the "conjunction") and in [7] (where it is called the "Kronecker product"). We note here that the graphs represented by the matrices $A \times B$ and $B \times A$ can be shown to be isomorphic, and that the graph represented by $A \times B$ depends only on the graphs represented by the matrices A and B and not on their labelling.

Let

$$L = \begin{bmatrix} U & V \\ W & X \end{bmatrix}, \quad H = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

We assume here, as we shall for the rest of this paper, that the diagonal blocks (U, X, A and D) are all square. We define the *partitioned tensor product of* L *and* H to be the matrix

$$\begin{bmatrix} U \times A & V \times B \\ W \times C & X \times D \end{bmatrix}$$

and denote it by $L \underline{\times} H$. Note that the value of this product depends on the partitioning of L and H. Where ambiguity as to the partitioning arises, we will indicate the intended partitioning by dotted lines.

Examples 2.3. We illustrate the above definition of the partitioned tensor product. Let

$$L = \left[\begin{array}{c|cc} 1 & 1 & 1 \\ \hline 1 & 1 & 0 \\ 1 & 0 & 1 \end{array}\right], \quad H = \begin{bmatrix} A & B \\ B^T & A \end{bmatrix}, \quad H^* = \begin{bmatrix} A & B^T \\ B & A \end{bmatrix}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

and B^T is just the transpose of B. Then H and H* are the adjacency matrices of the labelled graphs G(H) and G(H*) shown in Figure 1. $L \underline{\times} H$ and $L \underline{\times} H^*$ are the adjacency matrices of the graphs $G(L \underline{\times} H)$ and $G(L \underline{\times} H^*)$ respectively and these are shown in Figure 1 also.

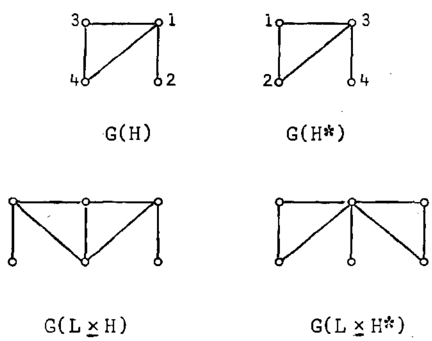

FIGURE 1

We now turn to the statement and proofs of our main results. Let

$$L = \begin{bmatrix} I_m & J \\ K & I_n \end{bmatrix}, \quad H = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad H^* = \begin{bmatrix} D & C \\ B & A \end{bmatrix}$$

where the matrices A and D are square while I_m (I_n) is the $m \times m$ ($n \times n$) identity matrix. We have the following result.

Theorem 2.4. *(a) If* $m = n$, $L \underline{\times} H$ *and* $L \underline{\times} H^*$ *are cospectral*

(b) If $m \neq n$, $L \underline{\times} H$ *and* $L \underline{\times} H^*$ *are cospectral iff* A *and* D *are.*

Proof. By 3.8 we have $\operatorname{tr}[(L \underline{\times} H)^r - (L \underline{\times} H^*)^r] = (m-n)(\operatorname{tr} A^r - \operatorname{tr} D^r)$ for $r = 1,2,3,\ldots$. Applying Lemma 2.1 the result is immediate.

Examples 2.5. By way of illustration of 2.4 we point out that the adjacency matrices of the last two graphs in Figure 1 have the form $L \underline{\times} H$ and $L \underline{\times} H^*$ with L having the form required by 2.4. Hence these two graphs are cospectral. In fact they are the smallest connected cospectral graphs (see [1]). As another example let L be the same matrix as in 2.3, with the same partitioning. Let H be the adjacency matrix of the labelled graph $G(H)$ in Figure 2. Then the graphs represented by the matrices $L \underline{\times} H$ and $L \underline{\times} H^*$ are shown in Figure 2, as $G(L \underline{\times} H)$ and $G(L \underline{\times} H^*)$ respectively.

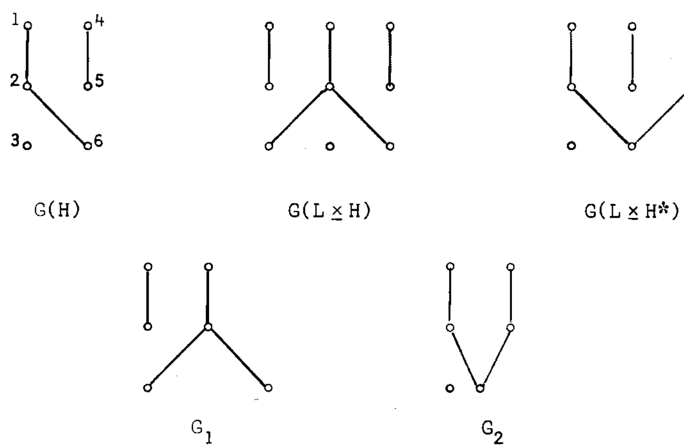

FIGURE 2

Since the characteristic polynomial of a graph is just the product of the characteristic polynomial of its components (see [8]) the graphs G_1 and G_2 obtained from $G(L \underline{\times} H)$ and $G(L \underline{\times} H^*)$ respectively by "cancelling" isomorphic components are cospectral. In fact they are the smallest cospectral forests [1].

Now let

$$K = \left[\begin{array}{cc|cc} A & 0 & B & 0 \\ 0 & D & 0 & 0 \\ \hline C & 0 & D & 0 \\ 0 & 0 & 0 & A \end{array}\right], \qquad K^* = \left[\begin{array}{cc|cc} D & 0 & C & 0 \\ 0 & A & 0 & 0 \\ \hline B & 0 & A & 0 \\ 0 & 0 & 0 & D \end{array}\right] :$$

Then if L is any matrix of the form required by 2.4, $L \underline{\times} K$ and $L \underline{\times} K^*$ are cospectral. But it is easy to show that $L \underline{\times} K$ is isomorphic to the matrix

$$\begin{bmatrix} L \underline{\times} H_1 & 0 \\ 0 & L \underline{\times} H_2 \end{bmatrix}$$

where

$$H_1 = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad \text{and} \quad H_2 = \begin{bmatrix} D & 0 \\ 0 & A \end{bmatrix}$$

In other words, considered as a multigraph, $L \underline{\times} K$ has two components whose adjacency matrices can be represented as $L \underline{\times} H_1$ and $L \underline{\times} H_2$. Now $L \underline{\times} H_2$ in turn can be shown to have two components with adjacency matrices $I_m \times D$ and $I_n \times A$. (We are stretching the definition of components here and above since they are not necessarily connected). Thus the characteristic polynomial of $L \underline{\times} K$ is $(L \underline{\times} H_1)(\lambda)A(\lambda)^n D(\lambda)^m$. Similarly we can show the characteristic polynomial of $L \underline{\times} K^*$ is $(L \times H_1^*)(\lambda)A(\lambda)^m D(\lambda)^n$. But $L \underline{\times} K$ and $L \underline{\times} K^*$ are cospectral so we have shown that

$$(L \underline{\times} H_1^*)(\lambda)A(\lambda)^{m-n} = (L \underline{\times} H_1)(\lambda)D(\lambda)^{m-n} \quad \ldots \ (1).$$

Suppose that the matrices $L \underline{\times} H_1$ and $L \underline{\times} H_1^*$ represent connected graphs. Then the matrices A and D represent induced subgraphs of both, and so by a well-known result (see e.g. [8]) the spectral radii of A and D are strictly less than the spectral radii of $L \underline{\times} H_1$ and $L \underline{\times} H_1^*$. So by (1) $L \underline{\times} H_1$ and $L \underline{\times} H_1^*$ have the same spectral radii. We summarize our conclusions as follows:

Theorem 2.6. *Let* L, H, m *and* n *be as in the statement of* 2.4. *Then*

$$(L \underline{\times} H^*)(\lambda)A(\lambda)^{m-n} = (L \underline{\times} H)(\lambda)D(\lambda)^{m-n}$$

and if $L \underline{\times} H$ *and* $L \underline{\times} H^*$ *represent connected graphs, they have the same spectral radius.*

Thus Theorem 2.6 enables one to construct pairs of non-cospectral (and so non-isomorphic) graphs with the same spectral radius.

Let

$$L_i = \begin{bmatrix} I_{m_i} & J_i \\ K_i & I_{n_i} \end{bmatrix} \quad i = 1,2; \qquad H = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Then we have the following:

Theorem 2.7. *Let* L_1 *and* L_2 *be cospectral. Then*

(a) *if* $m_1 = n_1$, $L_1 \underline{\times} H$ *and* $L_2 \underline{\times} H$ *are cospectral*

(b) *if* $m_1 \neq n_1$, $L_1 \underline{\times} H$ *and* $L_2 \underline{\times} H$ *are cospectral iff* A *and* D *are.*

Proof. By 3.9, $\operatorname{tr}[(L_1 \underline{\times} H)^r - (L_2 \underline{\times} H)^r] = (m_1 - m_2)(\operatorname{tr} A^r - \operatorname{tr} D^r)$ for $r = 1,2,\ldots$

and is zero for $r = 0$. Applying 2.1 the result follows at once.

The argument used to prove 2.6 can be extended to yield:

Theorem 2.8. *With notation and assumptions as in* 2.7 *we have*

$$(L_1 \underline{\times} H)(\lambda)A(\lambda)^{m_1-m_2} = (L_2 \underline{\times} H)(\lambda)D(\lambda)^{m_1-m_2}$$

and if $L_1 \underline{\times} H$ *and* $L_2 \underline{\times} H$ *represent connected graphs, they have the same spectral radius.*

Applications 2.9. The results given in 2.4 and 2.6 could, of course, be trivial in that the graphs represented by the matrices $L \underline{\times} H$ and $L \underline{\times} H^*$ may be isomorphic, and not just cospectral.

For completeness then, we outline a method for obtaining a number of distinct pairs of cospectral non-isomorphic graphs from any given graph G, based on 2.4.

Let d_i be the degree of the i^{th} vertex of G. Label G so that $d_i \geq d_{i+1}$ for $i = 1,2,\ldots,n-1$. Let $\underset{\sim}{a}$ denote the n-tuple of non-negative integers $(a_1,a_2,\ldots,a_n)$ where $a_1 = n$, $a_n = 0$ and $a_i \geq a_{i+1}$ for $i = 1,2,\ldots,n-1$. Let $\underset{\sim}{A}$ denote the set of all such n-tuples and for $\underset{\sim}{a}$ in $\underset{\sim}{A}$ let $s(\underset{\sim}{a})$ denote the sum of the entries of $\underset{\sim}{a}$. Take two copies of G, G_1 and G_2 say, labelled as described above. Let (i,j) denote an edge joining vertex i in G_1 to vertex $j(\text{mod } n)$ where $j \neq 0(\text{mod } n)$ and to vertex n otherwise. Put in $s(\underset{\sim}{a})$ edges as follows: $(1,1),(1,2)\ldots(1,a_1),(2,a_1+1),\ldots,(2,a_1+a_2),(3,a_1+a_2+1),$ $\ldots(m,s(\underset{\sim}{a}))$ where m is the greatest integer such that $a_m \neq 0$. Call this graph $G(\underset{\sim}{a})$ and let L be the same matrix as in 2.3. Then $L \underline{\times} G(\underset{\sim}{a})$ and $L \underline{\times} G^*(\underset{\sim}{a})$ (where the labelling and partitioning of the adjacency matrix of $G(\underset{\sim}{a})$ is such that the diagonal blocks are the adjacency matrices of G_1 and G_2) are cospectral by 2.4. But $L \underline{\times} G(\underset{\sim}{a})$ and $L \underline{\times} G^*(\underset{\sim}{a})$ are non-isomorphic as the maximum degree of a vertex in $L \underline{\times} G(\underset{\sim}{a})$ is d_1+2n, while for $L \underline{\times} G^*(\underset{\sim}{a})$ it is less than or equal to the maximum of the set $\{d_1+n, d_1+2(n-1)\}$, which is strictly less than d_1+2n. If $\underset{\sim}{b}$ is another element of $\underset{\sim}{A}$ then it can be shown by comparing degrees that $L \underline{\times} G(\underset{\sim}{a})$ and $L \underline{\times} G(\underset{\sim}{b})$ are non-isomorphic unless $\underset{\sim}{a} = \underset{\sim}{b}$. The same holds, naturally, for $L \underline{\times} G^*(\underset{\sim}{a})$ and $L \underline{\times} G^*(\underset{\sim}{b})$. Finally the maximum degrees of $L \underline{\times} G(\underset{\sim}{a})$ and $L \underline{\times} G^*(\underset{\sim}{b})$ are always different.

Thus we have at least as many pairs of non-isomorphic cospectral graphs as there are elements of $\underset{\sim}{A}$. This number can be shown to be $\binom{2n-2}{n-2}$ which is asymptotically $4^{n-1}/\sqrt{\pi n}$.

Let

$$M = \begin{bmatrix} O & J \\ K & O \end{bmatrix}, \quad N = \begin{bmatrix} O & B \\ C & O \end{bmatrix}, \quad N^* = \begin{bmatrix} O & C \\ B & O \end{bmatrix}.$$

Then one can easily show that, considered as a multigraph, $M \times N$ has two components with adjacency matrices $M \underline{\times} N$ and $M \underline{\times} N^*$. Let

$$M' = \begin{bmatrix} I_m & J \\ K & I_n \end{bmatrix}.$$

Then $M' \underline{\times} N = M \underline{\times} N$, and $M' \underline{\times} N^* = M \underline{\times} N^*$. So by 2.6 $M \underline{\times} N$ and $M \underline{\times} N^*$ have their characteristic polynomials differing only by a power of λ and therefore are cospectral if they have the same size.

Thus in general we can regard $M \underline{\times} N$ and $M \underline{\times} N^*$ as "cospectral but for zeros". In the special case where $K^T = J$, $C^T = B$ and all the matrices are (0-1) matrices the graphs represented by them are all bipartite. Thus we can show that the two components of the tensor product of two connected bipartite graphs are always "cospectral but for zeros" and are cospectral when they have the same number of vertices. (Of course these components may be isomorphic.)

Finally we point out that we have found examples of graphs, not capable of non-trivial representations as partitioned tensor products, cospectral to graphs which do admit such representations.

3. PROOFS

Our aim in this section is to prove Lemmas 3.8 and 3.9 which we used to derive 2.4 and 2.6 - 2.8. Throughout this section we let

$$H = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

and assume, as usual, that A and D are square.

We introduce some new notation. Let

$$I(A,D) = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}, \qquad P(B,C) = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}.$$

Thus we may write $H = I(A,D) + P(B,C)$.

We list the following properties of I and P.

<u>Lemma 3.1.</u> *For* $r = 1,2,3,\ldots$ *we have*

(a) $I(A,D)^r = I(A^r,D^r)$

(b) $P(B,C)^{2r} = I\left((BC)^r,(CB)^r\right)$

(c) $P(B,C)^{2r+1} = P\big((BC)^r B,(CB)^r C\big)$

(d) $I(A,D)P(B,C) = P(AB,DC)$

(e) $P(B,C)I(A,D) = P(BD,CA)$.

Proofs. (a), (d) and (e) are direct consequences of the definitions while (b) and (c) follow from (a), (d) and (e) and the observation that $P(B,C)^2 = I(BC,CB)$. Our assumption that H is partitioned so that A and D are square ensures that all the matrix products are defined.

We now note that for arbitrary $m \times m$ matrices X and Y $(X+Y)^r$ can be written as a sum of monomials in X and Y. Relative to some ordering we denote these monomials by $f_i^{(r)}(X,Y)$, $i = 1,2,\ldots,2^r$; e.g., $(X+Y)^2 = X^2+XY+YX+Y^2$ so we can define $f_1^{(2)}(X,Y) = X^2$, $f_2^{(2)}(X,Y) = XY$, $f_3^{(2)} = YX$ and $f_4^{(2)} = Y^2$. We can regard each monomial as a function in two variables X and Y and will denote it in this case simply as $f_i^{(r)}$. We will also use $f_1^{(r)}$ to denote X^r.

For convenience we assume that f denotes some fixed monomial $f_i^{(r)}$ from now to the end of Lemma 3.7. We now derive some properties of f.

Lemma 3.2.

$$f\big(I(A,D),P(B,C)\big) = \begin{bmatrix} g_{11}(A,B,C,D) & g_{12}(A,B,C,D) \\ g_{21}(A,B,C,D) & g_{22}(A,B,C,D) \end{bmatrix}$$

where the g_{ij} $(i = 1,2;\ j = 1,2)$ *are monomials in* A, B, C *and* D.

Proof. The result follows trivially from 3.1.

In the following we find it convenient to abbreviate $f\big(I(A,D),P(B,C)\big)$ and $g_{ij}(A,B,C,D)$ as $f(H)$ and $g_{ij}(H)$ respectively, since no ambiguity can arise.

Lemma 3.3. *Let*

$$M = \begin{bmatrix} U & V \\ W & X \end{bmatrix}$$

where U *and* X *are square. Then we have*

$$f\big(I(U\times A, X\times D),\ P(V\times B, W\times C)\big) = f\big(I(U,X),\ P(V,W)\big) \underline{\times} f\big(I(A,D),P(B,C)\big).$$

(Making the obvious adjustments in notation we write this as $f(M \underline{\times} H) = f(M) \underline{\times} f(H)$.)

Proof. For arbitrary matrices Q, R, S and T we have $(Q\times R)(S\times T) = QS \times RT$, if the products exist (see [5] p. 257). Since by 3.2 the g_{ij} are monomials this fact implies the assertion of the lemma.

Lemma 3.4. *Let*

$$M_i = \begin{bmatrix} U_i & V_i \\ W_i & X_i \end{bmatrix}, i = 1, 2$$

and let notation be as in 3.2 *and* 3.3 *otherwise. Then*

$$\operatorname{tr}[f(M_1 \underline{\times} H) - f(M_2 \underline{\times} H)] = \operatorname{tr} g_{11}(H)\big(\operatorname{tr} g_{11}(M_1) - \operatorname{tr} g_{11}(M_2)\big) + \operatorname{tr} g_{22}(H)\big(\operatorname{tr} g_{22}(M_1) - \operatorname{tr} g_{22}(M_2)\big).$$

Proof. By 3.3 we have $f(M_i \underline{\times} H) = f(M_i) \underline{\times} f(H)$ and so

$$\begin{aligned} \operatorname{tr} f(M_i \underline{\times} H) &= \operatorname{tr} \big(f(M_i) \underline{\times} f(H)\big) \\ &= \operatorname{tr} [g_{11}(M_i) \times g_{11}(H) + g_{22}(M_i) \times g_{22}(H)], i = 1,2. \end{aligned}$$

But $\operatorname{tr}(X \times Y) = \operatorname{tr} X \cdot \operatorname{tr} Y$ for arbitrary matrices X and Y (see [5] p. 258). Using this we obtain the statement of the lemma.

Let H^* be the matrix

$$\begin{bmatrix} D & C \\ B & A \end{bmatrix}.$$

Assume D is a $k \times k$ matrix and that A is $\ell \times \ell$. Let I_k and I_ℓ be the $k \times k$ and $\ell \times \ell$ identity matrices respectively. Let Q be the permutation matrix $P(I_k, I_\ell)$. Then if K, say, is any matrix of the same size and with same partition as H, $Q^{-1}KQ = K^*$.

Lemma 3.5. *With notation as in* 3.2 *we have* $g_{11}(H) = g_{22}(H^*)$ *and* $g_{12}(H) = g_{21}(H^*)$.

Proof. Since f is a monomial, $Q^{-1}f(H)Q = f(Q^{-1}HQ) = f(H^*)$. But by the definition of Q

$$Q^{-1}f(H)Q = \begin{bmatrix} g_{22}(H) & g_{21}(H) \\ g_{12}(H) & g_{11}(H) \end{bmatrix}.$$

Comparing this matrix with the one given for $f(H^*)$ by 3.2, the lemma follows immediately.

Lemma 3.6. *With notation as in* 3.3 *and* 3.5 *we have*

$$\operatorname{tr}[f(M \underline{\times} H) - f(M \underline{\times} H^*)] = \big(\operatorname{tr} g_{11}(M) - \operatorname{tr} g_{22}(M)\big)\big(\operatorname{tr} g_{11}(H) - \operatorname{tr} g_{22}(H)\big).$$

Proof. This follows from the application of the identities in 3.5 to Lemma 3.4.

Now let

$$L_i = \begin{bmatrix} I_{m_i} & J_i \\ K_i & I_{n_i} \end{bmatrix}, i = 1,2$$

where I_{m_i} (I_{n_i}) is just the $m_i \times m_i$ ($n_i \times n_i$) identity matrix. We define the *degree of* X in the monomial f to be the sum of the exponents of the powers of X in the expansion of f(X,Y) (e.g., the degree of X in $f(X,Y) = X^3YXY^2$ is four). Similarly we have of course the degree of Y in f. We call the sum of the degree of X and the degree of Y the *total degree* of f. The following properties of L_1 (and L_2) are essential to our main results.

Lemma 3.7. *Let the notation be as in* 3.2 *and* 3.5. *Let* s *be the degree of* Y *in* f, *let* t *be the total degree of* f. *Then*

(a) *if* $s \neq 0, \operatorname{tr} g_{11}(L_1) = \operatorname{tr} g_{22}(L_2)$,

(b) *if* $s = 0, \operatorname{tr} g_{11}(L_1) = m_1, \quad \operatorname{tr} g_{22}(L_1) = n_1$,

(c) *if* $s \neq 0, \operatorname{tr} g_{11}(L_1) = \operatorname{tr} g_{22}(L_2)$ *for all monomials* f *iff* L_1 *and* L_2 *are cospectral.*

Proof. Note that $f(L_1) = f\big(I(I_{m_1}, I_{n_1}), P(J_1,K_1)\big)$. Now the arguments of f commute, so $f(L_1)$ is just $P(J_1,K_1)^s$ when $s \neq 0$. Applying the identities in 3.1 we see $\operatorname{tr} g_{11}(L_1) = \operatorname{tr} g_{22}(L_1) = 0$ if s is odd. So assume $s = 2r$, $r \neq 0$. Again by 3.1 we have $g_{11}(L_1) = (J_1K_1)^r$ and $g_{22}(L_1) = (K_1J_1)^r$.

Now for any matrices X and Y we have tr (XY) = tr (YX) whenever the products are defined (see e.g., [4]). We have, then

$$\operatorname{tr} (XY)^n = \operatorname{tr} \big(X.(YX)^{n-1}Y\big) = \operatorname{tr} \big((YX)^{n-1}Y.X\big) = \operatorname{tr} (YX)^n.$$

Hence $\operatorname{tr} (J_1K_1)^r = \operatorname{tr} (K_1J_1)^r$ and so (a) holds for all $s \neq 0$, odd or even.

If $s = 0$, $f(L_1) = I(I_{m_1}, I_{n_1})$ and so $\operatorname{tr} g_{11}(L_1) = m_1$, $\operatorname{tr} g_{22}(L_1) = n_1$ and (b) is proved.

By 2.1, L_1 and L_2 are cospectral iff $m_1+n_1 = m_2+n_2$ and $\operatorname{tr} (L_1^t) = \operatorname{tr} (L_2^t)$ for $t = 1,2,3,\ldots$. It is easily shown that this is equivalent to the requirement that $\operatorname{tr} \big(P(J_1,K_1)^t\big) = \operatorname{tr} \big(P(J_2,K_2)^t\big)$ for $t = 1,2,3,\ldots$, i.e., that $\operatorname{tr} f(L_1) = \operatorname{tr} f(L_2)$ for all monomials f. As $\operatorname{tr} f(L_1) = \operatorname{tr} g_{11}(L_1) + \operatorname{tr} g_{22}(L_1)$, by (a) and (b) this implies $\operatorname{tr} g_{11}(L_1) = \operatorname{tr} g_{11}(L_2)$ and so (c) holds.

Lemma 3.8. *Let* H *and* H* *be as in* 3.5. *Let* $L = L_1$ *be as in* 3.7 *(with* $m = m_1$, $n = n_1$*). Then* $\operatorname{tr} [(L \underline{\times} H)^t - (L \underline{\times} H^*)^t] = (m-n)(\operatorname{tr} A^t - \operatorname{tr} D^t)$ *(and is zero for* $t = 0$*).*

Proof. $\operatorname{tr}\,(L \underline{\times} H)^t$ can be expressed as a sum of terms $\operatorname{tr} f(L \underline{\times} H)$ by our definition of the monomials f, and these monomials will all have total degree t. By 3.6 we have

$$\operatorname{tr}\,[f(L \underline{\times} H) - f(L \underline{\times} H^*)] = \big(\operatorname{tr} g_{11}(L) - \operatorname{tr} g_{22}(L)\big)\big(\operatorname{tr} g_{11}(H) - \operatorname{tr} g_{22}(H)\big) \;\ldots\; (1).$$

By 3.7 (a) the L.S. is zero if s, the degree of Y in $f(X,Y)$, is non-zero. If $s = 0$, $f(X,Y) = X^t$ and $g_{11}(H) = A^t$, $g_{22}(H) = D^t$, $g_{11}(L) = I_m$, $g_{22}(L) = I_n$. Substituting these values in (1) we arrive at the statement of the lemma.

Lemma 3.9. *Let* L_1, L_2 *be as in* 3.7 *with* L_1 *and* L_2 *cospectral. Let* H *be as in* 3.5. *Then* $\operatorname{tr}\,[(L_1 \underline{\times} H)^t - (L_2 \underline{\times} H)^t] = (m_1-m_2)(\operatorname{tr} A^t - \operatorname{tr} D^t)$ $\quad t = 1,2,3,\ldots$ *(and is zero for* $t = 0$*)*.

Proof. From 3.4 we have

$$\operatorname{tr}\,[f(L_1 \underline{\times} H) - f(L_2 \underline{\times} H)] = \operatorname{tr} g_{11}(H)\big(\operatorname{tr} g_{11}(L_1) - \operatorname{tr} g_{11}(L_2)\big)$$
$$+ \operatorname{tr} g_{22}(H)\big(\operatorname{tr} g_{22}(L_1) - \operatorname{tr} g_{22}(L_2)\big) \ldots \quad (2)$$

By 3.7 (c) we may assume without loss that $f(X,Y) = X^t$. Then we have $g_{11}(H) = A^r$, $g_{22}(H) = D^r$, $g_{11}(L_i) = I_{m_i}$, $g_{22}(L_i) = I_{n_i}$ $(i = 1,2)$. As L_1 and L_2 are cospectral $m_1+n_1 = m_2+n_2$, hence $m_1-m_2 = n_2-n_1$. Substituting these values in (2), we obtain the lemma.

Although, as stated in the introduction, we have only considered matrices with non-negative integral entries, Lemmas 3.8 and 3.9 will hold for matrices over more general rings, e.g., the complex numbers. We also mention that it is possible to generalize these results to matrices with a 3×3 partition, but the proofs, while conceptually the same, required more complicated notation.

REFERENCES

[1] C. Godsil and B. McKay, Some computational results on the spectra of graphs, *This volume.*

[2] F. Harary, *Graph Theory.* (Addison Wesley, 1969).

[3] F. Harary and G.W. Wilcox, Boolean operations on graphs, *Math. Scand.* 20 (1967), 41-51.

[4] F.E. Hohn, *Elementary Matrix Algebra.* 2nd edition (MacMillan, 1964).

[5] P. Lancaster, *Theory of Matrices.* (Academic Press, 1969).

[6] J.V. Uspensky, *Theory of Equations.* (McGraw-Hill, 1948).

[7] P.M. Weichsel, The Kronecker product of graphs, *Proc. Amer. Math. Soc.* 13 (1963), 47-52.

[8] R.J. Wilson, On the adjacency matrix of a graph, in *Combinatorics* (Proc. Conf. Comb. Maths., Oxford) (Inst. Math. App. 1972), 295-321.

Department of Mathematics,
University of Melbourne,
Parkville, VIC.