Euler-Maclaurin summation

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The Bernoulli number $B_s$ is the coefficient of $t^s/s!$ in the Taylor expansion of $t/(e^t-1)$. Examples: $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, $B_4 = -\frac{1}{30}$, $B_5 = 0$, $B_6 = \frac{1}{42}$, $B_7 = 0$, $B_8 = -\frac{1}{30}$.

**Theorem 1** Let $m, n \geq 1$ be integers. Let $f(t)$ be a real or complex-valued function defined on $0 \leq t \leq n$. If $f^{(2m)}$ is absolutely integrable on $(0, n)$, then

$$
\sum_{i=0}^{n} f(i) = \int_{0}^{n} f(x) \, dx + \frac{1}{2} (f(0) + f(n)) + \sum_{s=1}^{m-1} \frac{B_{2s}}{(2s)!} (f^{(2s-1)}(n) - f^{(2s-1)}(0)) + R_m, \tag{1}
$$

where

$$
|R_m| \leq 2 \left| \frac{B_{2m}}{(2m)!} \right| \int_{0}^{n} |f^{(2m)}(x)| \, dx \tag{2}
$$

$$
|R_m| \leq \left| \frac{B_{2m-2}}{(2m-2)!} \right| \int_{0}^{n} |f^{(2m-2)}(x)| \, dx \tag{3}
$$

Moreover, if $f^{(2m)}(t)$ does not change sign in $(0, n)$ then $R_m$ is bounded in absolute value by twice the first neglected term in (1).

**Notes**

1. “Absolutely integrable” just means that the absolute value has a finite integral.

2. Bound (3) is not much use for $m = 1$. In fact it seems that bound (2) is better most of the time.

3. Usually it will suffice to use

$$
\int_{0}^{n} |g(t)| \, dt \leq n \max_{0 \leq t \leq n} |g(t)|
$$

instead of evaluating the integral in $R_m$.

4. If you need to estimate an infinite sum, you can just let $n \to \infty$.

5. A useful bound is

$$
\left| \frac{B_{2m}}{(2m)!} \right| < \frac{10}{3(2\pi)^{2m}}
$$

for $m \geq 1$. The factor $\frac{10}{3}$ can be reduced to $\frac{13}{6}$ for $m \geq 2$. 

1
Example

Let’s estimate the sum
\[ \sum_{k=n}^{2n} \sqrt{k} \]
as \( n \to \infty \). To match this to the theorem, define \( f(t) = \sqrt{n + t} \).

Try \( m = 1 \). By Maple we have
\[ \int_0^n f(t) = \frac{3}{2}(2^{3/2} - 1)n^{3/2}. \]

We also have \( f(0) = O(n^{1/2}) \) and \( f(1) = O(n^{1/2}) \). The first two derivatives are \( f'(t) = \frac{1}{2}(n + t)^{-1/2} \) and \( f''(t) = -\frac{1}{4}(n + t)^{-3/2} \). Therefore, by (2) and the third note, \( R_1 = O(n^{-1/2}) \). (This also follows from the “Moreover” part of the theorem, since \( f''(t) < 0 \) for \( 0 < t < n \).)

Consequently,
\[ \sum_{k=n}^{2n} \sqrt{k} = \frac{3}{2}(2^{3/2} - 1)n^{3/2} + O(n^{1/2}). \]

References
