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AUTOMORPHISMS OF RANDOM GRAPHS WITH SPECIFIED VERTICES

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Conditions are found under which the expected number of automorphisms of a large random labelled graph with a given degree sequence is close to 1. These conditions involve the probability that such a graph has a given subgraph. One implication is that the probability that a random unlabelled k-regular simple graph on n vertices has only the trivial group of automorphisms is asymptotic to 1 as $n\to\infty$ with $3\leqq k=O(n^{1/2-\varepsilon})$. In combination with previously known results, this produces an asymptotic formula for the number of unlabelled k-regular simple graphs on n vertices, as well as various asymptotic results on the probable connectivity and girth of such graphs. Corresponding results for graphs with more arbitrary degree sequences are obtained. The main results apply equally well to graphs in which multiple edges and loops are permitted, and also to bicoloured graphs.

1. Introduction

One of our main aims is to show that the proportion of unlabelled k-regular graphs on n vertices which have no symmetries is $1+O(n^{-1})$ as $n\to\infty$ with $k\geqq 3$ fixed. For $k=3$ and $n\leqq 40$, this trend is supported by the data given by Robinson and Wormald [12]. Equivalent statements of this property are that the expected number of automorphisms of a random k-regular labelled graph is $1+O(n^{-1})$, or that the number of labelled k-regular graphs is asymptotic to $n!$ times the number of unlabelled k-regular graphs as $n\to\infty$ with $k\geqq 3$ fixed. Amongst other things, this establishes an asymptotic formula for the number of unlabelled k-regular graphs. Most of our results are actually derived in a setting which is much more general than this, and apply just as well to graphs or coloured graphs in which the minimum degree is 3 and the maximum degree does not increase too quickly with n (for example, with bounded maximum degree).

In the next Section we prove the central result, setting forth sufficient conditions for the expected number of automorphisms of a random labelled graph from an arbitrary class of graphs to be $1+o(1)$. In order to apply this result, it is necessary to

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have a suitable upper bound on the probability that one of the labelled graphs in the class under consideration contains a given set of edges. This is done in Section 3. One of the more interesting applications is the establishment of an asymptotic formula for the probability that a random unlabelled k-regular simple graph has a given connectivity and/or girth, with k fixed.

2. The main result

The definitions given here apply throughout this paper. A *graph* can have loops or multiple edges, but a *simple graph* cannot. The *edges* of a graph include its loops, if any. Multiple edges, which are usually treated as sets of parellel single edges, will instead be treated as single edges with a nonnegative integral *multiplicity*. A *subgraph* H of G has the same vertex set as G and each edge of H has at most the multiplicity of the corresponding edge in G. A *full subgraph* H of G is a subgraph such that each edge of non-zero multiplicity in H has exactly the same multiplicity as the corresponding edge of G. If G is *labelled*, its vertices are labelled with $1, \ldots, n$, in which case v_i denotes the vertex labelled i. Thus we regard all labelled n-vertex graphs as having the same vertex set. The *degree sequence* of G is then $(k_1, \ldots, \ldots, k_n)$ where k_i is the degree of v_i, each loop contributing 2 to the degree of its incident vertex. By an *automorphism* of G we mean a permutation of $\{v_1, v_2, \ldots, v_n\}$ which preserves G (so that edge multiplicities are preserved). Let $\mathcal{S}$ be a set of labelled graphs and denote by $\mathcal{S}_n$ the set of n-vertex graphs in $\mathcal{S}$. We require that all elements of $\mathcal{S}_n$ have the same degree sequence, denoted by $(k_{n,1}, \ldots, k_{n,n})$. The theorem in this section will deal with the expected number, denoted by $T(n)$, of non-trivial automorphisms of a graph chosen at random from $\mathcal{S}_n$. For this purpose, $\mathcal{S}_n$ is regarded as a probability space in which the probabilities of different graphs occurring need not be the same; the following theorem holds for any probability distributions.

We denote by $\delta=\delta(n)$ the minimum of $k_{n,j}$ for $j=1, \ldots, n$, and by $\Delta=\Delta(n)$ the maximum, and put $\beta=\beta(n)=1/(3\Delta+24)$. We only consider classes $\mathcal{S}$ for which $\delta(n)\geqq 1$ and $\Delta(n)$ is always finite. The number of $k_{n,j}$ which equal i is denoted by $n_i=n_i(n)$.

On various occasions we use the following implication of Stirling's formula for factorials:

$$(i/O(1))^i \leqq i! \leqq (iO(1))^i.$$

We also use the following elementary results.

Lemma 2.1. *The number of ways of arranging x indistinguishable pigeons in y distinctly numbered pigeonholes, each of unlimited capacity, is $\binom{x+y-1}{x}$, which is monotonically non-decreasing in both x and y for x, $y\geqq 0$.* ∎

Our pigeons will be edges and our pigeonholes will be pairs of vertices.

Lemma 2.2. *For non-negative integers x, y, a and b,*

$$\binom{x}{a}\binom{y}{b} \leqq \binom{x+y}{a+b}. \quad ∎$$

Lemma 2.3. *Let* $a \geqq 0$ *and* $x \geqq 1$ *be integers, and* $t > 0$. *Then, for* $0 \leqq b \leqq x$,

$$\frac{1}{t^b}\binom{a+b}{b} \leqq e^{O(x)} + \left(O\left(\frac{a}{xt}+\frac{1}{t}\right)\right)^x.$$

Proof. For non-triviality take $b > 0$. Observe that

$$\frac{1}{t^b}\binom{a+b}{b} \leqq \frac{1}{b!}\left(\frac{a+b}{t}\right)^b \leqq \left(\frac{O(a+b)}{bt}\right)^b.$$

Since $\binom{a+b}{c}/t^b$ is unimodal or monotonic increasing in b, the maximum for $0 \leqq b \leqq x$ occurs at $b = x$ or at the greatest b for which $(a+b)/tb > 1$. The first case gives the bound $(O(a+x)/tx)^x$. In the second, $(a+b)/b = t \pm (1/b)$ and so we have the bound $O(1)^b = \exp(O(x))$, as $x \geqq b$. ∎

An *acceptable* function for $\mathcal{S}$ is a function $f(n)$ such that for all n, $M \geqq 1$ and for every labelled graph H with n vertices and M edges, the probability that H is a full subgraph of a random member of $\mathcal{S}_n$ is at most $(f(n)/n)^M$. Here, and elsewhere, each edge is counted according to its multiplicity.

In the following theorem, $o()$ and $O()$ refer to the passage of n to infinity, where n only takes values for which $|\mathcal{S}_n| \neq 0$. The constant implicit in $O()$ depends only on the function $f(n)$, and not otherwise on the $k_{i,j}$ or $\mathcal{S}$.

Theorem 2.4. *Suppose that* $f(n) > 1$ *for all* n *and that* $f(n)$ *is acceptable for* $\mathcal{S}$.

(i) *If* $\delta \geqq 6$ *and* $f(n)^2 = o(n^{1-2/\delta})$, *or if* $3 \leqq \delta \leqq 5$ *and* $f(n)^{3+18\beta} = o(n^{2+6\beta-6/\delta})$, *then* $T(n) = (O(1)f(n))^{2\delta}/n^{\delta-2}$.

(ii) *If* $f(n)^{3\beta+1/2} = o(n^\beta)$, *and* $n_i f(n)^{3\beta i + i/2} = o(n^{\beta i + i/3})$ *for* $i \in \{1, 2\}$, *then* $T(n) \leqq z(n)$ *where* $z(n) = o(1)$ *is a function which depends only on* $f(n)$ *and not otherwise on* $\mathcal{S}$ *or the* $k_{i,j}$.

Proof. Clearly, $T(n)$ is the sum over all permutations σ of the probability that a random member G of $\mathcal{S}_n$ has σ as an automorphism. This probability can be obtained by summing, over all graphs H, the probability that H is the subgraph of G containing just those edges moved by σ, and that σ is an automorphism of H.

Let U and R be disjoint subsets of $\{v_1, \ldots, v_n\}$ with $|U| = 2u$, $|R| = r$, and and $2u + r > 0$. We first seek an upper bound on the expected number, $B(U, R)$, of automorphisms σ of a graph G chosen randomly from $\mathcal{S}_n$, such that U is the support of the 2-cycles of σ and R is the support of the cycles of length at least 3. Summing over all U and R and all u and r, with U and R not both empty, will give a bound on $T(n)$.

Let $G(U, R)$ denote the full subgraph of G with edge set consisting of precisely those edges of which at least one end is moved by σ. Clearly

$$B(U, R) = \sum_\sigma \sum_H \operatorname{Prob}(G(U, R) = H),$$

where the first sum is over all feasible σ and the second is over all graphs H with vertex set V and fixed by σ such that at least one end of every edge of H is moved by σ. Obviously we only need to consider graphs H whose vertex degrees agree with G on $U \cup R$.

Let $k(U)$ and $k(R)$ denote the mean degree of vertices in U and R, respectively (i.e. $k(U)$ is the mean of $k_{n,i}$ for all i such that $v_i \in U$). Since all $k_{n,i}$ are at least 1, so are $k(U)$ and $k(R)$. Put $k(\emptyset)=1$ for consistency. A *diagonal* of a $2j$-cycle of σ is a non-loop edge of H which has both ends in that 2-cycle and has its ends interchanged by σ^j. We define the following parameters for σ and H: σ has

l	3-cycles,
s	4-cycles,
h	6-cycles,

and H, given σ, has

w	diagonals of 2-cycles,
$2t$	diagonals of 4-cycles,
$3d$	diagonals of 6-cycles,
$2x$	edges with both ends in U other than diagonals of 2-cycles,
y	edges between U and R,
$3m$	edges with both ends in 3-cycles,
a	edges with both ends in R other than the $2t+3d+3m$ already counted.

By counting edge-ends we have:

$$2w+4x+y \leqq 2uk(U), \tag{2.1}$$

$$4t+6d+6m+2a+y \leqq rk(R), \tag{2.2}$$

and clearly

$$3l+4s+6h \leqq r. \tag{2.3}$$

The number of possibilities for a permutation σ of the type under consideration, given l, s and h, is at most

$$\frac{(2u)!\,r!}{u!\,2^u 3^l l!\,4^s s!\,6^h h!}$$

and hence at most

$$\frac{(2u)!\,r!}{u!\,l!}. \tag{2.4}$$

We next estimate the number of possible arrangements of the edges of H within $R \cup U$, given σ, by bounding the number of possible arrangements of the edges in each of the edge sets whose cardinalities are specified above. Since each such edge set is fixed by σ, each is determined by choosing which of the possible edge-orbits are present, and with what multiplicities. If i edge-orbits (counting multiplicities) are present in H amongst a set of j distinct possible edge-orbits (not counting multiplicities), then the number of possible arrangements of these edge-orbits of H is $\binom{j+i-1}{i}$ by Lemma 2.1 As a result we obtain the following bounds on the number of arrangements of the edges specified. In each case we note i and j. By Lemma 2.1, any overestimate of i or j yields a valid upper bound on the number of arrangements.

For the w diagonals in u 2-cycles, $i=w$ and $j=u$ and the bound is

$$\binom{u+w-1}{w}. \tag{2.5}$$

For the $2t$ diagonals in s 4-cycles, $i=t$ and $j=s$ and the bound is

$$\binom{s+t-1}{t}. \tag{2.6}$$

For the $3d$ diagonals in h 6-cycles, $i=d$ and $j=h$ and the bound is

$$\binom{h+d-1}{d}. \tag{2.7}$$

For the $2x$ non-diagonals with both ends in U, each edge-orbit contains 2 edges and the number of distinct possible edge-orbits is $2\binom{u}{2}+u$, so $i=x$ and $j=u^2$ and the bound is

$$\binom{u^2+x-1}{x}. \tag{2.8}$$

For the y edges between R and U, all edge-orbits have cardinality at least 4 and there are just $2ur$ distinct possible edges joining R and U, so $i\leqq y/4$ and $j\leqq ur/2$ and the bound is

$$\binom{\lfloor ur/2\rfloor+\lfloor y/4\rfloor-1}{\lfloor y/4\rfloor}. \tag{2.9}$$

For the $3m$ edges with both ends in 3-cycles, there are $\binom{3l}{2}+3l$ distinct possible edges in orbits of 3 each, so $i=m$ and $j=l(3l+1)/2$ and the bound is

$$\binom{l(3l+1)/2+m-1}{m}. \tag{2.10}$$

For the a other edges with both ends in R, there are $\binom{r}{2}+r$ distinct possible edges in orbits of cardinality at least 4, so $i\leqq a/4$ and $j\leqq r(r+1)/8$ and the bound is

$$\binom{\lfloor r(r+1)/8\rfloor+\lfloor a/4\rfloor-1}{\lfloor a/4\rfloor}. \tag{2.11}$$

The choices of the edges of H within $U\cup R$ determine the ends in U of the edges between $U\cup R$ and $V\setminus(U\cup R)$, for any graph whose degrees in $U\cup R$ are specified. To complete a choice of H, we therefore only have to decide for each remaining orbit of edges, which vertex fixed by σ is at the other end. Our bound on the number of possibilities for this vertex is n in each case. This yields the following bound on the numbers of arrangements of the specified edges.

For the $2uk(U)-2w-4x-y$ edges between U and $V\setminus(U\cup R)$, the orbits are of cardinality 2 and the bound is

$$n^{uk(U)-w-2x-y/2}. \tag{2.12}$$

For the $rk(R)-4t-6d-6m-2a-y$ edges between R and $V\setminus(U\cup R)$, in orbits of cardinality at least 3, the bound is

$$n^{(rk(R)-4t-2a-y)/3-2d-2m}. \tag{2.13}$$

The total number of edges in H is

$$M = 2uk(U)+rk(R)-w-2t-3d-2x-y-3m-a,$$

so that the probability that $G(U, R)$ is H is at most

$$(f(n)/n)^M, \tag{2.14}$$

by the hypotheses of the theorem.

The product of (2.12)—(2.14) is, using (2.2), at most

$$f(n)^{2uk(U)+rk(R)-w-2t-3d-2x-y-3m-a}\,n^{-uk(U)-rk(R)/2}. \tag{2.15}$$

We can now obtain an upper bound on $B(U, R)$ by summing the product of (2.4)—(2.11) and (2.15) over all $l, s, h, w, t, d, x, y, m$ and a satisfying (2.1)—(2.3). We obtain a bound on the general term and multiply by the number of terms afterwards.

Firstly consider all factors involving s, h, w, t and d. Those in (2.15) can be ignored since $f(n)>1$. The product of (2.5)—(2.7) is at most

$$\binom{s+t+u+w+d+h}{t+w+d}, \quad \text{by Lemma 2.2,}$$

$$\leqq 2^{s+t+u+w+d+h}.$$

Since $4s+6h\leqq r$ by (2.3), $4t+6d\leqq rk(R)$ by (2.2), and $2w\leqq 2uk(U)$ by (2.1), this is

$$e^{O(rk(R)+uk(U))}. \tag{2.16}$$

The factors involving x and y are

$$\binom{u^2+x-1}{x}\binom{\lfloor ur/2\rfloor+\lfloor y/4\rfloor-1}{\lfloor y/4\rfloor} f(n)^{-2x-y} \leqq \binom{u^2+\lfloor ur/2\rfloor+x+\lfloor y/4\rfloor}{x+\lfloor y/4\rfloor}\left(f(n)^2\right)^{-x-\lfloor y/4\rfloor}$$

using Lemma 2.2 and $f(n)>1$. Since $x+\lfloor y/4\rfloor\leqq uk(U)/2$ by (2.1), this is at most

$$\exp\left(O(1)uk(U)\right)+\left(\frac{O(u+r)}{f(n)^2}\right)^{uk(U)/2} \tag{2.17}$$

by Lemma 2.3, as $k(U)\geqq 1$.

The factors involving m, a and l are

$$\frac{1}{l!}\binom{\lfloor r(r+1)/8\rfloor+\lfloor a/4\rfloor-1}{\lfloor a/4\rfloor}\binom{l(3l+1)/2+m-1}{m} f(n)^{-3m-a} \leqq$$

$$\leqq \frac{1}{l!}\binom{\lfloor r(r+1)/8\rfloor+l(3l+1)/2+m+\lfloor a/4\rfloor}{m+\lfloor a/4\rfloor}\left(f(n)^3\right)^{-m-\lfloor a/4\rfloor}.$$

Put $\alpha=k(R)\beta=k(R)/(3\Delta+24)$. We consider two cases.

Case 1. $l\leqq \alpha r$.

Since $6m+2a\leqq rk(R)$ by (2.2) and $3m\leqq 3l\Delta/2\leqq 3\alpha r\Delta/2$, we have

$$m+a/4 \leqq r\left(k(R)+\alpha\Delta\right)/8 \leqq r\left(k(R)/6-\alpha\right).$$

So by Lemma 2.3, the above factor is at most

$$\exp\big(O(1)rk(R)\big)+\left[O(1)\left(\frac{r^2}{\lfloor k(R)/6-\alpha\rfloor rf(n)^3}+\frac{1}{f(n)^3}\right)\right]^{r(k(R)/6-\alpha)} \leqq \tag{2.18}$$

$$\leqq \exp\big(O(1)rk(R)\big)+\left(\frac{O(r+u)}{f(n)^3}\right)^{r(k(R)/6-\alpha)},$$

as $\alpha \leqq k(R)/8$.

Case 2. $l>\alpha r$.

Here we use $m+\lfloor a/4\rfloor \leqq rk(R)/6$, so that by Lemma 2.3 the contribution in this case is, for $r>0$, at most

$$\exp\big(O(1)rk(R)\big)+\frac{1}{(\alpha r)!}\left(\frac{O(r)}{f(n)^3}\right)^{rk(R)/6},$$

which is also at most (2.18).

Since each of the variables $l, s, h, w, t, d, x, y, m$ and a has at most $rk(R)+2uk(U)$ possible values, the number of possible sets of values is $\exp\big(O(rk(R)+uk(U))\big)$. Multiplying this by the factors remaining in (2.4) and (2.15), together with (2.16), (2.17) and (2.18), we now obtain

$$B(U,R)= \tag{2.19}$$

$$=\frac{(2u)!\,r!}{(u+r)^u}\left(\frac{O(1)f(n)^2}{n}\right)^{uk(U)+rk(R)/2}\left(\left(\frac{O(u+r)}{f(n)^2}\right)^{uk(U)/2}+1\right)\left(\left(\frac{O(u+r)}{f(n)^3}\right)^{r(k(R)/6-\alpha)}+1\right).$$

Here we have used $1/u!=\exp\big(O(u+r)\big)/(u+r)^u$.

To prove (i), we proceed by observing that the number of choices of the sets U and R given u and r is $\binom{n}{r}\binom{n-r}{2u}\leqq n^{r+2u}/(r!(2u)!)$. The expected number of automorphisms of G with u 2-cycles and r vertices in longer cycles, with $r+2u>0$, is at most this times (2.19), which is

$$(A_U^{uk(U)}+B_U^{uk(U)})(A_R^{rk(R)/6}+B_R^{rk(R)/6}), \tag{2.20}$$

where

$$A_U=O(1)(u+r)^{1/2-1/k(U)}f(n)/n^{1-2/k(U)},$$

$$B_U=O(1)f(n)^2/n^{1-2/k(U)},$$

$$A_R=O(1)(u+r)^{1-6\beta}f(n)^{3+18\beta}/n^{3-6/k(R)},$$

$$B_R=O(1)f(n)^6/n^{3-6/k(R)},$$

and $k(U)$ and $k(R)$ each have whatever values, between δ and Δ inclusive, maximise (2.20). Since $n^2/(u+r)>1$ for $n>1$, we have

$$n^{2/k(U)}(u+r)^{-1/k(U)}\leqq n^{2/\delta}(u+r)^{-1/\delta},$$

so that A_U is maximised for $k(U)=\delta$. Clearly, the same holds for B_U, A_R and B_R. Thus, putting $C=f(n)^2/n^{1-2/\delta}$, we have $B_U=O(1)C$, and $B_R=O(1)C^3$. Since $u+r\leqq n$ we also have $A_U=O(1)C^{1/2}$. Also, putting $D=f(n)^{3+18\beta}/n^{2+6\beta-6/\delta}$, we have $A_R=O(1)D$ as $u+r\leqq n$.

Since $C^{3/2+9\beta}/D=n^{(1-6\beta)(1-6/\delta)/2}$ and $\beta<1/24$, $C=o(1)$ implies $D=o(1)$ for $\delta\geqq 6$ and $D=o(1)$ implies $C=o(1)$ for $3\leqq\delta\leqq 5$. By the hypotheses of (i), therefore, C and D are both $o(1)$ in either case, and so A_U, B_U, A_R and B_R are all $o(1)$. Hence (2.20) attains its maximum possible value for n large by putting $k(U)=k(R)=\delta$. Thus $T(n)$ is at most the sum, over all u and r such that $2\leqq 2u+r\leqq n$ with $r\neq 1$ and $r\neq 2$, of $(A_U^{u\delta}+B_U^{u\delta})(A_R^{r\delta/6}+B_R^{r\delta/6})$. Since A_U, A_R, B_U and B_R are all $o(1)$, this sum is $O(1)$ times the sum of its two "worst" terms; i.e. the term with $u=1$ and $r=0$, and that with $u=0$ and $r=3$. Here and henceforth, $O(1)$ depends on the functions represented by $o(\cdot)$ in the statement of the theorem. We now have from (2.20) that

$$T(n) = A'_U+B'_U+A'_R+B'_R$$

where

$$A'_U = (O(1)f(n))^{\delta}/n^{\delta-2},$$

$$B'_U = (O(1)f(n))^{2\delta}/n^{\delta-2},$$

$$A'_R = (O(1)f(n))^{3\delta/2+9\beta\delta}/n^{3(\delta-2)/2},$$

and

$$B'_R = (O(1)f(n))^{3\delta}/n^{3(\delta-2)/2}.$$

Clearly $A'_U=O(1)B'_U$, and $A'_R=O(1)B'_R$ as $9\beta\leqq 3/11$ for $\Delta\geqq\delta\geqq 3$. Also, $B'_R=B'_U(O(1)C)^{\delta/2}$. As we have already seen, $C=o(1)$ and thus $B'_R=o(1)B'_U$. It follows that $T(n)=O(1)B'_U$, and (i) is established. Naturally, this argument is simpler in the special case of k-regular graphs, for which $k(U)=k(R)=k$.

To prove (ii) we first let u_i and r_i denote the numbers of vertices of degree i in U and R respectively. Then the number of choices of U and R with $|U|=2u$ and $|R|=r$ is at most $\prod\binom{n_i}{u_i}\binom{n_i}{r_i}$, where the product here and in what follows is for $i=1, \dots, \Delta$. Also, $2uk(U)=\sum iu_i$ and $rk(R)=\sum ir_i$. Hence, by (2.19), $T(n)$ is at most the sum over all u and r such that $1\leqq u+r\leqq n$ and $r\geqq 3$, of

$$(2.21)\qquad (2u)!\,(u+r)^{-u}\sum{}^{*}\Big(\prod\binom{n_i}{u_i}(A_U^i+B^i)^{u_i}\,r!\prod\binom{n_i}{r_i}(A_R^i+B^i)^{r_i}\Big),$$

where $\sum^*$ denotes the sum over all u_i and r_i such that $u_1+\dots+u_\Delta=2u$ and $r_1+\dots+r_\Delta=r$, and this time

$$A_U = (u+r)^{1/4}O(1)f(n)^{1/2}n^{-1/2},$$

$$B = O(1)f(n)n^{-1/2},$$

and

$$A_R = (u+r)^{1/6-\beta}O(1)f(n)^{3\beta+1/2}n^{-1/2} \leqq O(1)f(n)^{3\beta+1/2}n^{-1/3-\beta}.$$

To obtain (2.21) we also used $A_U^{2uk(U)}+B^{2uk(U)}\leqq\prod(A_U^i+B^i)^{u_i}$.

Since $r!\prod\binom{n_i}{r_i}\leqq\binom{r}{r_1,\dots,r_\Delta}\prod n_i^{r_i}$ and $(2u)!\prod\binom{n_i}{u_i}\leqq\binom{2u}{u_1,\dots,u_\Delta}\prod n_i^{u_i}$, (2.21) is at most

$$(2.22)\qquad (u+r)^{-u}\Big(\sum_{i=1}^{\Delta} n_i(A_U^i+B^i)\Big)^{2u}\Big(\sum_{i=1}^{\Delta} n_i(A_R^i+B^i)\Big)^{r}.$$

We have from the hypotheses of (ii) that $f(n)^{3\beta+1/2}$ is $o(n^{\beta})$, so certainly $nB^3=o(1)$, as $\beta\leqq 1/9$. Thus the sum of n_iB^i from $i=3$ to Δ is $o(1)$. Also, for $i=1$ and 2, n_iB^i is

$$O(1)\left(n_if(n)^{3\beta i+i/2}/n^{\beta i+i/3}\right)\left(f(n)^{i/2-3\beta i}/n^{i/6-\beta i}\right),$$

and the two factors here are both $o(1)$ by the hypotheses of (ii). For $i\geqq 3$, $n_iA_U^i(u+r)^{-1/2}$ is at most

$$n_i(u+r)^{(i-2)/4}O(1)^if(n)^{i/2}n^{-i/2}\leqq\left(O(1)f(n)^{1/2}\right)^i/n^{i/4-1/2}\leqq\left(O(1)f(n)^{1/2}n^{-1/12}\right)^i.$$

Since $f(n)^6=o(n)$, it follows that the sum of $n_iA_U^i(u+r)^{-1/2}$ from $i=3$ to Δ is $o(1)$. For $i=1$ and 2, $(u+r)^{-1/2}n_iA_U^i\leqq n_iB^i=o(1)$ as we have seen. Finally, $n_iA_R^i$ is $o(1)$ for $i=1$ and 2 by the second hypothesis of (ii), and its sum from $i=3$ to Δ is $o(1)$ by the first hypothesis. It follows that the sum of (2.22) or (2.21) over u and r such that $1\leqq u+r\leqq n$ is $o(1)$, and thus $T(n)=o(1)$. ∎

From the proof of Theorem 2.4 it is possible to obtain a bound on $T(n)$ in part (ii) as is done in part (i), if so desired. Also, (2.22) can be used to take account of n_i for $i\geqq 3$ to obtain a slightly stronger but more complicated variant of (ii).

3. Applications

To apply Theorem 2.4, we need an acceptable function $f(n)$. All our applications will be to classes $\mathcal{S}$ in which all the graphs in $\mathcal{S}_n$ have equal probabilities.

Estimates of such an $f(n)$ have been previously found (explicitly or implicitly) for many classes of graphs (see [2], [3], [6], [7], [8], [9] and [13] for example). In all of these except [7], [8] and [9], the maximum degree must be constant or very slowly increasing.

Here we will find an acceptable function $f(n)$ for a very general class of graphs which includes all those considered in the papers mentioned above. The proof technique is a generalization of that used in [7]. By "switching" the edges of a subgraph with other edges of the graph in almost all possible ways, we obtain a reasonable bound on the probability of that subgraph occurring in a random graph.

For each integer n, choose a partition π of $V=\{v_1,v_2,\dots,v_n\}$, integers $m_1\geqq 0$ and $m_2\geqq 1$, and a non-negative integer sequence $g_1,g_2,\dots,g_n$ (not all zero). The set $\mathcal{S}_n=\mathcal{S}_n(m_1,m_2,\pi,g_1,g_2,\dots,g_n)$ is defined to be the class of all graphs G with vertex set V such that

(i) the degree of v_i is g_i $(1\leqq i\leqq n)$, and

(ii) the multiplicity of an edge (possibly loop) vw of G is at most m_1 if v and w are in the same cell of π and at most m_2 otherwise.

For example, if $m_1=0$, $m_2=1$ and π is discrete (n cells) we obtain the class of simple graphs with degree sequence $g_1,g_2,\dots,g_n$. If $m_1=0$, $m_2=1$ and π has two cells, we obtain simple bipartite graphs.

If $\mathcal{S}_n\neq\emptyset$ (which we assume henceforth), $\mathcal{S}_n$ is made into a probability space by giving each graph equal probability.

Let L and H be graphs with vertex set V and degrees $l_1,l_2,\dots,l_n$ and $h_1,h_2,\dots,h_n$, respectively, where $h_i\leqq l_i\leqq g_i$ for $1\leqq i\leqq n$. Write $H\subseteq L$ if H is a subset of L. For $H\subseteq L$, $\mathcal{S}_n(L,H)$ is defined to be the set of all graphs $G\in\mathcal{S}_n$ such that, for each edge vw of non-zero multiplicity in L, the multiplicity of vw in G is the same as the multiplicity of vw in H.

For $v, w \in V$, define $H+vw$ to be the graph formed from H by increasing the multiplicity of the edge vw by one. Define $M_G = 1/2 \sum_{i=1}^{n} g_i$, $M_L = 1/2 \sum_{i=1}^{n} l_i$, $M_H = 1/2 \sum_{i=1}^{n} h_i$, $\Delta = \max_{1 \leq i \leq n} g_i$, and $m = \max \{m_1, m_2\}$.

Lemma 3.1. *Let* $H + v_i v_j \subseteq L$, *where* $i \neq j$. *Then*

$$((M_G - M_H)/m - 8\Delta^2) |\mathcal{S}_n(L, H+v_i v_j)| \leqq (g_i - h_i)(g_j - h_j) |\mathcal{S}_n(L, H)|.$$

Proof. Let K be the number of pairs (G_1, G_2), where $G_1 \in \mathcal{S}_n(L, H+v_i v_j)$, $G_2 \in \mathcal{S}_n(L, H)$ and G_2 can be obtained from G_1, via one of the operations shown in Figure 1 (for some x, y).

In Figure 1, the numbers on edges indicate their multiplicities, and edges not drawn as loops are not loops. Given G_1, it is clear that x and y can be chosen in at least $(M_G - M_H)/m - 8\Delta^2$ ways, no matter what π is. (Essentially, we can choose any edge xy in $G \setminus H$ unless it is either too close to v_i or v_j, or if the operation will violate π. The number eliminated by the latter restrictions is at most $8\Delta^2$.) Conversely, given G_2, x and y can be chosen in at most $(g_i - h_i)(g_j - h_j)$ ways. Therefore

$$K \geqq ((M_G - M_H)/m - 8\Delta^2) |\mathcal{S}_n(L, H+v_i v_j)|,$$

and

$$K \leqq (g_i - h_i)(g_j - h_j) |\mathcal{S}_n(L, H)|,$$

which imply the required inequality. ∎

Lemma 3.2. *Suppose that H has only loops, and let* $H + v_i v_i \subseteq L$. *Then*

$$((M_G - M_H)/m - 8\Delta^2) |\mathcal{S}_n(L, H+v_i v_i)| \leqq (g_i - h_i)(g_i - h_i - 1) |\mathcal{S}_n(L, H)|.$$

Proof. Let K be the number of pairs (G_1, G_2), where $G_1 \in \mathcal{S}_n(L, H+v_i v_i)$, $G_2 \in \mathcal{S}_n(L, H)$ and G_2 can be obtained from G_1 via one of the operations shown in Figure 2 (for some x and y).

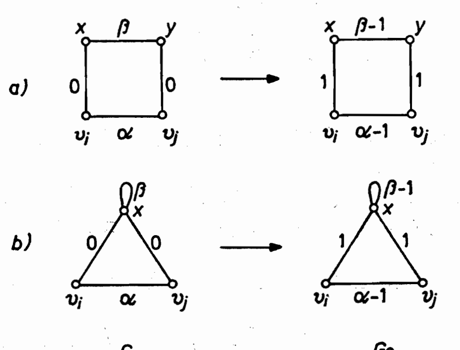

Fig. 1

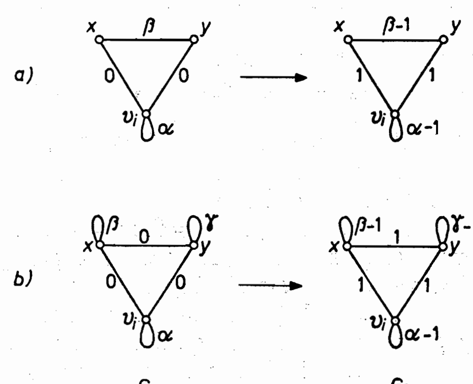

Fig. 2

As in the previous Figure, edges not drawn as loops are not loops. Given G_1 there are at least $(M_G-M_H)/m-8\Delta^2$ ways of choosing x and y. Conversely, given G_2 there are at most $(g_i-h_i)(g_i-h_i-1)$ ways of choosing x and y. Therefore

$$K \geqq \big((M_G-M_H)/m-8\Delta^2\big)|\mathcal{S}_n(L, H+v_iv_i)|,$$

and

$$K \leqq (g_i-h_i)(g_i-h_i-1)|\mathcal{S}_n(L, H)|,$$

which imply the required inequality. ∎

Theorem 3.3. *Suppose that* $M_G-M_L \geqq 8m\Delta^2$. *Then*

$$\frac{|\mathcal{S}_n(L, L)|}{|\mathcal{S}_n|} \leqq \frac{m^{M_L}\prod_{i=1}^{n} g_i^{[l_i]}}{(M_G-8\Delta^2 m)^{[M_L]}},$$

where $x^{[y]}=x(x-1)\cdots(x-y+1)$.

Proof. Repeated application of Lemmas 3.1 and 3.2 show that the expression on the right is an upper bound on $|\mathcal{S}_n(L, L)|/|\mathcal{S}_n(L, \emptyset)|$, where $\emptyset$ denotes the graph with no edges. This in turn is clearly an upper bound on $|\mathcal{S}_n(L, L)|/|\mathcal{S}_n|$. ∎

Corollary 3.4. *Let* $\Delta^2 m=o(M_G)$ *as* $n\to\infty$. *Then*

$$\frac{e\big(1+o(1)\big)mn\Delta^2}{M_G}$$

is an acceptable function for $\mathcal{S}$.

Proof. A simple application of Stirling's formula is that $x^{[y]} \geqq (x/e)^y$, if $x \geqq y \geqq 0$.

The probability that L is a full subgraph of a random graph in $\mathcal{S}_n$ is $|\mathcal{S}_n(L, L)|/|\mathcal{S}_n|$.

Suppose firstly that $M_G-M_L \geqq 8m\Delta^2$. Then, by Theorem 3.3,

$$\frac{|\mathcal{S}(L, L)|}{|\mathcal{S}_n|} \leqq \frac{m^{M_L}\Delta^{2M_L}}{(M_G-8\Delta^2 m)^{[M_L]}} \leqq \left(\frac{me\Delta^2}{M_G-8\Delta^2 m}\right)^{M_L} = \left(\frac{m\Delta^2 e\big(1+o(1)\big)}{M_G}\right)^{M_L}.$$

Suppose, on the other hand, that $M_G-M_L<8m\Delta^2$. Then, by changing the the multiplicities of one or more of the edges of L to zero, we obtain a subgraph L' of L such that L' has at most $M_G-8\Delta^2 m$ and at least $M_G-(8\Delta^2+1)m$ edges. Then

$$\frac{|\mathcal{S}(L, L)|}{|\mathcal{S}_n|} \leqq \frac{|\mathcal{S}(L', L')|}{|\mathcal{S}_n|} \leqq \left(\frac{m\Delta^2 e\big(1+o(1)\big)}{M_G}\right)^{M_G-(8\Delta^2+1)m} =$$

$$= \left(\frac{m\Delta^2 e\big(1+o(1)\big)}{M_G}\right)^{M_G} \leqq \left(\frac{m\Delta^2 e\big(1+o(1)\big)}{M_G}\right)^{M_L}. \quad ∎$$

The function $f(n)$ in Corollary 3.4 can be substituted into Theorem 2.4 to obtain a quite general bound on $T(n)$. Particularly interesting special cases are for regular graphs and for graphs with Δ bounded, so we examine the resulting bound on

$T(n)$ for these types of graphs in particular. We emphasise that since Corollary 3.4 is independent of π, the following four corollaries have equally valid formulations for simple graphs, general graphs, bicoloured graphs, or for graphs with any colour partitions, so long as m is uniformly bounded. It is likely that even the latter restriction can be weakened, but we have not done so.

Corollary 3.5. *Suppose $k(n)$ is any integer function satisfying $3 \leqq k(n) = O(n^{1/2-\varepsilon})$, for $\varepsilon > 0$. If $\mathcal{S}_n$ consists of the set of labelled $k(n)$-regular simple graphs on n vertices, then $T(n) = o(1)$.*

Proof. By Corollary 3.4 we can use $f(n) = O\big(k(n)\big)$. The claim now follows from Theorem 2.4(i). ∎

Corollary 3.5 has been independently discovered by Bollobás [5] for constant $k(n)$.

Corollary 3.6. *If $\mathcal{S}$ is such that m and Δ are bounded above and $\delta \geqq 3$, then $T(n) = O(n^{2-\delta})$.*

Proof. In this case $f(n) = O(\Delta^2)$ is bounded. ∎

Corollary 3.7. *If $\mathcal{S}$ is such that m and Δ are bounded and $n_i = o(n^{\beta i + i/3})$ for $i \in \{1, 2\}$, then $T(n) = o(1)$.*

Proof. This is from part (ii) of Theorem 2.4. ∎

Note that since the bounds in Theorem 2.4 are uniform over all appropriate degree sequences, the last three corollaries and all similar results also apply to classes $\mathcal{S}$ containing all graphs with any number of degree sequences. The following extends Corollary 3.6 in this way.

Corollary 3.8. *The expected number of non-trivial automorphisms of a labelled simple graph with a given degree sequence $k_1, \ldots, k_n$ in which each k_i is at least $\delta \geqq 3$ and at most Δ, is $O(n^{2-\delta})$ where $O(\,)$ denotes a bound depending only on Δ.* ∎

In order to study the probability of a random unlabelled graph having just the identity automorphism group we use the following simple result. For any graph G let $a(G)$ denote the order of the automorphism group of G. Let $\mathcal{U}$ be a set of unlabelled graphs on n vertices and let $\mathcal{L} = \mathcal{L}(\mathcal{U})$ be the set of labelled versions of graphs in $\mathcal{U}$. Let $\varepsilon_1 = \varepsilon_1(\mathcal{U})$ be the proportion of elements $G \in \mathcal{U}$ with $a(G) > 1$, and let $\varepsilon_2 = |\mathcal{L}|^{-1} \sum_{G \in \mathcal{L}} \big(a(G) - 1\big)$.

Lemma 3.9. $\varepsilon_1 \leqq 2\varepsilon_2/(1+\varepsilon_2) \leqq 2\varepsilon_2$.

Proof. $|\mathcal{L}| \leqq (1-\varepsilon_1)n!\,|\mathcal{U}| + \varepsilon_1 n!\,|\mathcal{U}|/2 \leqq \big(1-(\varepsilon_1/2)\big)n!\,|\mathcal{U}|$.
Also

$$\varepsilon_2 = |\mathcal{L}|^{-1}\left(\sum_{G \in \mathcal{U}} \frac{n!}{a(G)}\, a(G)\right) - 1 = n!\,\frac{|\mathcal{U}|}{|\mathcal{L}|} - 1.$$

The lemma follows. ∎

The *partition* of an unlabelled graph is the unordered multiset of degrees of its vertices. As usual, δ denotes the minimum degree and Δ the maximum.

Corollary 3.10. *Consider a partition P of n degrees in which $\delta \geqq 3$. The proportion of unlabelled simple graphs on n vertices with partition P which have at least one non-identity automorphism is $O(n^{2-\delta})$, where $O(\,)$ denotes a bound depending only on Δ.*

Proof. The result of Corollary 3.8 can be summed over all degree sequences obtainable by ordering the partition P. Lemma 3.9 now completes the proof, with $\mathscr{U}$ being the set of unlabelled graphs with partition P. ∎

One can alternatively obtain a result more general than Corollary 3.10 by using the full power of Theorem 2.4 and Corollary 3.4, instead of Corollary 3.8. But one of our main objects is the establishment of asymptotic formulae for the numbers of various types of unlabelled graphs, based on existing formulae for numbers of labelled graphs. In the latter formulae, Δ is usually bounded or increasing very slowly.

Lemma 3.11. *With $\mathscr{U}$, $\mathscr{L}$ and ε_1 as in Lemma 3.9, if $\varepsilon_1 = o(1)$ then $|\mathscr{U}| = |\mathscr{L}|(1+o(1))/n!$.*

Proof. $n!\,|\mathscr{U}| \geqq |\mathscr{L}| \geqq n!\,|\mathscr{U}|(1-\varepsilon_1)$. ∎

Corollary 3.10 and Lemma 3.11 can be used with the known asymptotic formula for the number of labelled simple graphs with given degree sequence (see Bender and Canfield [3]) to obtain the following asymptotic formula for the number of unlabelled graphs with a given partition. One merely has to divide the formula for labelled graphs by $n!$ and multiply by the appropriate multinomial to account for the number of ordered degree sequences corresponding to an unordered partition. Similar formulae are obtainable in the same way for multigraphs from [3] for pseudographs from [14] and for bipartite graphs with or without multiple edges from Békéssy et al. [1].

Corollary 3.12. *The number of unlabelled simple graphs with precisely $d(i)$ points of degree i for $i=3, \dots, \Delta$ is*

$$\frac{(2m)!\, e^{-\gamma^2-\gamma}(1+o(1))}{m!\, 2^m \prod_{i=3}^{\Delta} (d(i)!\,(i!)^{d(i)})}$$

as $n \to \infty$ where $m = 1/2 \sum_{i=3}^{\Delta} i d(i)$ is an integer, $n = \sum_{i=3}^{\Delta} d(i)$, $\gamma = \sum_{i=3}^{\Delta} i(i-1)d(i)/4m$, and $o(1)$ denotes a function depending on the $d(i)$ but whose convergence to O as $n \to \infty$ is uniform over all partitions as long as Δ is bounded. ∎

Amongst other things, this gives an asymptotic formula for the numbers of unlabelled r-regular simple graphs, with r fixed, agreeing with [5]. The same formula is in fact valid $r = o(n^{1/3})$, as is proved by the application of Corollary 3.5 to the labelled enumeration in [9]. A similar enumeration of unlabelled regular bipartite graphs with degree $o(n^{1/3})$ follows from [8].

Our final observation is the well-known trivial connection which translates many of the known properties of random labelled graphs to properties of random unlabelled graphs. Firstly, if $\mathscr{U}$ is a class of unlabelled graphs, let $\mathscr{U}_n$ denote the subclass of n-vertex graphs in $\mathscr{U}$.

Lemma 3.13. *Suppose $\mathscr{U}' \subseteqq \mathscr{U}$ and define $P(n) = |\mathscr{L}(\mathscr{U}'_n)|/|\mathscr{L}(\mathscr{U}_n)|$. Suppose further that $\varepsilon_1(\mathscr{U}_n) = o(1)P(n)$ as $n \to \infty$. Then $|\mathscr{U}'_n|/|\mathscr{U}_n| = P(n)(1+o(1))$ as $n \to \infty$.* ∎

Thus, for instance, if $\mathscr{U}$ is the set of k-regular simple graphs (for $k \geqq 3$ fixed) and $\mathscr{U}'$ is the subset of k-connected k-regular graphs, then we know by the results of [15] that $P(n)=1+o(1)$ in Lemma 3.13. Thus Corollary 3.8 and Lemma 3.9 imply that the proportion of unlabelled k-regular simple graphs on n vertices which are k-connected is asymptotic to 1 as $n \to \infty$ $(k \geqq 3)$. Similarly, the proportions of unlabelled k-regular simple graphs with given cyclic connectivity are obtained from [15], and with given girth from [14]. The k-connectivity and girth results also extend to graphs with arbitrary degree sequence s, as long as the degrees are in the range $3, \ldots, \Delta$ with Δ fixed.

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