Euler-Maclaurin summation

Brendan McKay, Oct 2004

The Bernoulli number B_s is the coefficient of $t^s/s!$ in the Taylor expansion of $t/(e^t-1)$. Examples: $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, $B_4 = -\frac{1}{30}$, $B_5 = 0$, $B_6 = \frac{1}{42}$, $B_7 = 0$, $B_8 = -\frac{1}{30}$.

Theorem 1 *Let $m, n \geq 1$ be integers. Let $f(t)$ be a real or complex-valued function defined on $0 \leq t \leq n$. If $f^{(2m)}$ is absolutely integrable on $(0, n)$, then*

$$\sum_{i=0}^{n} f(i) = \int_0^n f(x)\,dx + \tfrac{1}{2}(f(0) + f(n)) + \sum_{s=1}^{m-1} \frac{B_{2s}}{(2s)!}(f^{(2s-1)}(n) - f^{(2s-1)}(0)) + R_m, \quad (1)$$

where

$$|R_m| \leq 2\frac{|B_{2m}|}{(2m)!}\int_0^n |f^{(2m)}(x)|\,dx \qquad [2] \quad (2)$$

$$|R_m| \leq \frac{|B_{2m-2}|}{(2m-2)!}\int_0^n |f^{(2m-2)}(x)|\,dx \qquad [1] \quad (3)$$

Moreover, if $f^{(2m)}(t)$ does not change sign in $(0, n)$ then R_m is bounded in absolute value by twice the first neglected term in (1).

Notes

1. "Absolutely integrable" just means that the absolute value has a finite integral.
2. Bound (3) is not much use for $m = 1$. In fact it seems that bound (2) is better most of the time.
3. Usually it will suffice to use
$$\int_0^n |g(t)|\,dt \leq n \max_{0 \leq t \leq n} |g(t)|$$
instead of evaluating the integral in R_m.
4. If you need to estimate an infinite sum, you can just let $n \to \infty$.
5. A useful bound is
$$\frac{|B_{2m}|}{(2m)!} < \frac{10}{3(2\pi)^{2m}}$$
for $m \geq 1$. The factor $\frac{10}{3}$ can be reduced to $\frac{13}{6}$ for $m \geq 2$.

Example

Let's estimate the sum

$$\sum_{k=n}^{2n} \sqrt{k}$$

as $n \to \infty$. To match this to the theorem, define $f(t) = \sqrt{n+t}$.

Try $m = 1$. By Maple we have

$$\int_0^n f(t) = \tfrac{3}{2}(2^{3/2} - 1)n^{3/2}.$$

We also have $f(0) = O(n^{1/2})$ and $f(1) = O(n^{1/2})$. The first two derivatives are $f'(t) = \frac{1}{2}(n+t)^{-1/2}$ and $f''(t) = -\frac{1}{4}(n+t)^{-3/2}$. Therefore, by (2) and the third note, $R_1 = O(n^{-1/2})$. (This also follows from the "Moreover" part of the theorem, since $f''(t) < 0$ for $0 < t < n$.)

Consequently,

$$\sum_{k=n}^{2n} \sqrt{k} = \tfrac{3}{2}(2^{3/2} - 1)n^{3/2} + O(n^{1/2}).$$

References

[1] A. M. Odlyzko, Asymptotic Enumeration Methods, *in* Handbook of Combinatorics, Vol II, Elsevier, 1995.

[2] R. Wong, Asymptotic Approximation of Integrals, Academic Press, 1989.