An Optimal Dimensionality Sampling Scheme on the Sphere for Antipodal Signals in Diffusion Magnetic Resonance Imaging

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Contribution

We propose a sampling scheme on the sphere and spherical harmonic transform (SHT) for reconstruction of the diffusion signal in diffusion magnetic resonance imaging (dMRI) that:

- Requires the optimal number of samples $N_{\rm O}$.
- Achieves reconstruction accuracy on the order of 10^{-14}.
- Has rotationally invariant reconstruction accuracy.
- Allows fast SHT computation with lowest complexity.

It is the only sampling scheme with all these desirable properties.

Problem Background

dMRI is used to determine structure and connectivity of white matter in the brain.

Acquisition and reconstruction of diffusion signal from $\boldsymbol{q}$-space measurements, where $\boldsymbol{q}$ is the diffusion wave vector, is a central problem in dMRI. Samples are obtained on a single sphere or multiple spheres in $\boldsymbol{q}$-space.

Number of measurements (images) severely limited; a sampling scheme that allows accurate and computationally efficient reconstruction using minimum number of samples is desirable.

Mathematical Background

Single Sphere Sampling: Diffusion signal at fixed $\boldsymbol{q}$-space radius $d(\theta,\phi)$ is a signal on the sphere with co-latitude $\theta \in [0,\pi]$ and longitude $\phi \in [0,2\pi)$. Spherical harmonics $Y_\ell^m(\theta,\phi)$, degree $\ell \le 0$ and order $m \in [-\ell,\ell]$, form a complete orthonormal set of basis functions on the sphere.

Spherical harmonic (spectral domain) expansion:

$$d(\theta,\phi) = \sum_{\substack{\ell=0\\ \ell\,\text{even}}}^{L-1} \sum_{m=-\ell}^{\ell} (d)_\ell^m Y_\ell^m(\theta,\phi), \quad L \text{ odd}, \tag{1}$$

where $(d)_\ell^m$ denotes the spherical harmonic coefficient of degree ℓ and order m. The band-limit L depends on the q-space radius. Eq. (1) is known as the inverse SHT.

Spherical harmonic transform (SHT):

$$(d)_\ell^m \triangleq \int_{\mathbb{S}^2} d(\theta,\phi)\overline{Y_\ell^m(\theta,\phi)}\,\sin\theta\, d\theta\, d\phi. \tag{2}$$

The SHT is calculated numerically; for accurate reconstruction of $d(\theta,\phi)$, a sampling scheme must allow accurate computation of Eq. (2).

Antipodally symmetry: $d(\theta,\phi) = d(\pi-\theta,\phi+\pi)$. Since $Y_\ell^m(\theta,\phi) = Y_\ell^m(\pi-\theta,\pi+\phi)$ for even ℓ and $Y_\ell^m(\theta,\phi) = -Y_\ell^m(\pi-\theta,\pi+\phi)$ for odd ℓ, expansion of $d(\theta,\phi)$ (Eq. (1)) only includes even ℓ spherical harmonics.

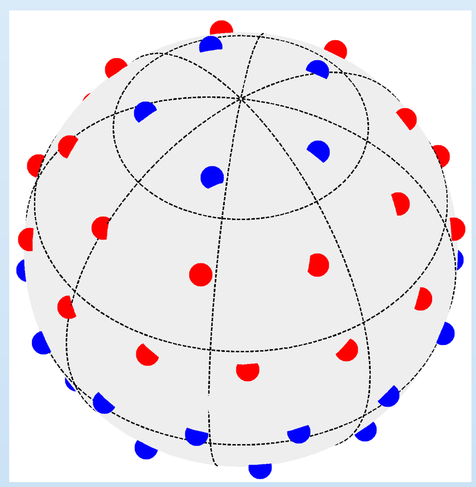
Optimal dimensionality: $N_{\rm O} = L(L+1)/2$ spherical harmonics in Eq. (1), the optimal dimensionality attainable by any sampling scheme that allows accurate computation of SHT.

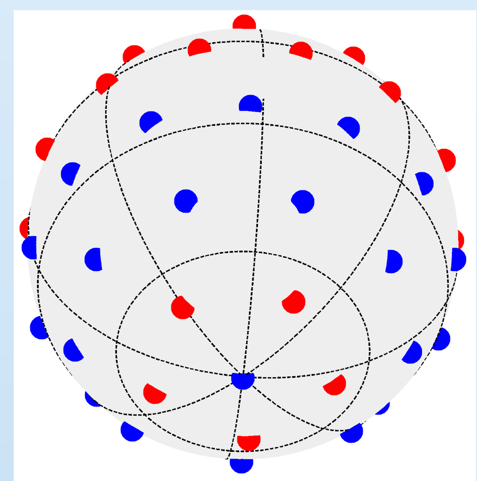
Comparison with Existing Schemes

Number of samples required by sampling schemes allowing accurate SHT computation:

Sampling Scheme	Number of Samples
Equiangular [1]	$2L^2$
Spherical Design [2]	$> L^2$
Optimal Dimensionality [3]	L^2
Proposed	$L(L+1)/2 = N_{\rm O}$

Less samples used by proposed scheme will reduce scan time required for accurate reconstruction of diffusion signal.





Proposed Sampling Scheme Structure

Customise [3], which has L iso-latitude rings composed of $2n+1$, $n = 0,1,\ldots,L-1$ samples, for antipodally symmetric signals:

- Place rings in pairs so antipodal to one another. With location of rings:

$$\boldsymbol{\theta} \triangleq [0,\ldots,\pi-\theta_{L-3},\theta_{L-3},\pi-\theta_{L-1},\theta_{L-1}]^T, \quad L \text{ odd}. \tag{3}$$

- k-th sample location, denoted by ϕ_k^n, in ring placed at θ_n:

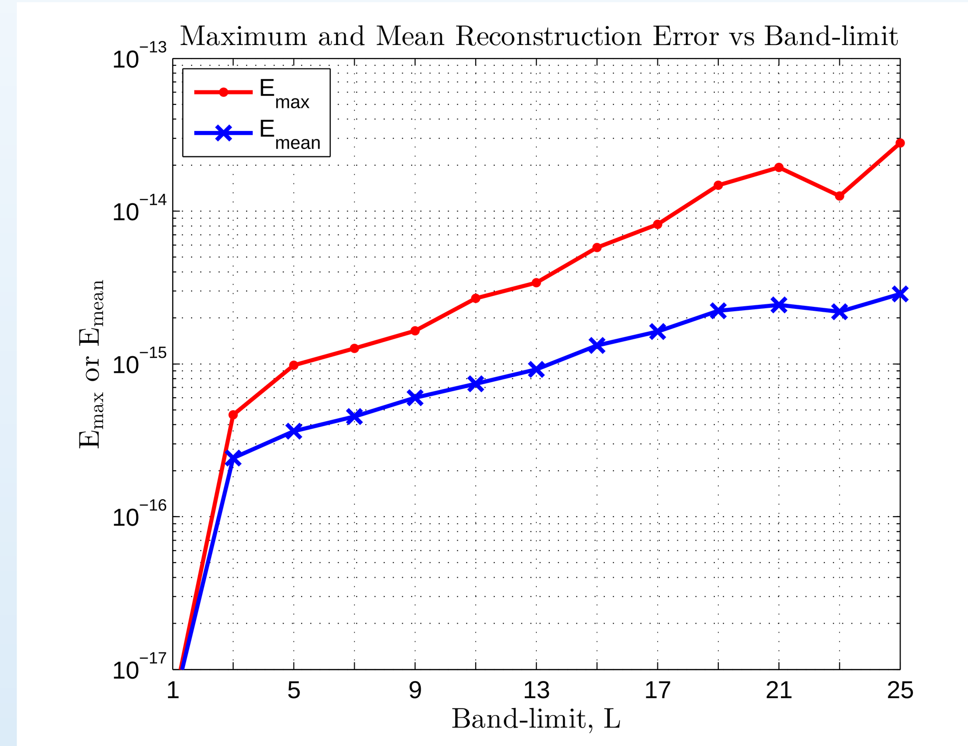
$$\phi_k^n \triangleq \begin{cases} \frac{2k\pi}{2n+1}, & n = 0,\,2,\,\ldots,\,L-1, \quad k \in [0,\,2n], \\ \frac{\pi(2k+1)}{2n+3}, & n = 1,\,3,\,\ldots,\,L-2, \quad k \in [0,\,2(n+1)]. \end{cases} \tag{4}$$

Samples in ring θ_n are antipodal to samples in θ_{n-1} for $n = 2,\,4,\,\ldots,\,L-1$; measurements only need to be taken over rings θ_n. Since measurements only taken over $(L+1)/2$ rings, total number of measurements is

$$\sum_{\substack{n=0\\ n\,\text{even}}}^{L-1} (2n+1) = \frac{(L+1)L}{2} = N_{\rm O}, \tag{5}$$

the optimal spatial dimensionality.

The proposed sampling scheme for $L = 7$ is shown above; locations where measurements are taken are blue and where antipodal symmetry is used to evaluate $d(\theta,\phi)$ are red.



Numerical Accuracy Experiment

Reconstruction accuracy for any band-limited antipodal signal is determined by obtaining:

1. Spherical harmonic coefficients of band-limited antipodally symmetric test signal $(f_{\rm t})_\ell^m$ for $0 < \ell < L$, ℓ even, $|m| \le \ell$ real and imaginary parts uniformly distributed $\in [-1,\,1]$.
2. $f_{\rm t}$ in spatial domain over proposed sampling grid using inverse SHT.
3. Spherical harmonic coefficients of the reconstructed signal $(f_{\rm r})_\ell^m$ using SHT
4. Maximum error $E_{\max}$ and mean error $E_{\rm mean}$:

$$E_{\max} \triangleq \max|(f_{\rm t})_\ell^m - (f_{\rm r})_\ell^m|, \tag{6}$$

$$E_{\rm mean} \triangleq \frac{1}{L^2}\sum_{\ell=0}^{L-1}\sum_{m=-\ell}^{\ell} |(f_{\rm t})_\ell^m - (f_{\rm r})_\ell^m|, \tag{7}$$

$E_{\max}$ and $E_{\min}$ are on order of numerical precision, showing proposed sampling scheme and SHT allows for accurate reconstruction of *any* band-limited antipodally symmetric signal.

Determining Ring Locations

Location of rings of iso-latitude samples determined using the following procedure:

1. Define a set of equiangular $(L+1)/2$ samples along θ:

$$\Theta = \left\{\frac{\pi(2t+1)}{2L-1}\right\}, \quad t = 0,1,\ldots,\frac{L-1}{2}. \tag{8}$$

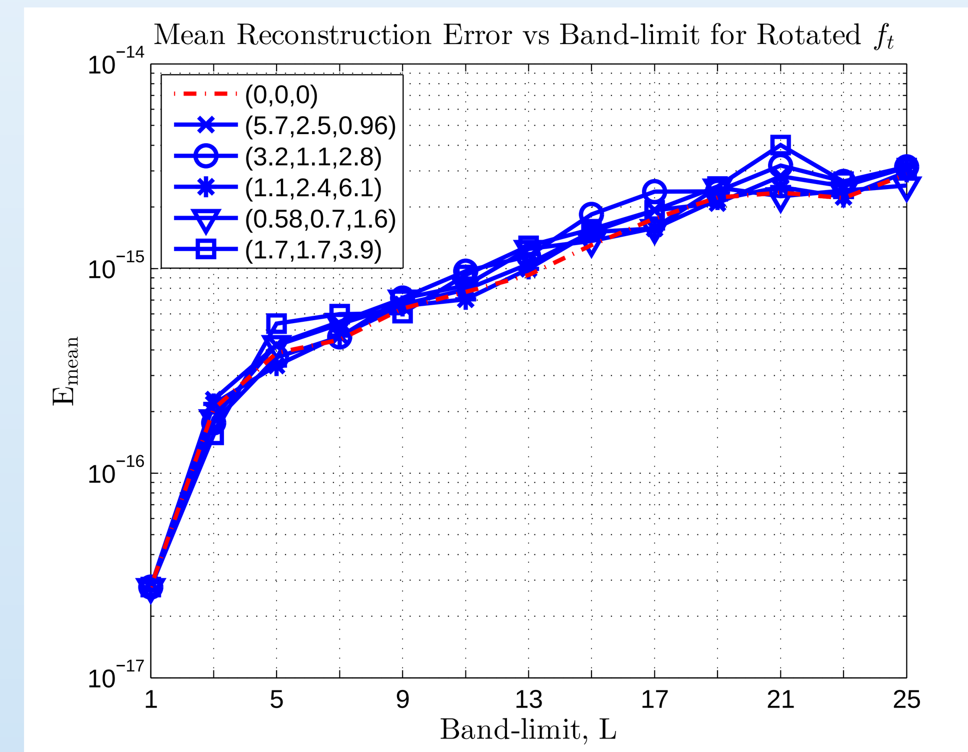
2. Choose $\theta_{L-1} = \frac{\pi(2\lfloor\frac{L-1}{2}\rfloor+1)}{2L-1}$ furthest from the poles, a natural choice for ring of $2L-1$ samples (largest number). Then $\theta_{L-2} = \pi - \frac{\pi(2\lfloor\frac{L-1}{2}\rfloor+1)}{2L-1}$ as antipodal.
3. For each $m = L-3,\ L-5,\ \ldots 2$, choose θ_m and $\theta_{m-1} = \pi - \theta_m$ from Θ, given in (8), that minimises the sum of condition numbers of the matrices $\mathbf{P}^m$ and $\mathbf{P}^{m-1}$.
4. Choose $\theta_0 = 0$ or $\theta_0 = \pi$.

Results in well-conditioned $\mathbf{P}_L^m$, allowing an accurate SHT.

Rotational Invariance Experiment

Numerical accuracy experiment performed on 5 rotated versions of $f_{\rm t}$; randomly obtain Euler angles (α,β,γ) from uniform distributions, where $\alpha,\gamma \in [0,2\pi)$ and $\beta \in [0,\pi]$.

$E_{\rm mean}$ is the same order of magnitude for all rotation angles, demonstrating accuracy does not depend on angle of rotation.



Spherical Harmonic Transform

$f(\theta,\phi)$ is a band-limited signal on the sphere, $\boldsymbol{\theta}^m \triangleq [\theta_{|m|},\,\theta_{|m|+1},\,\ldots,\,\theta_{L-1}]^T \subset \boldsymbol{\theta}$ and

$$\mathbf{g}_m \equiv G_m(\boldsymbol{\theta}^m) \triangleq [G_m(\theta_{|m|}),\,G_m(\theta_{|m|+1}),\,\ldots,\,G_m(\theta_{L-1})]^T,$$

with

$$G_m(\theta_n) \triangleq \int_0^{2\pi} f(\theta_n,\phi)e^{-im\phi}d\phi = 2\pi\sum_{\ell=|m|}^{L-1}(f)_\ell^m \widetilde{P}_\ell^m(\theta_n), \tag{9}$$

for each order $|m| < L$ and each $\theta_n \in \boldsymbol{\theta}$, where $\widetilde{P}_\ell^m(\theta_n) \triangleq Y_\ell^m(\theta_n,0)$ denotes scaled associated Legendre functions.

The spherical harmonic coefficients of order m can be recovered from (9) by setting up a system of linear equations,

$$\mathbf{g}_m = \mathbf{P}_L^m \mathbf{f}_m, \quad |m| \le L, \tag{10}$$

where $\mathbf{P}_L^m$ is a matrix of size $(L-m)\times(L-m)$ defined as

$$\mathbf{P}_L^m \triangleq 2\pi\begin{pmatrix} \widetilde{P}_{|m|}^m(\theta_{|m|}) & \widetilde{P}_{|m|+1}^m(\theta_{|m|}) & \cdots & \widetilde{P}_{L-1}^m(\theta_{|m|}) \\ \widetilde{P}_{|m|}^m(\theta_{|m|+1}) & \widetilde{P}_{|m|+1}^m(\theta_{|m|+1}) & \cdots & \widetilde{P}_{L-1}^m(\theta_{|m|+1}) \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{P}_{|m|}^m(\theta_{L-1}) & \widetilde{P}_{|m|+1}^m(\theta_{L-1}) & \cdots & \widetilde{P}_{L-1}^m(\theta_{L-1}) \end{pmatrix},$$

and $\mathbf{f}_m$ is a vector of $(f)_\ell^m$ of $|m| < L$,

$$\mathbf{f}_m = \left[(f)_{|m|}^m,\,(f)_{|m|+1}^m,\,\ldots,\,(f)_{L-1}^m\right]^T. \tag{11}$$

Computational complexity of SHT is $(O(L^4))$, much smaller least squares methods $(O(L^6))$ used by most sampling schemes.

Future Work

- Applying proposed sampling scheme to real dMRI images
- Extending scheme to multiple sphere sampling in q-space

References

[1] A. Daducci, J. D. McEwen, D. V. D. Ville, J. P. Thiran and Y. Wiaux, "Harmonic analysis of spherical sampling in diffusion MRI," *Proc. 19th Ann. Meet. Int. Soc. Magn. Reson. Med.*, Jun. 2011.

[2] E. Caruyer, C. Lenglet, G. Sapiro, and R. Deriche, "Design of multishell sampling schemes with uniform coverage in diffusion MRI," *Magn. Reson. Med.*, vol. 69, no. 6, pp. 1534–1540, Jun. 2013.

[3] Z. Khalid, R. A. Kennedy and J. D. McEwen, "An optimal-dimensionality sampling scheme on the sphere with fast spherical harmonic transforms," *IEEE Trans. Signal Process.*, vol. 62, no. 17, pp. 4597–4610, Sep. 2014.