DETERMINANTS AND RANKS OF RANDOM MATRICES OVER $\mathbb{Z}_m$

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Let $\mathbb{Z}_m$ be the ring of integers modulo $m$. The $m$-rank of an integer matrix is the largest order of a square submatrix whose determinant is not divisible by $m$. We determine the probability that a random rectangular matrix over $\mathbb{Z}_m$ has a specified $m$-rank and, if it is square, a specified determinant. These results were previously known only for prime $m$.

1. Introduction

Let $m$ be an integer. The $m$-rank of an integer matrix $A$ is the greatest integer $k$ such that $A$ has a $k \times k$ submatrix (not necessarily contiguous) whose determinant is nonzero (mod $m$), or 0 if there is no such submatrix. If $m$ is a prime, the $m$-rank is the usual rank over the field GF($m$). In this paper we investigate the $m$-rank when the entries are chosen at random, independently and uniformly, from $\mathbb{Z}_m = \{0, 1, \ldots, m-1\}$. Our results appear to be new except for the case when $m$ is a prime. For corresponding results when $A$ is constrained to be symmetric, see [3].

We begin with some notation. For integer $n \geq 0$ and indeterminate $q$, define $\Pi_n(q) = (1 - q)(1 - q^2) \cdots (1 - q^n)$. In particular, $\Pi_0(q) = 1$. For integers $0 \leq k \leq n$, define

$$\binom{n}{k} = \frac{\Pi_n(q)}{\Pi_k(q) \Pi_{n-k}(q)}.$$

The polynomials $[\binom{n}{k}]$ are called Gaussian coefficients or $q$-binomial coefficients and have many combinatorial interpretations. For example, $[\binom{n}{k}]$ is the number of sub-spaces of dimension $k$ in a vector space of dimension $n$ over a field of $q$ elements. Gaussian coefficients are also of interest as generalizations of ordinary binomial coefficients, since $[\binom{n}{k}] \rightarrow (\binom{n}{k})$ as $q \rightarrow 1$. Expositions of the theory of Gaussian coefficients can be found in [1], [2] and [5].

For integers $n \geq 1$, $\Delta \geq 0$, $0 \leq \delta \leq n$ and $m \geq 1$, define $P_{\delta,0}(n, m)$ to be the probability that a random $(n + \Delta) \times n$ matrix over $\mathbb{Z}_m$ has $m$-rank $n - \delta$. It will also be convenient to define $P_{\delta,0}(0, m) = 1$.
The value of $P_{\Delta, \delta}(n, m)$ has previously been determined for prime $m$, as shown by the following theorem [4, 6].

**Theorem 1.1.** Let $n \geq 0$, $\Delta \geq 0$, $0 \leq \delta \leq n$ and let $p$ be a prime. Define $q = 1/p$. Then

$$P_{\Delta, \delta}(n, p) = q^{\delta(\delta + \Delta)} \left[ \frac{n}{\delta} \right] \Pi_{n+\Delta}(q) \left/ \Pi_{\delta+\Delta}(q) \right..$$

Theorem 1.1 is also true if, instead of $\mathbb{Z}_p$ with $p$ prime, we use any field of $p$ elements, whether or not $p$ is prime. Note that Theorem 1.1 disproves the result claimed by [7].

When $m$ is not a prime, the evaluation of $P_{\Delta, \delta}(n, m)$ becomes more involved because we are no longer working over a field. However, it is not difficult to show that we can restrict our attention to the case when $m$ is a prime power. For $-1 \leq \delta \leq n$, define

$$Q_{\Delta, \delta}(n, m) = \sum_{j=\delta+1}^{n} P_{j, \delta}(n, m).$$

**Lemma 1.1.** Suppose $m = p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where $p_1, p_2, \ldots, p_k$ are distinct primes. Then

$$Q_{\Delta, \delta}(n, m) = \prod_{i=1}^{k} Q_{\Delta, \delta}(n, p_i^{\alpha_i}).$$

**Proof.** The $m$-rank of a random matrix over $\mathbb{Z}_m$ is less than $n - \delta$ if and only if the $p_i^{\alpha_i}$-rank is less than $n - \delta$ for $i = 1, 2, \ldots, k$. By the Chinese Remainder Theorem, the latter events are independent. \qed

2. The full rank case

In this section we consider the case $\delta = 0$, i.e., we consider the probability $P_{\Delta, 0}(n, p^m)$ that a random $(n + \Delta) \times n$ matrix over $\mathbb{Z}_{p^m}$ has full $p^m$-rank, where $p$ is a prime. Results for a general modulus $m = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ are easily deduced from the multiplicative property of $Q$ stated in Lemma 1.1.

The principal tool for this section and the next will be Gaussian elimination. We begin with a simple lemma which has enough generality to cover both cases.

**Lemma 2.1.** Let $A$ be an $N \times n$ integer matrix with rows $R_1, R_2, \ldots, R_N$. For some integers $i, j, \alpha$ where $1 \leq i, j \leq N$ and $i \neq j$, form the $N \times n$ matrix $A'$ from $A$ by executing the row-operation $R_i := R_i - \alpha R_j$. Then, for any integers $m \geq 1$ and $t \geq 1$, $A$ has a $t \times t$ submatrix with nonzero determinant mod $m$ if and only if $A'$ has such a submatrix.
**Proof.** Suppose that \( B = A[r_1, r_2, \ldots, r_t; c_1, c_2, \ldots, c_t] \) is such a submatrix of \( A \), where the notation indicates that \( B = (b_{uv}) \), where \( b_{uv} = a_{uv} \) for \( 1 \leq u, v \leq t \).

The determinant of \( B' = A'[r_1, r_2, \ldots, r_i] \) is the same as that of \( B \) if \( i, j \in \{r_1, r_2, \ldots, r_t\} \) or \( i \notin \{r_1, r_2, \ldots, r_t\} \). Suppose instead that \( r_1 = i \) but \( j \notin \{r_2, \ldots, r_t\} \). Define \( B'' = A'[j, r_2, \ldots, r_i, c_1, c_2, \ldots, c_t] \). Then we have \( \det B' = \alpha \) \( \det B'' \). Since \( \det B \neq 0 \pmod m \), we must either have \( \det B' \neq 0 \pmod m \) or \( \det B'' \neq 0 \pmod m \).

Lemma 2.1 can be used to derive a 3-term recurrence from which \( P_{\Delta,0}(n, p^\mu) \) can be determined, using the boundary conditions \( P_{\Delta,0}(0, p^\mu) = 1 \) (\( \mu \geq 1 \)) and \( P_{\Delta,0}(n, 1) = 0 \). Here and below we write \( q = 1/p \).

**Lemma 2.2.** If \( n > 0, \Delta \geq 0 \) and \( \mu \geq 0 \), then

\[
P_{\Delta,0}(n, p^{\mu+1}) = (1 - q^{n+\Delta}) P_{\Delta,0}(n-1, p^{\mu+1}) + q^{n+\Delta} P_{\Delta,0}(n, p^\mu). \tag{2.1}
\]

**Proof.** Let \( A \) be a random \((n+\Delta) \times n\) matrix over \( \mathbb{Z}_{p^{\mu+1}} \). There are two cases. With probability \( q^{n+\Delta} \), the first column of \( A \) is divisible by \( p \). In this case, we may obtain a random matrix \( A' \) by dividing the first column of \( A \) by \( p \) and adding random multiples of \( p^\mu \) to that column. Clearly \( A \) has full \( p^{\mu+1} \)-rank if and only if \( A' \) has full \( p^n \)-rank. The (conditional) probability of this is \( P_{\Delta,0}(n, p^\mu) \).

The remaining case, which occurs with probability \( 1 - q^{n+\Delta} \), is that the first column of \( A \) is not divisible by \( p \). Since \( p \) is prime, we can apply a row interchange (if necessary) and a sequence of row operations of the form considered by Lemma 2.1, until \( A \) is reduced to the form

\[
\begin{bmatrix}
b_1 & b_2 & \cdots & b_n \\
0 & & & A''
\end{bmatrix},
\]

where \( b_1 \neq 0 \pmod p \). By Lemma 2.1, \( A \) has full \( p^{\mu+1} \)-rank if and only if \( A'' \) has full \( p^{\mu+1} \)-rank. Since \( A'' \) is clearly a random \((n+\Delta-1) \times (n-1)\) matrix over \( \mathbb{Z}_{p^{\mu+1}} \), this happens with probability \( P_{\Delta,0}(n-1, p^{\mu+1}) \). The result follows.

From Lemma 2.2 we can obtain several explicit expressions for \( P_{\Delta,0} \) as sums of polynomials in \( q \).

**Theorem 2.1.** If \( n \geq 1, \Delta \geq 0 \) and \( \mu \geq 0 \), then

\[
P_{\Delta,0}(n, p^{\mu+1}) = \frac{\Pi_{\Delta+n}(q)}{\Pi_{\Delta}(q)} \sum_{k=0}^{\Delta} q^{k(\Delta+1)} \left[ \begin{array}{c} n+k-1 \\ k \end{array} \right] \tag{2.2}
\]

\[
= \frac{\Pi_{\mu+n}(q)}{\Pi_{\mu}(q)} \sum_{k=0}^{\Delta} q^{k(\mu+1)} \left[ \begin{array}{c} n+k-1 \\ k \end{array} \right] \tag{2.3}
\]

\[
= 1 - \frac{q^{(\Delta+1)(\mu+1)}}{\Pi_{\Delta}(q) \Pi_{\mu}(q)} \sum_{k=0}^{n-1} q^k \Pi_{\Delta+k}(q) \Pi_{\mu+k}(q) / \Pi_k(q). \tag{2.4}
\]
Proof. Expression (2.2) gives the correct values for $\mu = 0$ or $n = 1$. Furthermore, for $\mu \geq 1$,

$$\frac{\Pi_{\Delta+n}(q)}{\Pi_{\Delta}(q)} \mu \sum_{k=0}^\mu q^{k(\Delta+1)} \left[ \frac{n+k-1}{k} \right] - (1-q^{n+\Delta}) \frac{\Pi_{\Delta+n-1}(q)}{\Pi_{\Delta}(q)} \sum_{k=0}^\mu q^{k(\Delta+1)} \left[ \frac{n+k-2}{k} \right]$$

$$- q^{n+\Delta} \frac{\Pi_{\Delta+n}(q)}{\Pi_{\Delta}(q)} \sum_{k=0}^{\mu-1} q^{k(\Delta+1)} \left[ \frac{n+k-1}{k} \right]$$

$$= \frac{\Pi_{\Delta+n}(q)}{\Pi_{\Delta}(q)} \left( \sum_{k=0}^\mu q^{k(\Delta+1)} \frac{\Pi_{n+k-1}(q)}{\Pi_k(q)} - \sum_{k=0}^\mu (1-q^{n-1}) q^{k(\Delta+1)} \frac{\Pi_{n+k-2}(q)}{\Pi_k(q)} \right)$$

$$- q^{n-1} \sum_{k=0}^{\mu-1} q^{k(\Delta+1)} \frac{\Pi_{n+k-1}(q)}{\Pi_k(q)}$$

$$= \frac{\Pi_{\Delta+n}(q) q^{n-1}}{\Pi_{\Delta}(q) \Pi_{n-1}(q)} \left( \sum_{k=0}^\mu (1-q^k) q^{k(\Delta+1)} \frac{\Pi_{n+k-2}(q)}{\Pi_k(q)} - \sum_{k=1}^\mu q^{k(\Delta+1)} \frac{\Pi_{n+k-2}(q)}{\Pi_{k-1}(q)} \right)$$

$$= 0,$$

so (2.1) is satisfied as well. Equation (2.2) follows by induction.

To establish (2.4), note from (2.2) and (2.1) that

$$P_{\Delta,0}(n, p^{\mu+1}) - P_{\Delta,0}(n, p^\mu) = q^{\mu(\Delta+1)} \frac{\Pi_{\Delta+n}(q) \Pi_{n+\mu-1}(q)}{\Pi_{\Delta}(q) \Pi_{\mu}(q) \Pi_{n-1}(q)},$$

and

$$P_{\Delta,0}(n, p^{\mu+1}) - q^{n+\Delta} P_{\Delta,0}(n, p^\mu) = (1-q^{n+\Delta}) P_{\Delta,0}(n-1, p^{\mu+1}).$$

Eliminating $P_{\Delta,0}(n, p^\mu)$ yields

$$P_{\Delta,0}(n, p^{\mu+1}) = P_{\Delta,0}(n-1, p^{\mu+1}) - q^{n+\Delta+\mu+\Delta} \frac{\Pi_{n+\Delta-1}(q) \Pi_{n+\mu-1}(q)}{\Pi_{\Delta}(q) \Pi_{\mu}(q) \Pi_{n-1}(q)},$$

from which (2.4) follows by induction.

Noting that (2.4) is symmetric in $\Delta$ and $\mu$, (2.3) follows immediately from (2.2). $\Box$

Note that the identity (2.2) = (2.4) is also true if $\Delta$ is not an integer, provided that we interpret $\Pi_{x+1}(q)/\Pi_x(q) = (1-q^{x+1})(1-q^{x+2}) \cdots (1-q^{x+t})$ for integer $t \geq 0$. The proof is the same. One of the referees has noticed that the identities (2.2) = (2.3) = (2.4) can also be derived from Heine’s Transformation (see [8, eq. 4.7] and [1, p. 19]).

Comparison of (2.2) and (2.3), or examination of (2.4), reveals the following interesting symmetry, for which we do not have a direct combinatorial explanation.
Corollary 2.1. For $n \geq 0$, $\Delta \geq 0$ and $\mu \geq 0$, we have

$$P_{\Delta,0}(n, p^{\mu+1}) = P_{\mu,0}(n, p^{\Delta+1}).$$

Corollary 2.2. Let $A$ be a random $n \times n$ matrix over $\mathbb{Z}_p^n$. Then, for $0 \leq i \leq p^n - 1$,

$$\text{Prob}(\det A \equiv i \pmod{p^n}) = \begin{cases} 
q^{i - \frac{1}{p} \Pi_{n-k-1}(q)} & \text{for } i \neq 0, \gcd(i, p^n) = p^k; \\
1 - \frac{\Pi_{n+\mu-1}(q)}{\Pi_{\mu-1}(q)} & \text{for } i = 0.
\end{cases}$$

Proof. By multiplying the first row of $A$ by numbers prime to $p^n$, it is easy to show that two determinant values (mod $p^n$) are equally likely if they are divisible by the same powers of $p$. The corollary now follows from (2.3). □

The Chinese Remainder Theorem can be used to extend Corollary 2.2 to arbitrary moduli.

3. The general case

In this section we determine $P_{\Delta,0}(n, p^{\mu})$ where $p$ is prime. As in Section 2, the result for general modulus follows from Lemma 1.1.

In order to derive a recurrence for $Q_{\Delta,0}(n, p^{\mu})$, we need to generalize it. For $0 \leq d \leq n$, define $Q_{\Delta,0}^{(d)}(n, p^{\mu})$ to be the probability that an $(n + \Delta) \times n$ random matrix $A$ over $\mathbb{Z}_p^n$ has $p^{\mu}$-rank less than $n - \delta$, subject to the event that the first $n - d$ columns of $A$ are divisible by $p$. In particular, $Q_{\Delta,0}^{(n)}(n, p^{\mu}) = Q_{\Delta,0}(n, p^{\mu})$. Let $q = 1/p$ as before.

Lemma 3.1. Suppose that $\Delta \geq 0$, $\mu \geq 0$, $0 \leq \delta \leq n$ and $0 \leq d \leq n$. Then

$$Q_{\Delta,0}^{(d)}(n, p^{\mu}) = \begin{cases} 
0, & \text{if } \delta = n, \\
1, & \text{if } \delta < n, \mu + \delta - n + d \leq 0, \\
Q_{\Delta,0}^{(d)}(n, p^{\mu+\delta-n}), & \text{if } d = 0, \delta < n, \mu + \delta - n > 0, \\
q^{n+\Delta}Q_{\Delta,0}^{(d-1)}(n, p^{\mu}) + (1 - q^{n+\Delta})Q_{\Delta,0}^{(d-1)}(n - 1, p^{\mu}), & \text{otherwise.}
\end{cases}$$

(3.1)

(3.2)

(3.3)

(3.4)

Proof. (3.1) follows from the definition of $Q$. To obtain (3.2), note that, since at least $n - d$ columns of $A$ are divisible by $p$, any $(n - \delta) \times (n - \delta)$ submatrix of $A$ has at least $n - \delta - d$ columns divisible by $p$. To obtain (3.3), divide every matrix entry by $p$.

Under the stated conditions for (3.4), there are two possibilities. With probability $q^{n+\Delta}$, the $(n - d + 1)$th column is divisible by $p$. If not, we can choose an element which is not divisible by $p$ in the $(n - d + 1)$th column and perform one phase of Gaussian elimination, just as in Lemma 2.2. □
Our next task is the elimination of the variable $d$. For notational convenience, define $Q_{\Delta, \delta}(n, p^\mu) = 1$ for $\mu \leq 0$. The following theorem generalises Theorem 1.1.

**Theorem 3.1.** For $\Delta \geq 0$, $\mu \geq 1$, $n \geq 1$, and $-1 \leq \delta \leq n$,

$$Q_{\Delta, \delta}(n, p^\mu) = \sum_{t=\delta+1}^{n} q^{n+\Delta} \frac{\Pi_{t+\Delta}(q)}{\Pi_{t+\Delta}(q)} Q_{\Delta, \delta}(t, p^\mu + \delta - t).$$

**Proof.** Define

$$R^{(\delta)}(n, p^\mu) = \frac{\Pi_{\delta+\Delta}(q)}{\Pi_{n+\Delta}(q)} (1 - Q_{\Delta, \delta}(n, p^\mu)).$$

Equations (3.1)–(3.4) can now be written thus:

$$R^{(\delta)}(n, p^\mu) = \begin{cases} 1, & \text{if } \delta = n, \\ 0, & \text{if } \delta < n, \mu + \delta - n + d \leq 0, \\ R^{(\delta)}(n, p^\mu + \delta - n), & \text{if } d = 0, \delta < n, \mu + \delta - n > 0, \\ q^{n+\Delta} R^{(d-1)}(n, p^\mu) + R^{(d-1)}(n-1, p^\mu), & \text{otherwise}. \end{cases}$$

(3.5) (3.6) (3.7) (3.8)

In Fig. 1, $A$ is the line segment from $(\delta, 0)$ to $(\delta, \delta)$, $B$ is the semi-infinite ray $n = \mu + \delta + d$ ($d \geq 0$), and $C$ is the line segment from $(\delta + 1, 0)$ to $(\delta + \mu - 1, 0)$. $A$, $B$ and $C$ are places on the $(n, d)$ plane where (3.5), (3.6) and (3.7) are applicable.
Application of (3.8) to the evaluation of $R^{(n)}(n, p^\mu)$ corresponds to enumerating a family of paths $L = (n_0, d_0), (n_1, d_1), \ldots, (n_k, d_k)$, where $(n_0, d_0) = (n, n)$ and, for $1 \leq i \leq k$, either $(n_i, d_i) = (n_{i-1}, d_{i-1} - 1)$ or $(n_i, d_i) = (n_{i-1} - 1, d_{i-1} - 1)$. It is required that $(n_k, d_k)$ is the first point on $L$ which belongs to $A \cup B \cup C$. The weight of $L$ is defined to be $q^{\sum_{i=\ell(L)}^{n+\Delta}}$, where $I(L) = \{ i \mid 0 \leq i < k, n_i = n_{i+1} \}$.

Let $W_A$ be the total weight of all the paths whose last point belongs to $A$. For $\delta + 1 \leq t \leq \delta + \mu - 1$, let $W_t$ be the total weight of all paths whose last point is $(t, 0)$. Then, by (3.5)–(3.8),

$$R^{(n)}(n, p^\mu) = W_A + \sum_{t=\delta+1}^{\delta+\mu-1} W_t R^{(t)}(t, p^{\mu+\delta-t}).$$  \hspace{1cm} (3.9)

To determine $W_A$, notice from the diagram that it is independent of $\mu$. Thus, by (3.9), $W_A = R^{(n)}(n, p)$.

Next consider $W_t$. If $t > \min(\delta + \mu - 1, n)$ then clearly $W_t = 0$, so suppose that $t \leq \min(\delta + \mu - 1, n)$. Let $L = (n_0, d_0), \ldots, (n_n, d_n)$ be a path with $(n_n, d_n) = (t, 0)$, and let $i_1 < i_2 < \cdots < i_{n-t}$ be the values of $d_j$ for $j \in \{0, 1, \ldots, n-1\} - I(L)$. In other words, $i_1, i_2, \ldots, i_{n-t}$ are (in reverse order) the values of $d$ at the points from which $L$ moves down diagonally. The weight of $L$ is

$$q^{(n-i_{n-t})(n+\Delta)+\sum_{j=1}^{n-t}(i_j-i_{j-1}-1)} = q^{(n+\Delta)(n-t-(n-t+1)/2)-(i_1+i_2+\cdots+i_{n-t})},$$

Therefore, the total weight of all such paths is

$$W_t = q^{(n+\Delta)(n-t-(n-t+1)/2)} \alpha_{n,t}(q),$$

where

$$\alpha_{n,t}(q) = \begin{cases} 1, & \text{if } t = 0, \\ \sum_{1 \leq i_1 < \cdots < i_{n-t} \leq n} q^{-(i_1+i_2+\cdots+i_{n-t})}, & \text{if } 1 \leq t \leq n, \end{cases}$$

is the coefficient of $x^{n-t}$ in $\prod_{i=1}^{n} (1 + q^{-i}x)$

$$= q^{-(n-t)(n-t+1)/2+(n-t)} \left[ \begin{array}{c} n \\ t \end{array} \right],$$

by [5, Exercise 2.6.10(b)]. Therefore, $W_t = q^{t(t+\Delta)} \left[ \begin{array}{c} n \\ t \end{array} \right]$, and so

$$R^{(n)}(n, p^\mu) = R^{(n)}(n, p) + \sum_{t=\delta+1}^{\min(\delta+\mu-1, n)} q^{t(t+\Delta)} \left[ \begin{array}{c} n \\ t \end{array} \right] R^{(t)}(t, p^{\mu+\delta-t}).$$  \hspace{1cm} (3.10)

The theorem follows on applying the definition of $R$ and Theorem 1.1. \hspace{1cm} $\Box$

If care is taken to avoid unnecessary repetition of computation, either Lemma 3.1 or Theorem 3.1 can be used to compute $P_{\Delta,\delta}(n, p^\mu)$, using a number of arithmetic operations bounded by a polynomial in $n + \Delta$ and $\mu$.

We are now equipped to develop expressions for $Q_{\Delta,\delta}(n, p^\mu)$ and $P_{\Delta,\delta}(n, p^\mu)$.
Theorem 3.2. Let $\Delta \geq 0$, $\mu \geq 1$, $n \geq 1$ and $0 \leq \delta \leq n$. Then
\[
P_{\Delta, \delta}(n, p^\mu) = \Pi_n(q) \Pi_{n+\Delta}(q) \nonumber
\times \left( \sum_{A_{n-\delta}(\mu)} f(\alpha_1, \ldots, \alpha_r) - \sum_{B_{n-\delta}(\mu)} f(\alpha_1, \ldots, \alpha_r) \right) \tag{3.11}
\]
and
\[
Q_{\Delta, \delta}(n, p^\mu) = \Pi_n(q) \Pi_{n+\Delta}(q) \sum_{C_{n-\delta}(\mu)} f(\alpha_1, \ldots, \alpha_r), \tag{3.12}
\]
where
\[
f(\alpha_1, \ldots, \alpha_r) = q^{\sum_{i=1}^{r} \alpha_i + \delta} \Pi_{\alpha_1+\delta}(q) \Pi_{\alpha_2+\delta+\Delta}(q) \Pi_{\alpha_2-\alpha_1}(q) \cdots \Pi_{\alpha_r-\alpha_{r-1}}(q) \Pi_{n-\alpha_r}(q),
\]
\[
A_{n-\delta}(\mu) = \{(\alpha_1, \ldots, \alpha_r) \mid 0 \leq \alpha_1 \leq \cdots \leq \alpha_r \leq n-\delta, \quad r \geq 1,
\]
\[
\alpha_2 + \cdots + \alpha_r \leq \mu - r \leq \alpha_1 + \cdots + \alpha_r \leq \mu - 1, \nonumber
\]
\[
B_{n-\delta}(\mu) = \{(\alpha_1, \ldots, \alpha_r) \mid 0 \leq \alpha_1 \leq \cdots \leq \alpha_r \leq n-\delta, \quad r \geq 2,
\]
\[
\mu - r + 1 \leq \alpha_2 + \cdots + \alpha_r \leq \mu - 1 < \alpha_1 + \cdots + \alpha_r, \nonumber
\]
and
\[
C_{n-\delta}(\mu) = \{(\alpha_1, \ldots, \alpha_r) \mid 1 \leq \alpha_1 \leq \cdots \leq \alpha_r \leq n-\delta,
\]
\[
\alpha_2 + \cdots + \alpha_r \leq \mu - 1 < \alpha_1 + \cdots + \alpha_r. \nonumber
\]

Proof. Consider the computation of $Q_{\Delta, \delta}(n, p^\mu)$ by repeated application of Theorem 3.1, with the boundary conditions $Q_{\Delta, \delta}(n, p^\mu) = 1$ if $\mu <= 0$. We see that $Q_{\Delta, \delta}(n, p^\mu)$ thus has the form
\[
\sum_{(t_1, \ldots, t_r) \in T(n, \delta, \mu)} \frac{\Pi_{n+\Delta}(q)}{\Pi_{t+r+\Delta}(q)} \left[ \begin{array}{c} n \\ t_1 \\ \vdots \\ t_r \end{array} \right] q^{\sum_{i=1}^{r} t_i}, \tag{3.13}
\]
where $T(n, \delta, \mu)$ is the set of all possible sequences of values of the summation index $t$ (in Theorem 3.1). A particular vector $(t_1, \ldots, t_r)$ occurs if $n \geq t_1 \geq \cdots \geq t_r \geq \delta + 1$, $t_1 + \cdots + t_{r-1} \leq \mu + (r-1)\delta - 1$ (if $r \geq 2$) and $t_1 + \cdots + t_r \geq \mu + r\delta$. Equation (3.12) now follows on substituting $\alpha_i = t_{r-i+1} - \delta$ for $1 \leq i \leq r$.

To prove (3.11) now note that, for $0 \leq \delta \leq n$,
\[
T(n, \delta - 1, \mu) \setminus T(n, \delta, \mu) = \{(t_1, \ldots, t_r) \mid n \geq t_1 \geq \cdots \geq t_r \geq \delta,
\]
\[
t_1 + \cdots + t_{r-1} \leq \mu + r\delta - \delta - r \quad (r \geq 2),
\]
\[
\mu + r\delta - r \leq t_1 + \cdots + t_r \leq \mu + r\delta - 1, \nonumber
\]
and
\[
T(n, \delta, \mu) \setminus T(n, \delta - 1, \mu)
\]
\[
= \{(t_1, \ldots, t_r) \mid n \geq t_1 \geq \cdots \geq t_r \geq \delta, \quad r \geq 2, \quad t_1 + \cdots + t_r \geq \mu + r\delta
\]
\[
\mu + r\delta - \delta - r + 1 \leq t_1 + \cdots + t_{r-1} \leq \mu + r\delta - \delta - 1). \nonumber
\]
Since the summand in (3.13) is independent of $\delta$, we can find $P_{\Delta, \delta}(n, p^\mu) = Q_{\Delta, \delta - 1}(n, p^\mu) - Q_{\Delta, \delta}(n, p^\mu)$ by subtracting the sum over $T(n, \delta, \mu) \setminus T(n, \delta - 1, \mu)$ from the sum over $T(n, \delta - 1, \mu) \setminus T(n, \delta, \mu)$. Equation (3.11) now follows on substituting $\alpha_i = t_{r-i+1} - \delta$ for $1 \leq i \leq r$. 

A similar evaluation of $R^{(n)}(n, p^\mu)$ for $\delta = 0$ by applying (3.10) yields the following identity when compared to (2.2). It may also be proved by induction on $\mu$ from the $q$-Vandermonde identity ([5, Exercise 2.6.3(c)]).

**Corollary 3.1.** If $n \geq 1$ and $\mu \geq 1$, then

$$
\sum_{(\alpha_1, \ldots, \alpha_r) \in \mathcal{P}_n(\mu)} q^{\sum_{i=1}^{r} \alpha_i \beta_i} q_{\alpha_1} \cdots q_{\alpha_{r-1}} q_{\alpha_r} = q^\mu \left[ \begin{array}{c} n + \mu - 1 \\ \mu \end{array} \right],
$$

where

$$
\mathcal{P}_n(\mu) = \{ (\alpha_1, \alpha_2, \ldots, \alpha_r) \mid 1 \leq \alpha_1 \leq \cdots \leq \alpha_r \leq n, \alpha_1 + \alpha_2 + \cdots + \alpha_r = \mu \}.
$$

4. **Asymptotics and bounds**

Lemma 1.1 and Theorem 3.1 enable us to obtain various bounds on $Q_{\Delta, \delta}(n, m)$ and, using Corollaries 4.3 and 4.4 below, it is easy to deduce corresponding bounds on $P_{\Delta, \delta}(n, m)$ and $Q_{\Delta, \delta}(n, m) = \sum_{\delta=0}^{\infty} \delta P_{\Delta, \delta}(n, m)$. The last quantity is $n$ minus the average $m$-rank of random $(n + \Delta) \times n$ matrices over $\mathbb{F}_m$.

Throughout this section we assume that $m = p_1^{\mu_1}p_2^{\mu_2} \cdots p_k^{\mu_k}$ where $p_1 < p_2 < \cdots < p_k$ are distinct primes, $\mu_i \geq 1$ and $k \geq 1$. We define $q_j = 1/p_j$ for $j = 1, \ldots, k$ and $q_0 = \prod_{j=1}^{k} q_j$. If $k = 1$ we may write $m = p^\mu$, $q = 1/p$ for simplicity.

We also define $h = \prod_{j=1}^{k} (p_j/(p_j - 1))$. Although $h$ is unbounded, it increases very slowly. In fact, it may be shown that $h \leq e^\gamma \ln(4.44 \ln m)$, where $\gamma = 0.5772\ldots$ is Euler's constant.

Define

$$
f(\Delta, q, n, t) = \frac{\Pi_{n+\Delta}(q)}{\Pi_{\alpha}(q)} \left[ \begin{array}{c} n \\ t \end{array} \right] \quad \text{and} \quad \Pi_{\alpha}(q) = \prod_{j=1}^{\infty} (1 - q^j).
$$

The proof of the following lemma is straightforward and will be omitted.

**Lemma 4.1.** If $\Delta \geq 0$, $1 \leq t \leq n$ and $0 < q \leq \frac{1}{2}$, then

$$
1 \leq f(\Delta, q, n, t) \leq 1/\Pi_{\alpha}(q),
$$

$$
0 \leq f(\Delta, q, n + 1, t) - f(\Delta, q, n, t) \leq q^{n+1-t}/\Pi_{\alpha}(q),
$$

and $q^{(\alpha + \Delta)} f(\Delta, q, n, t)$ is a monotonic increasing function of $q$.

**Theorem 4.1.** $Q_{\Delta, \delta}(n, m)$ is a monotonic increasing function of $n \geq 1$, and a monotonic decreasing function of $\Delta \geq 0$, $\delta \geq 0$, $\mu \geq 1$ and prime $p_i$ ($j = 1, \ldots, k$).
Proof. By Theorem 3.1 and Lemma 4.1,
\[ Q_{\Delta, \delta}(n + 1, p^\mu) \geq \sum_{t = \delta + 1}^{n} q^{t(\mu+\Delta)} f(\Delta, q, n + 1, t)Q_{\Delta, \delta}(t, p^{\mu+\delta-t}) \]
\[ \geq \sum_{t = \delta + 1}^{n} q^{t(\mu+\Delta)} f(\Delta, q, n, t)Q_{\Delta, \delta}(t, p^{\mu+\delta-t}) \]
\[ = Q_{\Delta, \delta}(n, p^\mu), \]
so monotonicity in \( n \) follows from Lemma 1.1. Monotonicity in \( \Delta \) is obvious as adding a row to a matrix cannot decrease its \( m \)-rank. Monotonicity in \( \delta \) is also obvious, as \( Q_{\Delta, \delta}(n, m) - Q_{\Delta, \delta+1}(n, m) = P_{\Delta, \delta+1}(n, m) \geq 0 \). Monotonicity of \( Q_{\Delta, \delta}(n, p^\mu) \) in \( \mu \) follows by induction on \( \mu \) from Theorem 3.1, and monotonicity in \( p = 1/q \) follows from Theorem 3.1 and the last part of Lemma 4.1. □

Corollary 4.1. \( Q_{\Delta, \delta}(\infty, m) = \lim_{n \to \infty} Q_{\Delta, \delta}(n, m) \) exists.

The following theorem sharpens the monotonicity results of Theorem 4.1.

Theorem 4.2. If \( \Delta \geq 0, \delta \geq 0 \) and \( n \geq 1 \), then
\[ Q_{\Delta, \delta+1}(n, m) \leq \xi(\delta + 2)q^{\delta+\Delta+2}Q_{\Delta+1, \delta}(n - 1, m), \] \[ Q_{\Delta+1, \delta}(n, m) \leq \xi(\delta + \Delta + 2)q^{\delta+1+\Delta}Q_{\Delta, \delta}(n, m), \]
and
\[ Q_{\Delta, \delta}(n + 1, p^\mu) - Q_{\Delta, \delta}(n, p^\mu) \leq 2.20q^{n+1+\Delta}(\delta+\Delta)/(1-q), \]
where \( \xi(x) \) is the Riemann zeta function.

Proof. To prove (4.1) it is sufficient to prove by induction on \( \mu \) that
\[ Q_{\Delta, \delta+1}(n, p^\mu) \leq \left( \frac{q^{\delta+\Delta+2}}{1 - q^{\delta+2}} \right) Q_{\Delta+1, \delta}(n - 1, p^\mu). \] \[ (4.4) \]
Using the inequality \( \lfloor \frac{n}{t} \rfloor \leq \lfloor \frac{n-1}{t} \rfloor/(1 - q^{-1}) \) and Theorem 3.1, the induction hypothesis gives
\[ Q_{\Delta, \delta+1}(n, p^\mu) = \sum_{t = \delta + 1}^{n-1} q^{t(\mu+\Delta+1)} \frac{\Pi_{n+\Delta}(q)}{\Pi_{t+\Delta+1}(q)} \left[ \frac{n}{t + \frac{1}{2}} \right] Q_{\Delta, \delta+1}(t + 1, p^{\mu+\delta-t}) \]
\[ \leq \left( \frac{q^{\delta+\Delta+2}}{1 - q^{\delta+2}} \right) \sum_{t = \delta + 1}^{n-1} q^{t(\mu+\Delta+1)} \frac{\Pi_{n+\Delta}(q)}{\Pi_{t+\Delta+1}(q)} \left[ \frac{n - 1}{t} \right] Q_{\Delta+1, \delta}(t, p^{\mu+\delta-t}), \]
so (4.4) follows.

To prove (4.2) it is sufficient to prove by induction on \( \mu \geq 1 \) that
\[ Q_{\Delta+1, \delta}(n, p^\mu) \leq \left( \frac{q^{\delta+1}}{1 - q^{\delta+\Delta+2}} \right) Q_{\Delta, \delta}(n, p^\mu). \] \[ (4.5) \]
The proof of (4.5) is similar to that of (4.4), using Theorem 3.1 and the inequality
\[
\frac{\Pi_{n+\Delta+1}(q)}{\Pi_{t+\Delta+1}(q)} \leq \frac{\Pi_{n+\Delta}(q)}{\Pi_{t+\Delta}(q)(1-q^{t+\Delta+1})}.
\]

To prove (4.3), we have from Theorem 3.1 and Lemma 4.1 that
\[
Q_{\Delta,\delta}(n+1, p^\nu) - Q_{\Delta,\delta}(n, p^\nu) \leq \sum_{t=\Delta+1}^{\infty} q^{(t+\Delta)+n+1-t}/\Pi_{\omega}(q)
\]
\[
\leq q^{n+1+(\delta+1)(\delta+\Delta)} \sum_{j=1}^{\infty} q^{j-1}/\Pi_{\omega}(q)
\]
\[
\leq 2.20q^{n+1+(\delta+1)(\delta+\Delta)}/(1-q),
\]
where the constant 2.20 arises in the worst case \( q = \frac{1}{2} \).

Corollary 4.2.
\[
Q_{\Delta,\delta+1}(n, m) \leq \frac{\pi^4}{36} q_0^{2\delta+\Delta+3} Q_{\Delta,\delta}(n, m).
\]  

Proof. This is immediate from (4.1), (4.2), the monotonicity of \( Q_{\Delta,\delta}(n, m) \) in \( n \), and the fact that \( \zeta(\delta + \Delta + 2) \leq \zeta(\delta + 2) \leq \zeta(2) = \frac{1}{2}\pi^2 \).

Corollary 4.3. If \( \delta \geq 1 \) then
\[
\left(1 - \frac{\pi^4}{36} q_0^{2\delta+\Delta+1}\right) Q_{\Delta,\delta-1}(n, m) \leq P_{\Delta,\delta}(n, m) \leq Q_{\Delta,\delta-1}(n, m).
\]

Proof. This is immediate from Corollary 4.2 with \( \delta \) replaced by \( \delta - 1 \).

Corollary 4.4. \( \delta(\Delta, n, m) \) is a monotonic increasing function of \( n \geq 1 \), and a monotonic decreasing function of \( \Delta \geq 0 \), \( \mu_j \geq 1 \) and prime \( p_j \) (\( j = 1, \ldots, k \)). Also,
\[
Q_{\Delta,\delta}(n, m) \leq \delta(\Delta, n, m) \leq Q_{\Delta,\delta}(n, m)/\left(1 - \frac{\pi^4}{36} q_0^{\Delta+3}\right).
\]

Proof. This is immediate from Theorem 4.1 and Corollary 4.2, as
\[
\delta(\Delta, n, m) = \sum_{\delta=0}^{n} Q_{\Delta,\delta}(n, m).
\]

We now give some upper and lower bounds on \( Q_{\Delta,\delta}(n, m) \). Corresponding bounds on \( P_{\Delta,\delta}(n, m) \) and \( \delta(\Delta, n, m) \) may easily be deduced from Corollaries 4.3 and 4.4.
Theorem 4.3. If $\Delta \geq 0$, $\delta \geq 0$, $n \geq 1$ and $\tau = \sqrt{\delta(\delta + \Delta)}$, then

$$Q_{\Delta,0}(n, m) \leq 2.30h/m^{\Delta+1},$$  \hspace{1cm} (4.7)

$$Q_{\Delta,\delta}(n, m) \leq 12.09h_0^{\delta(\delta + \Delta+2)}/m^{\Delta+1},$$  \hspace{1cm} (4.8)

and

$$Q_{\Delta,\delta}(n, m) \leq h^{7.66}/m^{2\delta+\Delta+2\tau}.$$  \hspace{1cm} (4.9)

Also, if $n \geq \delta + 1$, then

$$Q_{\Delta,\delta}(n, m) \geq 1/m^{(\delta+1)(\delta + \Delta+1)}.$$  \hspace{1cm} (4.10)

Proof. The lower bound (4.10) is trivial, as

$$Q_{\Delta,\delta}(n, m) \geq Q_{\Delta,\delta}(\delta + 1, m) = 1/m^{(\delta+1)(\delta + \Delta+1)}.$$  

To prove (4.7), observe that from Theorem 2.1,

$$Q_{\Delta,0}(n, p^\mu) = 1 - P_{\Delta,0}(n, p^\mu) \leq q^{\mu(\Delta+1)} \sum_{j=0}^{n-1} q^j/\Pi_j(q) \leq q^{\mu(\Delta+1)}/\Pi_{n-1}(q) \leq q^{\mu(\Delta+1)}/\Pi_n(q).$$

Thus, from Lemma 1.1,

$$Q_{\Delta,0}(n, m) \leq c_0h/m^{\Delta+1},$$  \hspace{1cm} (4.11)

where

$$c_0 = \prod_{j=1}^{k} \prod_{r=2}^{\infty} (1 - q_j)^{-1} \leq \prod_{r=2}^{\infty} \prod_{\text{prime } p} (1 - p^{-r})^{-1} = \prod_{r=2}^{\infty} \xi(r) = c,$$

say, and computation shows that $c < 2.30$.

To prove (4.8), observe that for $\delta \geq 1$

$$Q_{\Delta,\delta}(n, m) \leq \zeta(\delta + 1)\zeta(\delta + \Delta + 1)q_0^{2\delta+\Delta+1}Q_{\Delta,\delta-1}(n, m),$$

from (4.1) and (4.2). Thus, by induction on $\delta$ we have

$$Q_{\Delta,\delta}(n, m) \leq \left(\prod_{j=1}^{\delta} \zeta(j+1)(\zeta(j+\Delta+1))\right)q_0^{\delta(\delta+\Delta+2)}Q_{\Delta,0}(n, m).$$

Thus, from (4.11),

$$Q_{\Delta,\delta}(n, m) \leq c^3hq_0^{\delta(\delta+\Delta+2)}/m^{\Delta+1},$$

where $c^3 < 12.09$.

To prove (4.9) it is sufficient to show that

$$Q_{\Delta,\delta}(n, p^\mu) \leq q^{(2\delta+\Delta+2\tau)\mu}/(1 - q)^\alpha,$$  \hspace{1cm} (4.12)
where $\alpha \leq 7.66$. Define
\[
K(n, \mu) = \begin{cases} 
1, & \text{if } \mu \leq 0, \\
\sum_{j=1}^{n} q^{(j-\tau)^2} K(j, \mu - j)/\Pi_{n-j}(q), & \text{if } \mu > 0.
\end{cases}
\]
Then, by induction on $\mu$, we have from Theorem 3.1 that
\[
Q_{\Delta, \delta}(n + \delta, p^n) \leq q^{(26 + \Delta + 2\tau)\mu} K(n, \mu),
\]
so it is sufficient to show that $K(n, \mu) \leq 1/(1 - q)^\alpha$. We shall only sketch the proof here.

Let $\sigma$ be an integer such that $-0.5 \leq \varepsilon = \tau - \sigma \leq 0.7$ and $\beta_1 = \theta q^{1+\varepsilon^2} < 1$, where $\theta = \sum_{j=0}^{\infty} q^{(j+2+2\varepsilon)/\Pi_j(q)}$. By induction on $\mu$ we find that $n \leq \sigma - 1$ implies that $K(n, \mu) \leq (\theta q^{1+\varepsilon^2})^{(\sigma - 1)/n}$. Thus, by induction on $\mu$, we have $K(\sigma, \mu) \leq f_0$, where
\[
f_0 = \begin{cases} 
1 + s/(1 - \beta_1), & \text{if } \varepsilon = 0, \\
\max_{j \geq 0} m_j, & \text{if } \varepsilon \neq 0,
\end{cases}
\]
where $m_0 = 1$, $m_{j+1} = \beta_0 m_j + 2 \beta_1 s$, $\beta_0 = q^e$ and $s = \sum_{j=1}^{\infty} q^{(j+\varepsilon^2)/\Pi_j(q)}$. Now, for all integers $j > 0$, we have $K(\sigma + j, \mu) \leq f_j$, where
\[
f_j = \max \left(1, \frac{\sum_{i=0}^{j-1} q^{(i-\varepsilon)^2} f_i/\Pi_{j-i}(q) + \sum_{i=1}^{\infty} q^{(i+\varepsilon)^2}/\Pi_{j+i}(q)}{1 - q^{(j-\varepsilon)^2}} \right)
\]
and $f_\infty = \lim_{j \to \infty} f_j$ satisfies
\[
f_\infty \leq \max \left(1, \frac{\sum_{i=0}^{j-1} q^{(i-\varepsilon)^2} f_i + \sum_{i=1}^{\infty} q^{(i+\varepsilon)^2}}{\Pi_\infty(q) - \sum_{i=1}^{\infty} q^{(i-\varepsilon)^2}} \right),
\]
for all $j \geq 3$.

Since $K(n, \mu)$ is a monotonic increasing function of $n$, we have the uniform bound $K(n, \mu) \leq f_\infty$. Moreover, using the result that $q^e + q^{(1-e)^2} \leq 1 + 27q/16$, it is easy to see that $f_0, f_1, \ldots, f_\infty$ are uniformly $1 + O(q)$, so $K(n, \mu) \leq 1/(1 - q)^\alpha$ for some constant $\alpha$. To show that we can take $\alpha \leq 7.66$, choose $\sigma$ such that $-\frac{1}{2} < \varepsilon = \tau - \sigma < \frac{1}{2}$ for $p \equiv 3$, and $\varepsilon_0 < \varepsilon = \tau - \sigma \leq 1 + \varepsilon_0$ for $p = 2$, where $\varepsilon_0 = -0.306 \ldots$ is defined by $\sum_{j=1}^{\infty} 2^{-(j+\varepsilon_0^2)/\Pi_{j-1}(\frac{1}{2})} = 1$. This concludes our sketch of the proof of (4.12). $\square$

We can now show that the convergence of $Q_{\Delta, \delta}(n, m)$ to $Q_{\Delta, \delta}(\infty, m)$ as $n \to \infty$ is rapid.
Corollary 4.5. If $\Delta \geq 0$, $\delta \geq 0$ and $n \geq 1$ then
\[ Q_{\Delta, \delta}(n + 1, m) - Q_{\Delta, \delta}(n, m) \leq 26.6kq^{n-\delta}q^{(\delta+1)(\delta+\Delta+1)}. \]

Proof. Suppose $n \geq \delta$, for otherwise the result is trivial. From Theorem 4.1 we have
\[ Q_{\Delta, \delta}(n + 1, m) - Q_{\Delta, \delta}(n, m) \leq \sum_{j=1}^{k} (Q_{\Delta, \delta}(n + 1, p_j^\mu) - Q_{\Delta, \delta}(n, p_j^\mu))Q_{\Delta, \delta}(n + 1, m/p_j^\mu) \]
so the result follows from (4.3) and (4.8).

From Theorem 1.1 and Corollary 4.3, the lower bound (4.10) is almost attained if $m$ is a large prime. On the other hand, if $\tau$ is a positive integer which divides $\mu$, $n \geq \delta + \tau$, and $m = p^\mu$ for prime $p$, then $Q_{\Delta, \delta}(n, m) \geq 1/m^{2\delta + \Delta + 2\tau}$. Thus, although the bounds (4.9) and (4.10) differ widely, the exponents of $m$ are the best possible. However, the exponent 7.66 of $h$ in (4.9) is not the best possible. From numerical evidence we conjecture that $\lim_{\mu \to \infty} Q_{\Delta, \delta}(\infty, p^\mu)p^{(2\delta + \Delta + 2\tau)\mu}$ is maximal when $\Delta = 0$ and $\delta = 2$ (if $p \leq 3$) or $\delta = 1$ (if $p \geq 5$). This leads to the following conjecture, in which the constant $\pi^2/36$ is best possible (since $\lim_{\mu \to \infty} Q_{0,1}(n, p^\mu)p^{4\mu} = \lfloor n^{\frac{1}{3}} \rfloor^2$).

Conjecture 4.1.
\[ Q_{\Delta, \delta}(n, m) \leq \frac{\pi^4}{36} h \max(h, 8.81)/m^{2\delta + \Delta + 2\tau} \]
\[ \leq \left( \frac{\pi^2 e^\gamma}{6 \ln(16 \ln m)} \right)^2 / m^{2\delta + \Delta + 2\tau}. \]

Table 1

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<th>$\delta = 0$</th>
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<th>$\delta = 3$</th>
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In Table 1 we give some values of $P_{0,\delta(\infty, m)}$ and $\delta(0, \infty, m)$. The notation "\((-9)1.23\)" means $1.23 \cdot 10^{-9}$.

References