The minimum number of vertices with girth 6 and degree set $D = \{r, m\}$

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Abstract

A $(D; g)$-cage is a graph having the minimum number of vertices, with degree set $D$ and girth $g$. Denote by $f(D; g)$ the number of vertices in a $(D; g)$-cage. In this paper it is shown that $f(\{r, m\}; 6) \geq 2(rm - m + 1)$ for any $2 \leq r < m$, and $f(\{r, m\}; 6) = 2(rm - m + 1)$ if either (i) $2 \leq r \leq 5$ and $r < m$ or (ii) $m - 1$ is a prime power and $2 \leq r < m$. Upon these results, it is conjectured that $f(\{r, m\}; 6) = 2(rm - m + 1)$ for any $r$ with $2 \leq r < m$.

Keywords: Cage; Girth; Degree set; Symmetric graph

1. Introduction

A $(v, g)$-cage is a graph having the minimum number of vertices, with valence $v$ and girth $g$. The existence of $(v, g)$-cages was proved by Erdős and Sachs in the early of 1960s [4]. A $(D; g)$-cage is a graph which has the minimum number of vertices, with degree set $D$ and girth $g$. It is obvious that the $(v, g)$-cage is a special case of the $(D; g)$-cage when $D = \{v\}$. Denote the number of vertices in the $(v, g)$-cage by $f(\{v\}, g)$, which has the following property.

Lemma 1 (Longyear [7] and Wong [8]). If $k = v - 1$ is a prime power, $f(\{v\}, 6) = 2(k^2 + k + 1)$.

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The existence of \((D; g)-cages\) has also been discussed in [1]. Denote the number of vertices in the \((D; g)-cage\) by \(f(D; g)\), which has the following properties.

**Lemma 2** (Downs et al. [3]). If \(D = \{a_1, \ldots, a_k\}\) with \(2 \leq a_1 < \cdots < a_k\) and \(g\) is a positive integer with \(g \geq 3\), then \(f(D; g) \geq f_0(D; g)\), where

\[
f_0(D; g) = \begin{cases} 
1 + \sum_{i=1}^{t'} a_k(a_i - 1)^{i-1} & \text{if } g = 2t + 1, \\
1 + \sum_{i=1}^{t-1} a_k(a_i - 1)^{i-1} + (a_1 - 1)^{t-1} & \text{if } g = 2t.
\end{cases}
\]

**Lemma 3** (Wong [7]). For any \(m \geq 3\) and \(g \geq 3\),

\[
f(\{2, m\}; g) = \begin{cases} 
m(g - 2) + 4 & \text{if } g \text{ is even,} \\
\frac{2}{m(g - 1) + 2} & \text{otherwise.}
\end{cases}
\]

Given a \((D; g)-cage\) with degree set \(D = \{r, m\}\) and girth \(g \leq 5\), much effort has been taken in the past decades. For girths 3 and 4, Chartrand et al. [1] have shown \(f(D; 3) = 1 + a_k\) for \(D = \{a_1, a_2, \ldots, a_k\}\) and \(f(\{r, m\}; 4) = r + m\) for any \(r \leq r < m\). For girth 5, Downs et al. [3] have shown \(f(\{3, m\}; 5) = 3m + 1\) for any \(m \geq 4\), Limaye and Sarvate [5] have shown \(f(\{4, m\}; 5) = 4m + 1\) for any even \(m \geq 6\), we [8] have shown \(f(\{4, m\}; 5) = 4m + 1\) for any integer \(m \geq 5\), \(f(\{5, m\}; 5) = 5m + 1\) for any \(m \geq 6\). However, for the case where girth is 6, to the best of our knowledge, not much progress has been achieved.

In this paper, we deal with a \((D; g)-cage\) where the degree set \(D = \{r, m\}\) and the girth is 6. Our major contribution is to give a new lower bound for \(f(\{r, m\}; 6)\), which is shown to be tight if either (i) \(2 \leq r \leq 5\) and \(r < m\), or (ii) \(m - 1\) is a prime power and \(2 \leq r < m\).

The remainder of the paper is organized as follows. Section 2 provides the lower bound and upper bound for \(f(\{r, m\}; 6)\), and Section 3 concludes the paper.

### 2. The bounds for \(f(\{r, m\}; 6)\)

Let \(u\) be a vertex in a graph, \(d(u)\) be the degree of \(u\) and \(N(u)\) be the set of neighboring vertices of \(u\) in the graph.

#### 2.1. A lower bound for \(f(\{r, m\}; 6)\)

Following Lemma 2, it is easy to derive \(f(\{r, m\}; 6) \geq 1 + mr + (r - 1)^2\). In the following we improve this lower bound by Theorem 1.

**Theorem 1.** For any \(2 \leq r < m\), \(f(\{r, m\}; 6) \geq 2(rm - m + 1)\).
Proof. Let $H_{r,m}$ be a graph with degree set $D = \{r, m\}$ and girth 6. For a given vertex $u \in V(H_{r,m})$ with $d(u) = m$, we distinguish its neighboring vertices into two cases: (1) there is a vertex $w \in N(u)$ with $d(w) = m$; (2) $d(w) = r$ for every vertex $w \in N(u)$. We deal with Case 1 APPrst.

Case 1: There are two vertices $u_0, w_0 \in V(H_{r,m})$ such that $d(u_0) = d(w_0) = m$ and $u_0$ is adjacent to $w_0$. Denote
$$N(u_0) = \{w_0, u_1, u_2, \ldots, u_{m-1}\},$$
$$N(w_0) = \{u_0, w_1, w_2, \ldots, w_{m-1}\}.$$
Since the girth of $H_{r,m}$ is 6, we have
$$N(u_i) \cap N(u_j) = \{u_0\}, \quad 1 \leq i < j \leq m-1,$$
$$N(w_i) \cap N(w_j) = \{w_0\}, \quad 1 \leq i < j \leq m-1,$$
$$N(u_i) \cap N(w_j) = \emptyset, \quad 1 \leq i, j \leq m-1.$$
Therefore, we have
$$|V(H_{r,m})| \geq 2 + 2(m-1) + \sum_{1 \leq i \leq m-1} (d(u_i) - 1) + \sum_{1 \leq j \leq m-1} (d(w_i) - 1)$$
$$\geq 2 + 2(m-1) + 2(m-1)(r-1)$$
$$= 2 rm - m + 1 + 2(m - r)$$
$$> 2 rm - m + 1.$$
We then proceed Case 2.

Case 2: For any given vertex $u \in V(H_{r,m})$ with $d(u) = m$ and $d(w) = r$ for all $w \in N(u)$. Let $v_0 \in V(H_{r,m})$ be a vertex with $d(v_0) = m$. Denote
$$N(v_0) = \{v_1, v_2, \ldots, v_m\},$$
$$N(v_i) = \{v_0, v_i, \ldots, v_i, v_{r-1}\}, \quad 1 \leq i \leq m,$$
$$N_e = \{v_0\} \cup N(v_0) \cup N(v_1) \cup \cdots \cup N(v_m),$$
$$N_u = V(H_{r,m}) - N_e,$$
$$E_{vu} = \{e_{i,j} = (v_i, v_j) : v_i \in N_e \text{ and } v_j \in N_u\}.$$Then,
$$|N_e| = 1 + mr,$$
$$|E_{vu}| = \sum_{1 \leq i \leq m, 1 \leq j \leq r-1} (d(v_{i,j}) - 1) \geq m(r - 1)^2.$$
Consider the degree of a vertex $w \in N_u$; it can be classified into two subcases: either (2.1) $d(w) = r$ for all $w \in N_u$, or (2.2) there is a vertex $w$ with $d(w) = m$.  

Case 2.1: For any vertex \( w \in N_u \), \( d(w) = r \). Then,
\[
|N_u| \geq m(r-1)^2/r = m(r-2) + m/r > 1 + m(r-2),
\]
\[
|V(H_{r,m})| = |N_v| + |N_u| > 1 + mr + 1 + m(r-2) = 2rm - m + 1.
\]

Case 2.2: There is a vertex \( w \in N_u \) with \( d(w) = m \). Denote
\[
S_m = \{w_j: d(w_j) = m \text{ and } w_j \in N_u\},
\]
\[
|S_m| = s,
\]
\[
N(w_j) = \{w_{j,1}, w_{j,2}, \ldots, w_{j,m}\}, \quad 1 \leq j \leq s.
\]
We have
\[
|N(w_j) \cap N(v_i)| \leq 1, \quad 1 \leq j \leq s, \quad 1 \leq i \leq m,
\]
\[
|N(w_j) \cap N_v| \leq m, \quad 1 \leq j \leq s.
\]
Let
\[
|N(w_j) \cap N_v| = y_j, \quad 1 \leq j \leq s,
\]
\[
y_i = \max\{y_j: 1 \leq j \leq s\},
\]
\[
u_0 = w_t, \quad y = y_i.
\]
Denote
\[
N(u_0) = \{u_1, u_2, \ldots, u_m\}.
\]
Without loss of generality, we assume that
\[
u_i = v_{i,1}, \quad 1 \leq i \leq y.
\]
Denote
\[
N(u_i) = \begin{cases} 
\{u_0, v_{i,1}, v_{i,2}, \ldots, v_{i,r-2}\} & \text{if } 1 \leq i \leq y, \\
\{u_0, u_{i,1}, \ldots, u_{i,r-1}\} & \text{if } y + 1 \leq i \leq m.
\end{cases}
\]
We have
\[
N(u_i) \cap N(u_j) = \{u_0\}, \quad 1 \leq i < j \leq y.
\]
Let
\[
x = |(N(u_{y+1}) \cup \cdots \cup N(u_m)) \cap (N(v_{y+1}) \cup \cdots \cup N(v_m))|.
\]
Since the girth is 6, any of these \( x \) vertices and \( u_1, \ldots, u_y \) do not have a common neighbor. Now, we consider the number of vertices in \( S_m \). If \( |S_m| = s = 1 \) (see Fig. 1), then
\[
x \leq (m - y)(r - 1)
\]
Fig. 1. $H_{3,6}$ with $y = 3$ and $x = 6$.

$$|N_u| \geq 1 + y(r - 2) + (m - y)r - x + x(r - 2)/r$$
$$= 1 + yr - 2y + mr - yr - x + x - 2x/r$$
$$\geq 1 + m(r - 2) + 2(m - y) - 2(m - y)(r - 1)/r$$
$$= 1 + m(r - 2) + 2(m - y) - 2(m - y) + 2(m - y)/r$$
$$= 1 + m(r - 2) + 2(m - y)/r$$
$$\geq 1 + m(r - 2),$$

$$|V(H_{r,m})| = |N_v| + |N_u| \geq 1 + mr + 1 + m(r - 2) = 2(rm - m + 1).$$

Otherwise ($|S_m| = s \geq 2,$

$$|N_u| \geq 1 + m(r - 2) + 2s(m - y)/r$$

$$|V(H_{r,m})| = |N_v| + |N_u| \geq 1 + mr + 1 + m(r - 2) = 2(rm - m + 1).$$

2.2. An upper bound for $f(\{r,m\};6)$

We now consider the upper bound of $f(\{r,m\};6)$, which is stated by the following theorem.

**Theorem 2.** If $k = m - 1$ is a prime power with $2 \leq r < m$, then $f(\{r,m\};6) \leq 2(rm - m + 1)$.

**Proof.** By Lemma 1, we have $f(\{m\},6) = 2(k^2 + k + 1) = 2(m^2 - m + 1)$ for a $(m,6)$-cage. Let $H_m$ be an $(m,6)$-cage constructed in [7]. The $2(k^2 + k + 1)$ vertices in $H_m$ are arranged as in Fig. 2. The set $N_0$ consists of vertices $v_1, v_2, \ldots, v_k, v_1', \ldots, v_k', v_{21}, \ldots, v_{2k}, \ldots, v_{kk}$. The set $N_0$ can be defined similarly. Since $k$ ($= m - 1$) is a prime power, there must exist a complete set of mutually orthogonal Latin squares $\{L_2, L_3, \ldots, L_k\}$ with elements $1, 2, \ldots, k$ (see [2, p. 167, Theorem 5.2.4]).
Let

\[
L_1 = \begin{bmatrix}
1 & 2 & \ldots & k \\
1 & 2 & \ldots & k \\
. & . & \ldots & . \\
. & . & \ldots & . \\
1 & 2 & \ldots & k
\end{bmatrix}
\]

and

\[
L_t = \begin{bmatrix}
L_{11}^t & L_{12}^t & \ldots & L_{1k}^t \\
L_{21}^t & L_{22}^t & \ldots & L_{2k}^t \\
. & . & \ldots & . \\
. & . & \ldots & . \\
L_{k1}^t & L_{k2}^t & \ldots & L_{kk}^t
\end{bmatrix}, \quad t = 2, 3, \ldots, k,
\]

where \(L_{11}^1 = 1, L_{12}^1 = 2, \ldots, L_{1k}^1 = k\). The vertices of sets \(N_v\) and \(N_u\) are joined together according to the following rule:

\[
v_{pq} \sim u_{1q}, u_{2q}, \ldots, u_{kq} \quad (p, q = 1, 2, \ldots, k),
\]

where \(a \sim b\) means that there is an edge in the graph between \(a\) and \(b\). Since \(L_2, L_3, \ldots, L_k\) are mutually orthogonal Latin squares, it follows that \(H_m\) has girth 6 and valence \(m = k + 1\). For completeness, here we use an example to illustrate the construction.
Assume that \( m = 4 \), then \( k (= m - 1 = 3) \) is a prime power. We have

\[
L_1 = \begin{bmatrix}
1 & 2 & 3 \\
1 & 2 & 3 \\
1 & 2 & 3 \\
\end{bmatrix}, \quad L_2 = \begin{bmatrix}
1 & 2 & 3 \\
2 & 3 & 1 \\
3 & 1 & 2 \\
\end{bmatrix}, \quad L_3 = \begin{bmatrix}
1 & 2 & 3 \\
2 & 3 & 1 \\
\end{bmatrix}.
\]

The edges in graph \( H_4 \) are given by the following table:

\[
\begin{array}{cccc}
v_{11} \sim u_{11} & v_{12} \sim u_{12} & v_{13} \sim u_{13} & \\
u_{21} & u_{22} & u_{23} & \\
u_{31} & u_{32} & u_{33} & \\
v_{21} \sim u_{11} & v_{22} \sim u_{12} & v_{23} \sim u_{13} & \\
u_{22} & u_{23} & u_{21} & \\
u_{33} & u_{31} & u_{32} & \\
v_{31} \sim u_{11} & v_{32} \sim u_{12} & v_{33} \sim u_{13} & \\
u_{23} & u_{21} & u_{22} & \\
u_{32} & u_{33} & u_{31} & \\
\end{array}
\]

Let \( t = r - 1 \), \( k = m - 1 \) and \( G_{r,m} \) be the subgraph of \( H_m \) induced by the vertices in the set

\[
\{ v_0, v_1, v_{11}, \ldots, v_{1k}, v_2, v_{21}, \ldots, v_{2k}, \ldots, v_t, v_{t1}, \ldots, v_{tk}, \\
u_0, u_1, u_{11}, \ldots, u_{1k}, u_2, u_{21}, \ldots, u_{2k}, \ldots, u_t, u_{t1}, \ldots, u_{tk} \}.
\]

Following the above construction rules, the resulting graphs \( H_4 \) and \( G_{3,4} \) for \( r = 3 \) and \( m = 4 \) are shown in Fig. 3.

Since \( G_{r,m} \) has girth 6, degree set \( D = \{ r, m \} \), and \( |V(G_{r,m})| = 2(rm - m + 1) \) vertices, we have \( f(\{ r, m \}; 6) \leq 2(rm - m + 1) \) for a prime power \( m - 1 \) and any \( r \) with \( 2 \leq r < m \). The theorem then follows. \( \square \)

We have already discussed the case where \( m - 1 \) is a prime power. However, if \( m - 1 \) is not a prime power, it is much harder to deal with. Here we only deal with the case for \( r \leq 5 \) through the construction of a \((D;6)\)-cage with a degree set \( D = \{ r, m \} \), \( r = 3, 4, 5 \) and \( m > r \). We have the following theorem.

**Theorem 3.** For \( r = 3, 4, 5 \) and any \( m > r \), \( f(\{ r, m \}; 6) \leq 2(rm - m + 1) \).

**Proof.** For \( m = 4, 5, 6 \), it is obvious that \( r = m - 1 \) is a prime power, and \( f(\{ r, m \}; 6) \leq 2(rm - m + 1) \) by Theorem 3. For \( r = 3, 4, 5 \) and \( m \geq 7 \), a graph \( G_{r,m} \) is constructed as follows.

Denote by \( e_{x,y} \) an edge in \( G_{r,m} \) between vertices \( v_x \) and \( v_y \), \( 0 \leq x, y \leq 2(r - 1)m + 1 \). Then,

\[
V(G_{r,m}) = \{ v_0, v_1, \ldots, v_{2(r-1)m+1} \}.
\]
Following the above construction rules, the resulting graphs $G_{4,7}$ and $G_{5,7}$ for $r = 4, 5$ and $m = 7$ are shown in Fig. 4. Since graph $G_{r,m}$ has a degree set $D = \{r, m\}$, girth 6 and $|V(G_{r,m})| (= 2(rm - m + 1))$ vertices, we have $f(\{r, m\}; 6) \leq 2(rm - m + 1)$. The theorem then follows. □
2.3. A tight bound for $f(\{r;m\};6)$

When $r = 2$, $f(\{2;m\};6) = 2m + 2$ for any $m > 2$ by Lemma 3. Following Theorems 1–3 and Lemma 3, we have

**Theorem 4.** $f(\{r;m\};6) = 2rm - m + 1$, if either (i) $2 \leq r \leq 5$ and $r < m$, or (ii) $m - 1$ is a prime power and $2 \leq r < m$.

3. Conclusions

In this paper, we have shown that $f(\{r;m\};6) \geq 2rm - m + 1$ for any $r$ with $2 \leq r < m$, and $f(\{r;m\};6) = 2rm - m + 1$ if either (i) $2 \leq r \leq 5$ and $r < m$ or (ii) $m - 1$ is a prime power and $2 \leq r < m$. Upon these results, we have the following conjecture.
Conjecture. For any integer \( r \) with \( 2 \leq r < m \), \( f(\{r,m\}; 6) = 2(rm - m + 1) \).

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