It can be checked that matrix $A_{11} = \begin{bmatrix} 0.5 & 0.1 \\ 0.2 & 0.5 \end{bmatrix}$ has all eigenvalues inside the unit circle, namely $\lambda_1 = 0.6414$ and $\lambda_2 = 0.3586$. Then, one obtains

$$\begin{aligned} A_2 =& [0.2] + [1 \quad 1] \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.5 & 0.1 \\ 0.2 & 0.5 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 0.3 \end{bmatrix} \\ =& [0.9826]. \end{aligned}$$

From this it follows that the system is asymptotically stable.

Similarly, matrix $A_{22} = [0.2]$ has all eigenvalues inside the unit circle. One can then calculate matrix A_1

$$\begin{aligned} A_1 &= \begin{bmatrix} 0.5 & 0.1 \\ 0.2 & 0.5 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.3 \end{bmatrix} (1 - [0.2])^{-1} [1 \quad 1] \\ &= \begin{bmatrix} 0.5 & 0.1 \\ 0.575 & 0.875 \end{bmatrix}. \end{aligned}$$

Next, one easily finds that the system is asymptotically stable since eigenvalues of matrix A_1 are $\lambda_1 = 0.3831$ and $\lambda_2 = 0.9919$. □

IV. Concluding Remarks

Necessary and sufficient conditions for stability and asymptotic stability of a system described by the positive 2-D Roesser model (1) are given. Conditions are significantly simpler than known conditions for stability of the system described by regular 2-D Roesser model, one can check stability of positive 2-D system using well-known methods for 1-D systems. We hope that the conditions will stimulate research on synthesis of systems described by positive 2-D systems, for instance filters for image processing.

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Permutation Routing in All-Optical Product Networks

Weifa Liang and Xiaojun Shen

***Abstract*—In this brief, we study the permutation routing for an all-optical product network. We show a lower bound on the number of wavelengths needed to implement any permutation with one round. We also present efficient routing algorithms for two models, the wavelength nonconversion and conversion models, respectively.**

***Index Terms*—Algorithm design and analysis, all-optical networks, graph theory, permutation routing, product networks, wavelength assignment.**

I. Introduction

In this brief, we design routing algorithms for an *all-optical network* [1], [9], [14] and [21]. This kind of network offers the possibility of interconnecting hundreds to thousands of users, covering local to wide area, and providing capacities exceeding substantially those a conventional network can provide. The network promises data transmission rates several orders of magnitude higher than the current electronic network. The key to high speed in the network is to maintain the signal in optical form rather than electronic form. The high bandwidth of the fiber-optic links is utilized through *wavelength-division multiplexing* (WDM) technology which supports the propagation of multiple laser beams through a single fiber-optic link provided that each laser beam uses a distinct optical wavelength. Each laser beam, viewed as a carrier of signal, provides transmission rates of 2.5–10 Gbps. The major applications of this network are found in video conferencing, scientific visualization, real-time medical imaging, supercomputing and distributed computing [21]. A comprehensive overview of the physical principles and applications of this technology can be found in the books by Green [9] and McAulay [14]. Following [4], [8] and [15], we shall model the all-optical network as a *directed symmetric* graph, where every two adjacent vertices have a pair of opposite directed edges.

The Network Models: An all-optical network consists of vertices (nodes, stations, processors, etc.), interconnected by point-to-point fiber-optic links. Each fiber-optic link supports a given number of wavelengths. The vertex may be occupied either by terminals, switches, or both. *Terminals* send and receive signals. *Switches* direct the input signals to one or multiple output links. There are several types of optical switches. Among them, the *elementary switch* is capable of directing coming signals from a link to one or more output links. The elementary switch, however, cannot differentiate between different the incoming wavelengths along the same link. Rather, the entire signal is directed to the same output(s) [5], [1] and [20]. The *generalized switch*, on the other hand, is capable of switching coming signals based on their wavelengths [1] and [20]. This kind of switch splits coming signals with different wavelengths to different streams and directs them to separate outputs, using acousto–optic filters. In both cases, different messages passing through a common link must use different wavelengths. Unless otherwise specified, in this brief we will adopt generalized switches for the proposed routing algorithms.

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Besides the different choice of switches, there are different wavelength assignment polices for routing paths. In most cases, only a unique wavelength is allowed for each routing path. We call this WDM model the *wavelength nonconversion model*. If different wavelengths are allowed for different segments of a routing path, the model is called *wavelength conversion model*. (It is called the λ-*routing model* in [6].)

Previous Related Work: Optical routing in an undirected network G was considered by Raghavan and Upfal [20]. They proved an $\Omega(1/\beta^2)$ lower bound on the number of wavelengths needed to implement any permutation with one round, where β is the edge expansion of G (which is defined later). They also presented an algorithm which implements any permutation on bounded degree graphs using $O(\log^2 n/\log^2 \lambda)$ wavelengths within one round with high probability, where n is the number of nodes in G and λ is the second largest eigenvalue (in absolute value) of the transition matrix of the standard random walk on G. For degree d arrays, they presented an algorithm with an $O(dn^{1/d}/\log n)$ worst case performance. Aumann and Rabani [4] presented a near optimal implementation algorithm for bounded degree networks within one round. Their algorithm needs $O(\log^2 n/\beta^2)$ wavelengths. For any bounded dimension array, any given number of wavelengths, and an instance I, Aumann and Rabani [4] presented an algorithm which implements the I within $O(\log n \log |I| T_{\rm opt}(I))$ rounds, where $T_{\rm opt}(I)$ is the minimum number of rounds necessary to implement I. Kleinberg and Tardos [12] later obtained an improved bound of $O(\log n)$ on the approximation of the number of rounds required to implement I. Rabani [19] further improved the result in [12] to $O(poly \log \log n T_{\rm opt}(I))$. Pankaj [16] considered the permutation routing in hypercubes, shuffle-exchanges, and De Bruijn networks. He showed that permutation routing within one round can be achieved with $O(\log^2 n)$ wavelengths, while Aggrawal *et al.* [1] showed that $O(\log n)$ wavelengths are sufficient for the permutation routing in this case. Pankaj [16], [17] proved an $\Omega(\log n)$ lower bound on the number of wavelengths needed for any permutation routing with one round on a bounded degree network. Aumannn and Rabani [4] demonstrated that the permutation routing in hypercubes can be done with a constant number of wavelengths. Gu and Tamaki [10], [11] further showed it suffices for implementing any permutation in a directed symmetric hypercube using two wavelengths and in an undirected hypercube using eight wavelengths. Liang and Shen [13] showed that the permutation routing in a cube-connected cycle (CCC) can be achieved with one round with $2\lfloor \log n \rfloor$ wavelengths, which is almost tight, following Pankaj's proof [16], [17]. Barry and Humblet [5], [6] gave bounds for routing in passive (switchless) and λ-networks. An almost matching upper bound is presented later in [1]. Peiris and Sasaki [18] considered bounds for elementary switches. The connection between the packet routing and the optical routing is also addressed in [1]. The integral multicommodity flow problem related to the optical routing has been discussed in [4], [2], [3]. A comprehensive survey for optical routing appears in [8].

Our Results: In this brief we consider the permutation routing issue in an all-optical product network whose topology is a *direct product* of two networks. This kind of network includes many well known networks such as hypercubes, meshes, tori, etc. We first show a lower bound on the number of wavelengths needed for implementing any permutation with one round in a product network; then present permutation routing algorithms for such a network under two models, the wavelength nonconversion and conversion models, respectively.

II. Preliminaries

In this section, we define necessary notations and explain basic notions for later use. A directed graph is called a *directed symmetric* graph $G=(V, E)$ with $|V|=n$ and $|E|=m$, if edge $\langle u, v\rangle \in E$ if and only if $\langle v, u\rangle \in E$.

The *direct product* of two undirected graphs $G=(V_1, E_1)$ and $H=(V_2, E_2)$, denoted by $G \times H$, is defined as follows. The vertex set of $G\times H$ is the Cartesian product $V_1 \times V_2$. There is an edge between two vertices (v_1, v_2) and (v_1', v_2') in $G\times H$ if either $v_1 = v_1'$ and $(v_2, v_2') \in E_2$ or $v_2 = v_2'$ and $(v_1, v_1') \in E_1$. The graphs G and H are the *factors* of $G \times H$. Notice that $G\times H$ consists of $|V_2|$ copies of G connected by $|V_1|$ copies of H, arranged in a grid-like fashion. Each copy of H is called a *row* and each copy of G is called a *column* in the grid. An edge in $G\times H$ between (v_1, u) and (v_2, u) is called a G-*edge* if $(v_1, v_2) \in E_1$, and an edge of $G\times H$ between (v, u_1) and (v, u_2) is called an H-*edge* if $(u_1, u_2) \in E_2$.

A *request* in an all optical network $G(V, E)$ is an ordered pair of vertices (u, v) in G which corresponds to a message to be sent from u to v. u is the *source* and v is the *destination*. An *instance* I is a set of requests. If $I=\{(i, \pi(i)) \mid i \in V\}$, then the routing is called permutation routing, where π is a vertex permutation in G.

Let $P(x, y)$ denote a directed path in G from x to y. A *routing* in G for an instance I is a set of directed paths $\mathcal{R}(I)=\{P(x, y) \mid (x, y) \in I\}$. An instance I is said to be *implemented* if there is a routing path for every request in I, and the routing path has been assigned wavelengths that complies with routing policy defined by the model.

The *conflict graph*, associated with a permutation routing $\mathcal{R}=\{P(i, \pi(i)) \mid i \in V, i$ is the source and $\pi(i)$ is the destination of $i\}$ in a directed or undirected graph $G(V, E)$, is an undirected graph $G_{\mathcal{R}, \pi}=(V', E')$, where each directed (undirected) routing path in $\mathcal{R}$ is a vertex in V' and there is an edge in E' if the two corresponding paths in $\mathcal{R}$ share at least a common directed (or undirected) edge in G.

The *edge-expansion* $\beta(G)$ of $G(V, E)$ is the minimum, over all subsets S of vertices, $|S| \le n/2$, of the ratio of the number of edges leaving S to the size of S $(\subset V)$. A *bisection* of a graph $G(V, E)$ is defined as follows: Let $P=(V_1, V_2)$ be a vertex partition which partitions V into two disjoint subsets V_1 and V_2 such that $|V_1| = \lceil |V|/2 \rceil$ and $|V_2| = \lfloor |V|/2 \rfloor$. The *bisection problem* is to find such a vertex partition (V_1, V_2) in G that $|C|$ is minimized, where $C=\{(i, j) \mid i \in V_2, j \in V_1$, and $(i, j) \in E\}$. Let $c(G)=|C|$. The bisection concept for undirected graphs can be extended to directed graphs. For the directed version, $C=\{\langle i, j\rangle \mid i \in V_2, j \in V_1$, and $\langle i, j\rangle \in E\}$.

Let $\mathcal{R}$ be a permutation routing for π in G. We define $C(e, \mathcal{R} = \{P(i, \pi(i)) \mid i \in V, e \in P(i, \pi(i))$, and $P(i, \pi(i)) \in \mathcal{R}\}$ to be the *congestion* of edge e. Then, the congestion problem for π in G is to find a permutation routing $\mathcal{R}$ such that $\max_{e\in E}\{|C(e, \mathcal{R})|\}$ is minimized. Let $congest(G, \pi) = \min_{\mathcal{R}} \max_{e\in E} \{|C(e, \mathcal{R})|\}$. The congestion of G, $congest(G)$, is defined as $congest(G) = \max\{congest(G, \pi) \mid \pi$ is a permutation$\}$.

III. Routing Algorithms for Product Networks

In this section, we design permutation routing algorithms for product networks. We first show a lower bound on the number of wavelengths needed to implement any permutation in such a network with one round. We then present routing algorithms for the two models.

A. Lower Bounds on the Number of Wavelengths

Lemma 1: Given an all-optical network $G(V, E)$, we assume that $c(G)$ is the number of edges in its bisection, and $w_{\min}$ is the number of wavelengths needed to implement any permutation in G with one round. We have $w_{\min} \ge congest(G) \ge (n-1)/2c(G)$.

Proof: Following the congestion definition and the optical routing rule that different signals through a single link must be assigned different wavelengths, it follows that $w_{\min} \geq congest(G)$.

Let (V_1, V_2) be a bisection in G with $|V_1| \geq |V_2|$. Assume there is a permutation π which permutes the vertices in V_2 to the vertices in V_1. The congestion of G for π is $congest(G, \pi) \geq \lfloor n/2 \rfloor / c(G) \geq (n-1)/2c(G)$. Since $congest(G) \geq congest(G, \pi)$, the lemma then follows. ■

Notice that Lemma 1 always holds no matter whether the all-optical network uses the wavelength conversion model or not.

Lemma 2: Let $G \times H$ be a directed symmetric product network. Let $c(G')$ be the number of edges in a bisection of a graph G' and $V(G')$ be the vertex set of G'. Suppose that d_G and d_H are the maximum in-degrees (out-degrees) of G and H. Then, 1) $c(G \times H) \leq \min\{|V(H)|c(G), |V(G)|c(H)\}$ if both $|V(G)|$ and $|V(H)|$ are even; 2) $c(G \times H) \leq \min\{|V(G)|c(H) + d_H|V(G)|/2 + c(G), |V(H)|c(G)\}$ if $|V(G)|$ is even and $|V(H)|$ is odd; 3) $c(G \times H) \leq \min\{|V(H)|c(G) + d_G|V(H)|/2 + c(H), |V(G)|c(H)\}$ if $|V(G)|$ is odd and $|V(H)|$ is even; 4) $c(G \times H) \leq \min\{|V(G)|c(H) + \lfloor |V(G)|/2 \rfloor d_H + c(G), |V(G)|c(G) + \lfloor |V(G)|/2 \rfloor d_G + c(H)\}$ otherwise.

Proof: We first consider case 1) where both $p = |V(G)|$ and $q = |V(H)|$ are even. Let $V(G) = \{v_1, v_2, \ldots, v_p\}$ be the vertex set of G. Suppose $(V_1(G), V_2(G))$ is a bisection of G, where $V_1(G) = \{v_1, v_2, \ldots, v_{p/2}\}$ and $V_2(G) = \{v_{p/2+1}, \ldots, v_p\}$. We replace each vertex in G by the graph H, and establish a connection between the two corresponding vertices in the two copies of H if there is an edge in G. This results in the graph W. Now, we partition the vertex set of W into two disjoint subsets of equal size according to the bisection of G. Thus, the number of edges of W in this partition is $|V(H)|c(G)$. Similarly, there is another partition based on the bisection of H. The number of edges in this latter partition is $|V(G)|c(H)$. Thus, $c(G \times H) \leq \min\{|V(H)|c(G), |V(G)|c(H)\}$.

We then consider case 2) where $p = |V(G)|$ is even and $q = |V(H)|$ is odd. If we use only the G-edges for the vertex partition of W, then the number of edges in W in this partition is $|V(H)|c(G)$, following the above discussion. We now consider another vertex partition of W, which consists of both H- and G-edges as follows. Let $(V_1(H), V_2(H))$ be a bisection of H, where $|V_1(H)| = \lfloor |V(H)|/2 \rfloor$ and $|V_2(H)| = \lceil |V(H)|/2 \rceil$. We replace each vertex in H with the graph G, we then obtain a vertex partition $(V(W_1), V(W_2))$ in W with $|V(W_1)| = \lfloor |V(H)|/2 \rfloor |V(G)|$ and $|V(W_2)| = \lfloor |V(H)|/2 \rfloor |V(G)| + |V(G)|$. The number of edges in W in this partition is no more than $|V(G)|c(H)$. It is clear that this vertex partition is not a bisection of W. To obtain a vertex partition of W with equal size based on this partition, it proceeds as follows. Let $v \in V_2(H)$, there is a corresponding G in W for v, partition the vertices in this G copy into two equal sizes, using the bisection of G, and let (V_1^{GW}, V_2^{GW}) be the resulting vertex partition of the G copy. Now there is another vertex partition $(V(W_1) \cup V_1^{GW}, V(W_2) \cup V_2^{GW} - V(G))$ for W with $|V(W_2) \cup V_2^{GW} - V(G)| - |V(W_1) \cup V_1^{GW}| = 0$, and the number of edges in this partition is no more than $|V(G)|c(H) + d_H|V(G)|/2 + c(G)$. While the bisection of W is such a partition that has the minimum number of edges, therefore, $c(G \times H) \leq \min\{|V(H)|c(G), |V(G)|c(H) + d_H|V(G)|/2 + c(G)\}$. The other two cases can be dealt similarly, omitted. ■

Having Lemmas 1 and 2, the following theorem immediately follows.

Theorem 1: Let $W = G \times H$ be a directed symmetric product network. Let $c(X)$ be the number of edges in a bisection of graph X. Suppose that d_G and d_H are the maximum in-degrees (out-degrees) of G and H. Then, a lower bound on the minimum number of wavelengths $w_{\min}(W)$ for implementing any permutation in W with one round is as follows.

1) $$\max\left\{\frac{|V(G)|}{2c(G)} - 1, \frac{|V(H)|}{2c(H)} - 1\right\}$$

if both $|V(G)|$ and $|V(H)|$ are even.

2) $$\max\left\{\frac{|V(H)|}{2c(H) + d_H + 2c(G)/|V(G)} - 1, \frac{|V(G)|}{2c(G)} - 1\right\}$$

if $|V(G)|$ is even and $|V(H)|$ is odd.

3) $$\max\left\{\frac{|V(G)|}{2c(G) + d_G + 2c(H)/|V(H)} - 1, \frac{|V(H)|}{2c(H)} - 1\right\}$$

if $|V(G)|$ is odd and $|V(H)|$ is even.

4) $$\max\left\{\frac{|V(G)|}{2c(G) + d_G + 2c(H)/|V(H)|} - 1, \frac{|V(H)|}{2c(H) + d_H + 2c(G)/|V(G)} - 1\right\}$$

otherwise.

B. A Routing Algorithm on the Packet-Passing Model

Let $W = G \times H$ and π be a permutation in W. Assume the factors G and H of W are equipped with routing algorithms. We are interested in designing a routing algorithm for W by using the routing algorithms for G and H as *subroutines*. The permutation routing on a directed, product network W has been addressed by Baumslag and Annextein [7] for the packet-passing model. Their algorithm consists of three phases. 1) Route some set of permutations using G-edges only; 2) Route some set of permutations using H-edges only; 3) Route some set of permutations using G-edges only. Since the product network is a symmetric network, the three phases can be applied alternatively, i.e., by first routing on H, followed by G, and followed by H.

Now, for the given permutation π, consider a naive routing method in which each source in a column is routed to its destination row in the column, followed by routing each source in a row to its destination column in that row. This routing fails to produce edge-disjoint paths because there may exist a case where several sources in the same column have their destinations in a common row. If we are allowed to use an initial extra phase, this congestion problem can be solved easily. That is, we first "rearrange" each column so that each row consists of a set of sources whose destinations are in distinct columns. After that, a permutation of each row is required to get each source to its correct column after the rearrangement. Once all the sources are in their correct columns, a final permutation of each column suffices to get each source to its correct destination. This final phase is indeed a permutation since the destinations of all sources are distinct. Thus, the aim of the first phase is to find a set of sources P_R, one per column, such that every source in P_R has its destination in a different column, for every row R. To this end, a bipartite graph $G_B(X, Y, E_B)$ is constructed as follows. Let X and Y represent the set of columns in W. There is an edge between $x_i \in X$ and $y_j \in Y$ for each source in column i whose destination is in column j. Since π is a permutation, it follows that G_B is a regular, bipartite multigraph. Thus, G_B can be decomposed into a set of edge disjoint perfect matchings, and the destinations of the sources included in a single perfect matching are in distinct columns. Therefore, for every row R, the sources included in a single perfect matching form the set P_R. Each set P_R, is "lifted" to row R during the first phase of the algorithm. Since each source is included in precisely one perfect matching, the mapping of sources in a column during the first phase is indeed a permutation of the column.

C. Routings on the Wavelength Conversion Model

The algorithm in the previous section can be expressed in a different way. That is, the permutation π can be decomposed into three permutations $\sigma_i, i = 1, 2, 3$, where σ_1 and σ_3 are permutations in the columns of W and σ_2 is a permutation in the rows of W. For example, let v be a source in W at the position of row i_1 and column j_1 and $\pi(v) = u$ be the destination of v at the position row i_2 and column j_2. Suppose v has been "lifted" to the position of row i'_1 and column j_1 in the first phase. Then, $\sigma_1(i_1, j_1) = (i'_1, j_1)$, followed by the second phase of σ_2 such that $\sigma_2\sigma_1(i_1, j_1) = \sigma_2(i'_1, j_1) = (i'_1, j_2)$, and followed by the third phase of σ_3 such that $\sigma_3\sigma_2\sigma_1(i_1, j_1) = \sigma(i'_1, j_2) = (i_2, j_2)$. In the light of this observation, we now present a permutation routing algorithm for π in an all-optical symmetric product network W, based on the wavelength conversion model. We start with the following major theorem.

Theorem 2: Given permutation routing algorithms for networks G and H, there is a permutation routing algorithm for the product network $G \times H$. The number of wavelengths for any permutation with one round is at most $\max\{2w(G), w(H)\}$ if $w(G) \leq w(H)$; or $\max\{2w(H), w(G)\}$ otherwise, where $w(X)$ is the number of wavelengths needed to implement any permutation in network X with one round.

Proof: The permutation routing algorithm presented by Baumslag and Annexstein [7] is for the packet-passing model, which is totally different from the WDM model that we are concerned here. In their model, the path length was the major concern. Here we are not interested in the number of edges on a path, but in the number of wavelengths needed to avoid interference among the paths. To make their algorithm work for the WDM model, some modifications are necessary. As we can see, implementing the permutation routing in W for π can further be decomposed into three permutations $\sigma_i, 1 \leq i \leq 3$. Without loss of generality, assume $w(G) \leq w(H)$. Following the three phases of the above algorithm, permutation σ_1 can be implemented with $w(G)$ wavelengths [in other words, all routing paths can be colored with $w(G)$ colors]; permutation σ_2 can be implemented with $w(H)$ wavelengths [all routing paths can be colored with $w(H)$ colors, and the colors for σ_1 can be re-used here]; and permutation σ_3 can be implemented with $w(G)$ wavelengths [all routing paths can be colored with $w(G)$ colors, and the colors for σ_1 cannot be used here].

Now, for a given request $(i, \pi(i))$, let (u_1, v_1) and (u_2, v_2) be the positions of i and $\pi(i)$ in W. Then, the routing path L_i for the request $(i, \pi(i))$ consists of three routing segments $L_{i,1}$, $L_{i,2}$ and $L_{i,3}$ which correspond to σ_1, σ_2, and σ_3, where $L_{i,1}$ contains the vertices in column v_1 of W and consists of G-edges only; $L_{i,2}$ contains the vertices in row u'_1 of W and consists of H-edges only, assuming that $\sigma_1(i)$ is at row u'_1 and column v_1 in W; and $L_{i,3}$ contains the vertices in column v_2 of W and consists of G-edges only. L_i can be further simplified, and make it become a simple path L'_i after removing the cycles it contains. Let $L'_{i,j}$ be the corresponding segment of $L_{i,j}, 1 \leq j \leq 3$. L'_i is then assigned three different wavelengths for each of its segments, which are the wavelengths for $L_{i,j}$ originally, $j = 1, 2, 3$. That is, each L'_i for a request $(i, \pi(i))$ can be implemented with at most three wavelengths. Notice that the wavelengths used for $L_{i,1}$ may not be used for $L_{i,3}$ due to that both use the G-edges of W. Therefore, to implement any permutation in W with one round, at least $2w(G)$ wavelengths are needed. In summary, implementing any permutation π in W with one round can be done if $\max\{2w(G), w(H)\}$ wavelengths are available in W. ■

The following corollary can be derived directly, by Theorem 2.

Corollary 1: I) For a directed symmetric hypercube H_q of 2^q vertices, there is a permutation algorithm for implementing any permutation with one round if two wavelengths are available. II) For a directed symmetric $l \times h$ mesh M with $l \leq h$ and $n = lh$, there is a permutation algorithm for implementing any permutation with one round, if there are $\max\{l, \lfloor h/2 \rfloor\}$ wavelengths available. In particular, when $l = \sqrt{n/2}$ and $h = 2n$, the algorithm needs at most $\sqrt{n/2}$ wavelengths, which is almost optimal in terms of the wavelengths used.

Proof: I) Since $H_q = K_2 \times H_{q-1}$ and $w(K_2) = 1$, $w(H_q) = \max\{2w(K_2), w(H_{q-1})\}$ by Theorem 2. While $H_{q-1} = K_2 \times H_{q-2}$, it is easy to show $w(H_q) = 2$ by induction on q.

II) Assume $l \leq h$ and $M = L_l \times L_h$, where L_i represents a chain of i vertices. Obviously $w(L_i) = \lfloor i/2 \rfloor$. $w(M) = \max\{2w(L_l), w(L_h)\} \leq \max\{l, \lfloor h/2 \rfloor\}$ by Theorem 2. When $M = \sqrt{n/2} \times \sqrt{2n}$, we have $w(M) = \max\{2w(L_{\sqrt{n/2}}), w(L_{\sqrt{2n}})\} = \sqrt{n/2}$, and by Theorem 2 again, the lower bound of the number of wavelengths for any permutation in M within one round is $\Omega(\sqrt{n})$ due to $c(M) = \sqrt{n/2}$, so, this bound is almost tight. ■

D. Routings on the Wavelength Non-Conversion Model

In this section, we study the permutation routing in W for the wavelength nonconversion model in which every routing path is assigned a single wavelength. For convenience, we only consider the case of $w(G) \leq w(H)$. The case of $w(H) \leq w(G)$ can be dealt with similarly, and omitted.

Following the proof of Theorem 2, a routing path L'_i for every $i \in V(W)$ consists of three segments $L'_{i,k}$, and each segment has been assigned a wavelength (a color), $k = 1, 2, 3$, based on the wavelength nonconversion model. Let γ_i be the color (wavelength) of $L'_{i,k}$. Then, $(\gamma_1, \gamma_2, \gamma_3)$ is the ordered color tuple of L'_i. We treat the tuple $(\gamma_1, \gamma_2, \gamma_3)$ as a coordinate point in a three-dimensional Cartesian coordinate system. Assume that each coordinate point in the system has been assigned a unique label, i.e., L'_i is assigned a wavelength numbered by the label of $(\gamma_1, \gamma_2, \gamma_3)$. Then, the total number of coordinate points for all routing paths in W for a permutation is $w(G) \times w(H) \times w(G) = w(G)^2 w(H)$. However, such wavelength assignment is not valid for those routing paths that have the same labels, because some of them sharing common edges will be assigned the same wavelength. We use an example (see Fig. 1) to illustrate this case. For the given permutation π, consider two routing paths L'_i and L'_j which have identical color tuple, where L'_i starts from i, goes through y', $\pi(j)$, x, x', and ends at $\pi(i)$, L'_j starts from j, goes through x', $\pi(i)$, y, y', and ends at $\pi(j)$. Clearly L'_i and L'_j share two common segments which are from y' to $\pi(j)$ and from x to $\pi(i)$. It is obvious that L'_i and L'_j cannot be assigned the same wavelength on the WDM model.

To cope with this case, the following approach is applied. Let $l_{\max}(G)$ be the number of edges in the longest routing path in G and $\mathcal{R}_l = \{L'_i | L'_i$ is labeled by l in the Cartesian coordinate system$\}$. Then, the set of routing paths for π is $\mathcal{R} = \bigcup_{l=1}^{w(G)^2 w(H)} \mathcal{R}_l$. For each $\mathcal{R}_l$, an auxiliary graph $G_l = (V_l, E_l)$ which is a subgraph of the conflict graph in W, is constructed as follows. Every vertex in V_l corresponds to a routing path in $\mathcal{R}_l$. There is an edge in E_l between two vertices if the two corresponding routing paths share at least one common edge in W, $1 \leq l \leq w^2(G)w(H)$. Then, we have

Lemma 3: Let $G_l = (V_l, E_l)$ be defined as above, then the maximum degree of G_l is $l_{\max}(G)$.

Proof: Let L'_i and L'_j be the corresponding routing paths of two vertices in G_l. We know that L'_k consists of three segments $L'_{k,1}$, $L'_{k,2}$, and $L'_{k,3}$, $k = i$ or $k = j$. By the definition of $\mathcal{R}_l$, $L'_{i,p}$ and $L'_{j,p}$

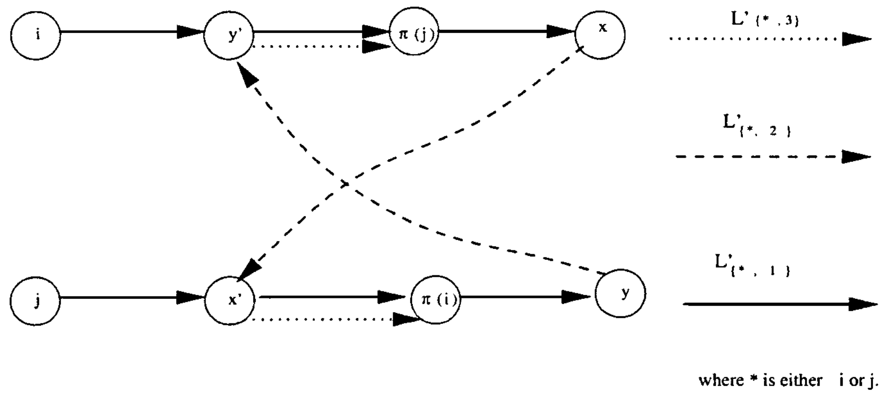

Fig. 1. An example.

are edge disjoint for all p, $1 \leq p \leq 3$. Notice that $L'_{i,1}$ and $L'_{j,3}$ (similarly $L'_{j,1}$ and $L'_{i,3}$) may not be edge disjoint (see Fig. 1). Since the number of edges in any routing path is no greater than $l_{\max}(G)$, there are at most $l_{\max}(G)$ other routing paths sharing common edges with L'_i. Therefore, the maximum degree of G_l is $l_{\max}(G)$. ■

Now, the graph G_l is colorable with $l_{\max}(G)+1$ colors such that the adjacent vertices are colored with different colors. This coloring can be done in polynomial time by a greedy approach. Thus, for each $\mathcal{R}_l$, we can assign $l_{\max}(G)+1$ wavelengths for the routing paths in it such that those paths sharing common edges are assigned different wavelengths, and there are $w(G)^2 w(H)$ $\mathcal{R}_l$s. Therefore, the total number of wavelengths required for any permutation within one round is $(l_{\max}(G)+1)w(G)^2 w(H)$, which is formally described as follows.

Theorem 3: Given permutation routing algorithms for networks G and H, there is a permutation routing algorithm for the product network $W = G \times H$. The number of wavelengths for any permutation in W with one round is $(l_{\max}(G)+1)w(G)^2 w(H)$ if $w(G) \leq w(H)$; or $(l_{\max}(H)+1)w(H)^2 w(G)$ otherwise, where $w(X)$ is the number of wavelengths needed to implement any permutation in network X with one round, and $l_{\max}(X)$ is the number of edges in the longest routing path in X.

Theorem 3 is only suitable for those kinds of product networks in which the number of wavelengths needed to implement any permutation with one round in its factor networks is small (constant or logarithmic of the problem size). Otherwise, it may not be good. Consider the following an example.

Let $|V(G)| = p$, $|V(H)| = q$, and $n = pq$. If both $w(H)$ and $w(G)$ are linear functions of the vertex sizes of G and H, i.e., $w(G) = ap$ and $w(H) = bq$ where a and b are constants with $0 < a, b < 1$. Without loss of generality, we further assume that $w(G) \leq w(H)$. Then, following Theorem 3, it needs $(l_{\max}(G)+1)w(G)^2 w(H) = (l_{\max}(G)+1)(a^2 b)pn = cn^{1+\alpha} > n$ wavelengths to implement any permutation with one round. Actually, any permutation can be implemented in any network with one round with n wavelengths, where $p = n^\alpha$ and $0 < \alpha \leq 1$. To cope with this case, we present another permutation routing algorithm for it. We start with the following lemma.

Lemma 4 [22]: Let $G_B(X, Y, E)$ be a bipartite graph such that for every subset S of X, we have $|N(S)| \geq |S|$, where $N(S)$ is the subset of Y that are adjacent to vertices in S. Then, G_B has a perfect matching of size $\min\{|X|, |Y|\}$.

Suppose $p \leq q$. Our idea comes from [22]. Each time we select p sources and their destinations such that these sources are in distinct rows and their destinations are in distinct columns. Such sources can be found through finding a perfect matching in a bipartite graph $G_B = (X, Y, E)$ where X is the set of rows and Y is the set of columns. There is an edge connecting $x \in X$ and $y \in Y$ if there is a source in row x whose destination is in column y. Clearly G_B is a bipartite multigraph, the degree of every vertex in X of G_B is q, and the degree of every vertex in Y of G_B is p. Since any subset $S \subseteq X$ and $|N(S)| \geq |S|$, there is a perfect matching in G_B by Lemma 4. By deleting this matching, we can find the next perfect matching in the remaining graph, and so on. As a result, G_B is decomposed into q edge disjoint perfect matchings. Since the routing paths in a perfect matching are edge disjoint, they can be assigned the same wavelength. So, we have

Lemma 5: Given a directed symmetric network $G \times H$ with $|V(G)| = p$, $|V(H)| = q$ and $p \leq q$, there is an algorithm for implementing any permutation in $G \times H$ with one round if q wavelengths are available.

If there are $\max\{w(G), w(H)\}$ wavelengths available for every fiber-optic link in $G \times H$, then, we have the following theorem.

Theorem 4: Let $G \times H$ be a directed symmetric network. On the wavelength nonconversion model, if there are permutation algorithms for implementing any permutation in G and H with one round with $w(G)$ and $w(H)$ wavelengths respectively, then there is a permutation algorithm for implementing any permutation in $G \times H$ within $|V(H)|/\max\{w(H), w(G)\}$ $(\leq 2c(H)+1)$ rounds with $\max\{w(H), w(G)\}$ wavelengths if $|V(G)| \leq |V(H)|$, or within $|V(G)|/\max\{w(H), w(G)\}$ $(\leq 2c(G)+1)$ rounds with $\max\{w(H), w(G)\}$ wavelengths, where $w(G)$ and $w(H)$ are the linear functions of their sizes and $c(X)$ is the number of edges in a bisection of X.

Proof: We only consider the case $|V(G)| \leq |V(H)|$. The analogous case $|V(H)| < |V(G)|$ is omitted. By Lemma 1, $w(H) \geq (|V(H)|-1)/2c(H)$, we have $|V(H)| \leq 2w(H)c(H)+1$. According to Lemma 5, the number of rounds needed is at most

$$\frac{|V(H)|}{\max\{w(H), w(G)\}} \leq \frac{2w(H)c(H)+1}{\max\{w(H), w(G)\}}$$

in order to implement any permutation in $G \times H$ with $\max\{w(H), w(G)\}$ wavelengths. That is, the number of rounds is at most $|V(H)|/w(H) \leq (2w(H)c(H)+1)/w(H) \leq 2c(H)+1$ if $w(G) \leq w(H)$; or $|V(H)|/w(G) \leq (2w(H)c(H)+1)/w(G) \leq 2c(H)+1$ otherwise. ■

IV. Conclusions

In this brief, we have shown a lower bound on the number of wavelengths required for routing any permutation in an all-optical product network with one round. We also have presented efficient routing algorithms for two models, the wavelength nonconversion and conversion models, respectively.

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Robust Control for Markovian Jump Linear Discrete-Time Systems With Unknown Nonlinearities

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***Abstract*—In this brief, we investigate the H_∞ control problem for a class of nonlinear discrete-time systems with Markovian jump parameters. The jump parameters considered here is modeled by a discrete-time Markov process. Our attention is focused on the design of state feedback controller such that both stochastic stability and a prescribed H_∞ performance are required to be achieved when the real system under consideration is affected by both unknown nonlinearity and norm-bounded real time-varying uncertainties. Sufficient conditions are proposed to solve the above problem, which are in terms of a set of solutions of linear matrix inequalities (LMIs).**

***Index Terms*—Discrete-time systems, Markovian jump parameters, norm-bounded uncertainties, Riccati-like inequalities.**

I. INTRODUCTION

Recently, stochastic linear uncertain systems have been studied extensively, in particular, the linear uncertain systems with Markovian jump parameters case, see, for example, [13], [15], [23] and [24]. A great amount of progress has been made in extending some of the results of the class of linear systems such as the stability, the stability robustness, the controllability, the observability, etc. to the class of linear systems with Markovian jump parameters. More recently, the H_∞ control problem for this class of systems in continuous case has been considered by De Souza and Fragoso [22], Shi and Boukas [16] which established the conditions guaranteeing the disturbance rejection in both cases of finite and infinite horizons. The counterpart of H_∞ control for discrete-time Markovian linear systems has been tackled by Fragoso *et al.*, [8], Boukas and Shi [5].

In this brief, we will investigate the problems of robust stability and robust H_∞ control of discrete-time Markovian linear systems with both unknown nonlinearity and norm-bounded real time-varying parameter uncertainties. The motivation for us to consider these kinds of uncertainties is that, in general, many uncertainties in real physical systems can be described by the above-mentioned uncertainty forms, which have been widely used in the design of robust control and filtering, see, e.g., [14], [17]–[19], [10]–[12].

The brief is organized as follows. In Section II, we give a brief description of the class of nonlinear discrete-time system with Markovian jump parameters and present some preliminary results. In Section III, we present our main results in this brief, i.e., robust stochastic stability and robust H_∞ control. The controller we design will achieve a prescribed H_∞ disturbance attenuation for all admissible uncertainties and unknown nonlinearity. It has been shown that the above problems have solutions if certain sets of coupled discrete Riccati-like inequalities have solutions.

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