A General Approach for All-to-All Routing in Multihop WDM Optical Networks

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***Abstract*—WDM optical networks provide unprecedented high speed and reliability for message transfer among the nodes. All-to-all routing is a fundamental routing problem in such networks and has been well studied on single hop WDM networks. However, the number of wavelengths to realize all-to-all routing on the single hop model typically is very large. One way to reduce the number of wavelengths is to use k-hop routing, in which each routing path consists of k segments and each segment is assigned a different wavelength, where k usually is a small constant. Because of the complexity of design and analysis for such a routing problem, only few papers discussed and proposed all-to-all routing by $k \geq 2$ hops. However, the proposed algorithms are usually exceeding complicated even for ring topologies. Often, an ad hoc approach is employed to deal with each individual topology.**

In this paper we propose a generic method for all-to-all routing in multi-hop WDM networks, which aims to minimize the number of wavelengths. We illustrate the approach for several optical networks of commonly used topology, including lines, rings, tori, meshes, and complete binary trees. For each case an upper bound on the number of wavelengths is obtained. The results show that this approach produces clear routing paths, requires less wavelengths, and can easily incorporate load balancing. For simple topologies such as lines and rings, this approach easily produces the same bounds on the number of wavelengths that were hard-obtained previously. Moreover, this general approach provides a unified routing algorithm for any d-dimensional torus, which seems impossible to obtain by the previous approach.

***Index Terms*—All-to-all routing, gossiping, multihop routing algorithms, network design, optical networks, robust routing protocol, WDM routing.**

I. Introduction

THE emerging *Wavelength-Division Multiplexing* (WDM) optical networks can provide capacities exceeding substantially those of conventional networks. Such networks promise data transmission rates several orders of magnitude higher than current electronic networks. This opens the opportunity for many real-time applications such as video conferencing, scientific visualization, real-time medical imaging, high speed supercomputing, and distributed computing [1], [26], [28]. The key to high speed in the network is to maintain the signal in optical form rather than traditional electronic form. The high bandwidth of fiber-optic links is utilized through the WDM technology which supports the propagation of multiple laser beams through a single fiber-optic link provided that each laser beam uses a distinct optical wavelength. Thus, optical communication introduces a new routing environment with new characteristics different from traditional one. By nature, packet routing algorithms are ill-designed for this setting. It is necessary to develop new routing methodologies for optical network communication.

WDM optical networks can be classified into two categories: *switch-less* (also called *broadcast-and-select*) and *switched*. Each of these can further be classified as either *single-hop* (also called all-optical) or *multihop* [23]. In switch-less networks, the transmission from each node is broadcast to all other nodes in the network. At each node, the desired signal is extracted from all the broadcast signals. The switch-less networks are practically important since the whole network can be constructed out of passive optical components, which are reliable and easy to operate. However, switch-less networks suffer from severe limitations when extended to wide area networks. Indeed, it has been proved in [1] that switch-less networks require a large number of wavelengths even for simple traffic patterns. Therefore, optical switches are required to build large scale networks. A switched optical network consists of nodes interconnected by point-to-point optic communication links. Each fiber-optic link supports a given number of wavelengths. Switches at each node direct their input signals to one or more output links. Each link consists of a pair of uni-directional links [26]. In this paper we consider switched networks in which signals from different applications may travel on the same communication link into a node and then exit along different outgoing links. The only constraint is that any two paths in the network sharing a common optical link must be assigned with different wavelengths. In switched networks it is allowed to "reuse wavelengths", thus, achieving a dramatic reduction on the number of required wavelengths with respect to switch-less networks [1].

All-optical (also called single hop) networks are networks where the information is transmitted in the form of light from source to its final destination without being converted to electronic form. Maintaining the signal in optical form allows the network to reach high speed since there is no overhead on conversions to and from the electronic form.

All-to-all routing (also called gossiping) is to disseminate a unique message from each node to every other node. This is a fundamental problem in multiprocessor systems and telecommunication networks that need to collect information about other nodes in the network regularly in order to manage

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network resources efficiently. The need also arises in many applications in the fields of parallel and distributed computing [7], [13], [10]. Therefore, the problem has been extensively studied in the literature in both traditional electronic networks [12], [18], [19] and optical networks [2], [5], [6], [27], [24], [25], [17]. However, most studies in optical networks focus on the single hop model, which usually requires a large number of wavelengths to be used. Using k (≥ 2) hops routing can greatly reduce the bound on the number of wavelengths required.

A. Related Work

All-to-all routing on a single hop model has been studied for rings, tori, meshes, hypercubes, and trees of rings [2], [4]–[6], [24], [27], [29], [30]. Beauquieer [2] showed that the number of wavelengths needed in a d-dimensional torus with n nodes in each dimension is at most $n^{d+1}/8$ when n is even. Bermond *et al.* [5], [6] showed that the number of wavelengths needed in a ring of n nodes is $\lceil (1/2)\lfloor n^2/4 \rfloor \rceil$, and in a hypercube of 2^d nodes is 2^{d-1}. Schroder *et al.* [27] considered some special product graphs with the following results. When n is odd, the number of wavelengths needed in a $n \times n$ torus is $(1/8)n(n^2-1)$ when $n \geq 3$; the number of wavelengths needed in a $s \times t$ torus or mesh with $2 \leq s$ and $2s \leq t$ is $\lceil s/2 \lfloor t/2 \rfloor \lceil t/2 \rceil \rceil$ or $s/2 \lfloor t/2 \rfloor \lceil t/2 \rceil$. Beauquier [2] showed that the number of wavelengths needed for a d-dimensional torus with n nodes in each dimension is either $n^{d+1}/8$ or no greater than $(n+1)^{d+1}/8$ depending on whether or not n is even, and the number of wavelengths needed for a d-dimensional mesh with n nodes in each dimension is either $n^{d+1}/4$ if n is even, or no more than $(n+1)^{d+1}/4$ otherwise. Narayanan *et al.* [24] considered all-to-all routing for a family of chordal rings of degree 4 by presenting an approximation algorithm that requires only 1.013 times the theoretic optimal bound. Zhang and Qiao [30] addressed the problem of scheduling all-to-all personalized connections in WDM rings. For a given number of wavelengths K and a number of transceivers per node T, they showed the lower bound on the schedule length, which is a function of the parameters K and T. They also presented a scheduling algorithm for the problem, which delivers a solution almost matching the lower bound.

Because of the complexity of design and analysis, only few papers [17], [25] discussed the multi-hop models. Opatrny [25] studied the *uniform* all-to-all routing problem for a symmetric directed ring. The numbers of wavelengths are $((n+3)/3)\sqrt{((n+16)/5)} + n/4$, $(n/2)\sqrt[3]{(\lceil n/4 \rceil + 4)/5}$, and $((n+16)/2)\sqrt[4]{\lceil n/4 \rceil + 8}$ for a uniform 2-hop, 3-hop and 4-hop model respectively, where a uniform routing is such a routing that each node communicates directly with an equal number of nodes at the same distance as any other node. Comellas *et al.* [9] further considered the uniform routing in a torus for some special cases. Independently, Gu and Peng [17] also studied the all-to-all routing problem in several specific mulit-hop WDM optical networks including lines, rings, 2-D square tori, and 3-D square tori. They showed that, to realize all-to-all routing in a network of N nodes under the k-hop model, it requires $O(N^{1+1/k})$ wavelengths in a ring, while it needs $O(N^{1+1/2k})$ or $O(N^{1+1/3k})$ wavelengths in a 2-D torus or a 3-D torus, respectively. Since they dealt with each case with topology-specific ad hoc design, their routing algorithms are exceedingly complicated. This makes it prohibitively difficult to extend their approaches to solve the problem in high dimensional tori or other topological structures. In contrast to their approaches, our approach is much simpler and generic, which is suitable for the architectures including lines, rings, d-dimensional square tori with any fixed d ($d \geq 1$), rather than $d \leq 3$, and complete binary trees. It must also be mentioned that our solution for complete binary trees partially answers an open problem given in [17], that is, how to efficiently realize all-to all routing in a tree topology, in terms of network resource (wavelengths) consumption.

B. Major Results of This Paper

A major result of this paper is a generic method for realizing all-to-all routing for a given network with k-hop routing. By this method, the original network is first partitioned into a number of subnetworks of roughly equal size such that the nodes in each subnetwork are connected. A super network whose topology is similar to the original network but with a smaller size is then constructed, which takes the subnetworks as its nodes. There is a super link between two nodes if there are any links between the two subnetworks. Each subnetwork is either the same type of network as the original one or a network which can be induced from the original network by removing some links from it. As a result, all-to-all routing for the original network is realized in two phases. In the first phase it realizes all-to-all routing for the super network under a single hop model, and in the second phase it realizes all-to-all routing in each subnetwork under a $(k-1)$-hop model, recursively, assuming that the solutions for the original network and the subnetworks under a single hop model are known in advance. We will illustrate this generic method for several commonly used topologies in later sections. A brief summary of the bounds on the number of wavelengths required is as follows.

For a line or a ring of N nodes, the number of wavelengths used is no more than $(1/4)kN^{1+1/k} + o(N^{1+1/k})$ or $(1/2^{2+1/k})kN^{1+1/k} + o(N^{1+1/k})$ respectively. For a d-dimensional torus or mesh of N nodes with the same number of nodes in each dimension, the number of wavelengths needed is no more than $(1/2^{2+1/k})kN^{1+1/kd} + o(N^{1+1/kd})$ or $(1/4)kN^{1+1/kd} + o(N^{1+1/kd})$, where $d \geq 2$. For a complete binary tree of N nodes, the number of wavelengths used is no more than $2^{(k-1/2^{k-1})}N^{1+1/2^{k-1}} + o(N^{1+1/2^{k-1}})$.

Although these bounds for rings, 2-D tori, and 3-D tori match with that in [17], our algorithms are much simpler, clearer, and more general, as shown in later sections. Especially, our methods produces a unified routing algorithm for any d-dimensional torus, which seems impossible to obtain by the previous approach. Clearly, the previously hard-obtained bounds on 2-D and 3-D tori now become only special cases in our general solution when $d = 2$, 3 respectively.

C. Paper Organization

The remainder of the paper is organized as follows. Section II introduces necessary notations and concepts. Since the generic method has been presented in the introduction, the following sections present detailed algorithms for several commonly used topologies. Section III deals with all-to-all routing for a line of N nodes. Section IV deals with all-to-all routing for a ring of N nodes. Section V deals with all-to-all routing for a d-dimensional torus or mesh of N $(= n^d)$ nodes with $d \geq 2$. Section VI studies all-to-all routing in a complete binary tree. The conclusion is given in Section VII.

II. Preliminaries

A WDM optical network can be represented by a *symmetric* directed graph $G = (V, E)$ with node set V representing the nodes of the network and edge set E representing optical links. Moreover, edge $\langle u, v\rangle \in E$ if and only if edge $\langle v, u\rangle \in E$.

A *request* is an ordered pair (u, v) of nodes in G which corresponds to a message to be sent from u to v. An *instance* I is a collection of requests. Given an instance I in the network, an *optical routing problem* is to determine a path through the network and assign a wavelength to each request in I, so that no two requests whose paths share a link are assigned the same wavelength. Since the cost of an optical switch is proportional to the number of wavelengths it can handle, it is paramount to determine paths and wavelengths so that the total number of wavelengths required is minimized. Thus, an optical routing problem contains the related tasks of *route assignment* and *wavelength assignment*. A routing R for a given instance I is a set of paths $\{P(x, y) \mid (x, y) \in I\}$, where $P(x, y)$ is a path from x to y in the network. By representing a wavelength by a color, the wavelength assignment can be seen as a coloring problem where one color is assigned to all the edges of a path. We say that the coloring of a given set of paths is *conflict-free* if any two paths sharing a link are assigned different colors.

Given an instance I and a routing R for a graph G, the *wavelength index* of the routing R, denoted by $\vec{w}(G, I, R)$, is the minimum number of colors needed for a conflict-free assignment to paths given by R. The *edge-congestion* or *load* of the routing R for I, denoted by $\vec{\pi}(G, I, R)$, is the maximum number of paths that share a common link. The *optimal wavelength index* $\vec{w}(G, I)$, and the *optimal load* $\vec{\pi}(G, I)$ for the instance I in G are the minimum values over all possible routings for the instance I in G. It is easy to see that $\vec{w}(G, I, R) \geq \vec{\pi}(G, I, R)$ for every routing R, thus, $\vec{w}(G, I) \geq \vec{\pi}(G, I)$.

Given a routing R for an instance I in G, the *routing graph* $G_R = (V, E_R, I)$ [11] is defined as follows. There is an edge $(u, v) \in E_R$ if there is a directed path in R from u to v, $u \in V$ and $v \in V$. Let $L(e, G, I, R)$ be the number of wavelengths assigned to $e \in E$ by routing R for an instance I in G. The link load is defined $L(G, I, R) = \max_{e\in E}\{L(e, G, I, R)\}$. The *link load* $L(G, I)$ of G for an instance I is defined as $L(G, I) = \min_{R\in\mathcal{R}}\{L(G, I, R)\}$, where $\mathcal{R}$ is the set of all possible routings for I.

Let I be an instance in G. A *k-hop solution* [22], [23] of I is a routing R and an assignment of wavelengths to the paths in R such that 1) it is conflict-free, i.e., any two paths of R sharing a directed link have different wavelengths; 2) for each request (u, v) in I, a path from u to v in R can be obtained by concatenation of at most k paths in R. In other words, in a *k-hop model* with $k \geq 2$, the signal must be converted into electronic form $k - 1$ times before reaching its destination. This conversion slows down the transmission, but reduces the number of wavelengths needed significantly [23].

An all-to-all routing instance I_A is an instance in a network that consists of all ordered pairs of nodes. For any given directed symmetric graph G, the question of whether $\vec{w}(G, I_A) = \vec{\pi}(G, I_A)$ is still open [15]. It is known to be true only for some specific networks, which include rings [6], trees [14], square tori or meshes [2], [27], d-dimensional tori or meshes with each dimension containing the same number of nodes [27], [2], hypercubes [6], and the Cartesian product of complete graphs [2].

Despite the fact that the single hop solution for all-to-all routing in a network is desirable, it uses an unrealistically large number of wavelengths to realize the instance I_A, while the number of wavelengths in a real network is very limited. To reduce the number of wavelengths needed, one possible way is to realize I_A on a k-hop model with $k \geq 2$. There are number of different routing solutions for I_A on a k-hop model. Some of them have poor performance. For example, consider a ring of n nodes numbered from 0 to $n - 1$. One possible 2-hop solution for I_A in the ring is $R = \{p_{0,i} \mid 1 \leq i \leq n-1\} \cup \{p_{i,0} \mid 1 \leq i \leq n-1\}$, in which path $p_{0,i}$ is the shortest path from node 0 to node i and path $p_{i,0}$, from node i to node 0. Any request from node i to node j in I_A can be obtained by a concatenation of paths $p_{i,0}$ and $p_{0,j}$. Thus, a conflict-free assignment of colors to all the paths in R uses $\lceil (n-1)/2\rceil$ colors. However, this solution suffers load imbalance and poor fault-tolerance, because node 0 is a bottleneck whose failure will result in the entire network broken down. For better fault-tolerance and load balancing, Narayanan *et al.* [24] proposed a *uniform routing* in a ring, in which each node can communicate directly with the *same number* of nodes and at *the same* distance along the ring as any other node. Thus, the routing graph derived from the uniform routing for I_A is a regular graph in which every node has the same degree.

Inspired by the definition of uniform routing, in this paper we propose a *robust routing* which is a fault-tolerant and load balanced routing. However, we do not require the routing graph derived by the robust routing to be a regular graph because the requirement of regular graphs is too strict, and impossible in some cases. Instead, we only require that the work load at each node be roughly balanced. More specifically, for a given instance I in G, let D be a function of the size of G, we focus on finding a robust routing R for I in G on a k-hop model such that (i) D is minimized and the degree of every node in the routing graph $G_R(V, E_R, I)$ is between $D - \rho$ and $D + \rho$, where ρ is either a small constant or $\rho = o(D)$; (ii) the link load $L(G, I, R)$ is minimized. Condition (i) aims to minimize and balance the node load in routing R, so that the difference of work load between any two nodes is no more than 2ρ; condition (ii) intends to minimize the number of wavelengths associated with each link in G, thereby reducing the cost of optical-to-electronic switches or electronic-to-optical switches. In particular, we will focus on devising k-hop robust routing algorithms for I_A in several popular optical networks including lines, rings, d-dimensional tori or meshes with n nodes in each dimension, and complete binary trees.

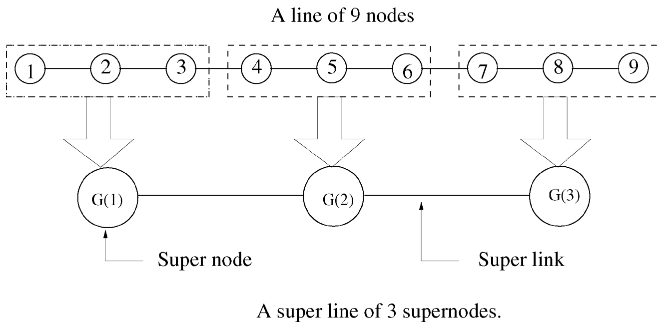

Fig. 1. A super line derived from a line of 9 nodes.

In the remaining discussion, we define $w_X(N;k)$ to be the number of wavelengths for realizing all-to-all routing in a network X of N nodes using k-hop routing, where $X = \{\text{line, ring, star, and complete binary tree}\}$.

III. All-to-All Routing for a Line of N Nodes

In this section we investigate all-to-all routing for a line of N nodes. We first obtain a solution for 2-hop routing and then extend the result to the k-hop model, $k \geq 2$.

A. The 2-Hop Model

All-to-all routing for a line under a single hop model has been studied and the minimum number of wavelengths needed is $w_{line}(N;1) \leq N^2/4$ [5]. We consider this problem on the 2-hop model. The idea is to partition a line into several roughly equal segments. Thus, every segment forms a group and contains a roughly equal number of nodes. We treat each group as a *supernode*. Thus, a *super line* consists of the supernodes with two adjacent supernodes connected by a super link. Fig. 1 illustrates the construction of a super line for a line of 9 nodes.

The algorithm for 2-hop routing consists of two phases. In the first phase it realizes all-to-all routing in the super line on a single hop model, and in the second phase it realizes all-to-all routing within each group (in a line) on a single hop model also. Thus, any routing path consists of at most two segments obtained in the two phases, and each segment is assigned a different wavelength. Note that in the second phase, the same set of wavelengths will be used for all-to-all routing in every group due to the fact that all groups are disconnected. Fig. 2 illustrates the two phases of the proposed algorithm for a line of 9 nodes. We need to say a few words about the all-to-all routing in the super line. The super line can be treated as $l\ (= 3)$ networks under the single hop model logically, since each supernode contains l nodes. In this example node 1 in supernode $G(1)$, node 4 in supernode $G(2)$, and node 7 in supernode $G(3)$ form a network. Similarly, node 2 in $G(1)$, node 5 in $G(2)$ and node 8 in $G(3)$ form a network; node 3 in $G(1)$, node 6 in $G(2)$ and node 9 in $G(3)$ also form a network. Realizing all-to-all routing in the super line is to realize all-to-all routing in these three networks under a single hop model. Since these networks share some links, each one needs a different set of wavelengths.

Now let us consider group $G(1)$ containing nodes 1, 2, and 3. After the first phase of the algorithm, node 1 received messages from nodes 4 and 7, node 2 received messages from nodes 5 and 8, and node 3 received messages from nodes 6 and 9. Obviously, after the second phase, each node in $G(1)$ received the messages

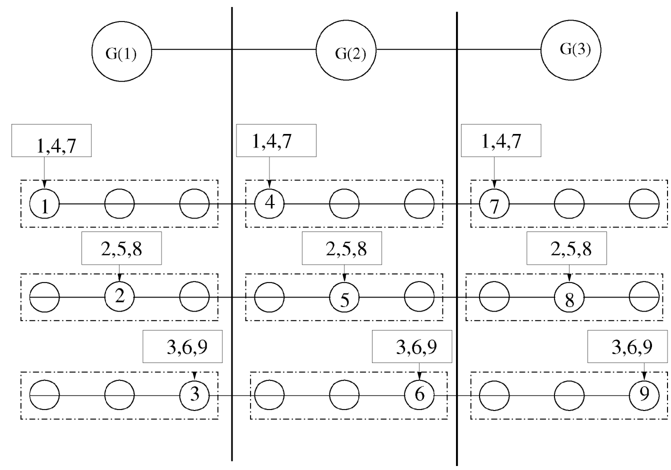

Fig. 2. Two phases for realizing all-to-all routing in a line of 9 nodes on the 2-hop model.

from all other nodes in the original network. The same is true for any other node.

Fig. 2 shows an ideal case where N is divisible by l, $N = ql$. In this ideal case, it is easy to estimate the number of wavelengths: Phase 1 needs $lq^2/4$ wavelengths, and Phase 2 needs $ql^2/4$ wavelengths since each node has q messages to send in the second phase. Therefore, the total number of wavelengths will be

$$w_{line}(N;2) \leq l \cdot \frac{q^2}{4} + q \cdot \frac{l^2}{4} = \frac{ql}{4}(q+l) = \frac{N}{4}(q+l)$$

$w_{line}(N;2)$ will be minimized when $q = l$. Therefore, $w_{line}(N;2) \leq (1/2)N^{3/2}$ when $\sqrt{N}$ is an integer.

Now let us consider the general case where $\sqrt{N}$ is not an integer. If $\sqrt{N}$ is not an integer, we can find an integer K such that $K^2 < N < (K+1)^2$. Let $N = K^2 + r$, where $1 \leq r \leq 2K$. We distinguish two cases, $1 \leq r \leq K$, and $K < r \leq 2K$.

(i) $1 \leq r \leq K$. In this case we divide the N nodes into K groups, each of which contains at most $(K+1)$ nodes. There are $(K+1)$ networks in the super line, each containing K nodes except one that contains r nodes. If $r = K$, then,

$$\begin{aligned} w_{line}(N;2) &\leq \frac{(K+1)K^2}{4} + \frac{K(K+1)^2}{4} \\ &= \frac{K^3}{2} + \frac{3K^2}{4} + \frac{K}{4} \\ &= \frac{K^3}{2} + O(K^2) \\ &= \frac{N^{3/2}}{2} + O(N) \\ &= \frac{N^{3/2}}{2} + o(N^{3/2}). \end{aligned} \tag{1}$$

If $r < K$, there are r groups containing $(K+1)$ nodes each and $(K-r)$ groups containing K nodes each. For this case we add a *dummy* node to each of these $(K-r)$ groups and select a real node to take care of this dummy node. Now we still have $(K+1)$ networks in the super line and each one contains K nodes. Of course, one network contains r real nodes and $(K-r)$ dummy nodes. The function of each dummy node is to receive messages from other nodes in the network, and send no messages. The real node that takes care of a dummy node will do actual work for the dummy node as well as the work in its own network. Because a different network uses a different set of wavelengths. This can be implemented without problem. Thus, the total number of wavelengths needed will be the same as that of (1).

(ii) $K < r \leq 2K$. In this case we divide the nodes into $(K+1)$ groups with each group containing no more than $(K+1)$ nodes. Because $K < r \leq 2K$, there are r' $(= r - K)$ groups that contain $(K+1)$ nodes each and $(K - r') + 1$ groups that contain K nodes each, where $0 < r' \leq K$. By adding dummy nodes, we obtain $(K+1)$ networks in the super line and each network contains $(K+1)$ nodes. The number of wavelengths required is estimated as follows.

$$\begin{aligned} w_{line}(N;2) &\leq \frac{(K+1)(K+1)^2}{4} + \frac{(K+1)(K+1)^2}{4} \\ &= \frac{K^3}{2} + \frac{3K^2}{2} + \frac{3K}{2} + 1 \\ &= \frac{K^3}{2} + O(K^2) \\ &= \frac{N^{3/2}}{2} + o(N^{3/2}). \end{aligned} \tag{2}$$

We can see that in both cases (i) and (ii), we have $w_{line}(N;2) \leq (1/2)N^{3/2} + o(N^{3/2})$. We therefore have the following theorem.

Theorem 1: There is an algorithm for realizing all-to-all routing for a line of N nodes with 2-hop routing. The number of wavelengths used is no more than $N^{3/2}/2 + o(N^{3/2})$. Moreover, if $\sqrt{N}$ is an integer, $w_{line}(N;2) \leq N^{3/2}/2$.

Proof: This follows from the above discussion. ■

Note that the fundamental difference between this method and that of in [17] is that our method does not pre-determine the value l, but uses an equation to optimally determine l and q.

Remark: When we actually implement 2-hop routing, we need to add dummy nodes and distribute them along the line evenly. Very fine analysis shows that the even distribution will further reduce the number of wavelengths needed, although this reduction will not change the order stated in Theorem 1. Interested readers can refer to the paper in [20].

B. The k-Hop Model With $k > 2$

We now extend the approach for 2-hop routing to k-hop routing. The idea behind the algorithm is similar to that for 2-hop routing. That is, in the first phase the algorithm realizes all-to-all routing in the super line on a single hop model, and in the second phase the algorithm realizes all-to-all routing in each group on a $(k-1)$-hop model, recursively.

Theorem 2: There is an algorithm for realizing all-to-all routing for a line of N nodes under a k-hop model such that the number of wavelengths used is no more than $(1/4)kN^{1+1/k} + o(N^{1+1/k})$. Moreover, if $N^{1/k}$ is an integer, then, it is no more than $(1/4)N^{1+1/k}$.

Proof: The theorem has been proved for $k = 2$ in Section III.A. Now we prove this theorem by induction on k. We assume that $N^{1/k}$ is an integer first. Let $N = p^k$. We divide N nodes into p groups with each containing p^{k-1} nodes exactly. Then, the number of wavelengths needed in the first phase is $p^{k-1} \cdot p^2/4$, and the number of wavelengths needed in the second phase is $p \cdot (1/4)(k-1) \cdot (p^{k-1})^{k/(k-1)}$ by induction. Therefore, the total number of wavelengths will be

$$w_{line}(N;k) \leq \frac{1}{4}(k-1)p^{k+1} + \frac{1}{4}p^{k+1} = \frac{1}{4}kN^{1+1/k}.$$

So, the theorem is correct.

Now let us consider the case where $N \neq p^k$. For this case we have an integer p such that $p^k < N < (p+1)^k$. By adding dummy nodes at each recursion, we get a new line that has $N_1 = (p+1)^k$ nodes. Because the work done by a dummy node is less than the work required for each real node, we have $w_{line}(N;k) \leq w_{line}(N_1;k) \leq (1/4)kN_1^{(k+1)/k}$. Since k is a fixed constant, we have

$$\begin{aligned} N_1^{k+1/k} &= (p+1)^{k+1} \\ &= p^{k+1} + (k+1)p^k + \binom{k+1}{2}p^{k-1} + \cdots + p + 1 \\ &= p^{k+1} + O(p^k) \\ &= p^{k+1} + o(p^{k+1}). \end{aligned}$$

Therefore, $w_{line}(N;k) \leq \left(\frac{1}{4}\right)kp^{k+1} + o(p^{k+1}) = \left(\frac{1}{4}\right)kN^{1+1/k} + o(N^{1+1/k})$. ■

IV. All-to-All Routing for a Ring of N Nodes

In this section we study all-to-all routing for a ring of N nodes using k-hop routing. The idea is similar to that for the line. That is, the nodes in the ring are partitioned into a number of groups such that each group contains equal number of nodes roughly and the subgraph induced by the nodes in a group is a line. Each group is treated as a *supernode*, and there is a *super link* between two supernodes if there is any link between them. Thus, a *super ring* consists of the supernodes and super links. The algorithm consists of two phases. In the first phase it realizes all-to-all routing for the super ring under the single hop model, and in the second phase it realizes all-to-all routing in each group (in a line) under the $(k-1)$-hop model recursively. Thus, we have the following theorem.

Theorem 3: There is an algorithm for realizing all-to-all routing in a ring of N nodes for k-hop routing. The number of wavelengths needed is no more than $(1/2^{2+1/k})kN^{1+1/k} + o(N^{1+1/k})$.

Proof: Assume that N is divisible by l and $N = ql$. Then, we divide N nodes into q groups with each group containing l nodes. The total number of wavelengths required for realizing all-to-all routing in a ring, $w_{ring}(N;k)$, is as follows.

The number of wavelengths in the first phase is $l\lceil q^2/8 \rceil$, and the number of wavelengths in the second phase is $q \cdot (1/4)(k-1)l^{1+1/(k-1)}$. Thus,

$$\begin{aligned} w_{ring}(N;k) &\leq l\lceil \frac{q^2}{8} \rceil + \frac{1}{4}q(k-1)l^{1+1/(k-1)} \\ &= \frac{1}{8}ql(q + 2(k-1)l^{1/(k-1)}) \end{aligned} \tag{3}$$

assuming $q^2/8$ is an integer.

$w_{ring}(N;k)$ is minimized when $l = (N/2)^{1-1/k}$ and $q = 2(N/2)^{1/k}$. Therefore, assuming $l = p^{k-1}$, when $N = 2p^k$, we have $q = 2p$. Then,

$$w_{ring}(N;k) \leq \frac{1}{8}ql(q+2(k-1)l^{1/(k-1)}) = \frac{1}{2}k\left(\frac{N}{2}\right)^{1+1/k}.$$

If $q^2/8$ is not an integer, then the number of wavelengths in $w_{ring}(N;k)$ is increased by at most $l = o((N/2)^{1-1/k})$, following formula (3).

When $N \neq 2p^k$, we can use dummy nodes and arguments similar to those used for lines to show $w_{ring}(N;k)$ will increase in the order of $o(N^{1+1/k})$. Therefore, the theorem follows. ■

V. All-to-All Routing on a Square Mesh or Torus

In this section we first deal with the all-to-all routing problem in a $n \times n$ square torus or mesh. We then address all-to-all routing in a d-dimensional torus or mesh with n nodes in each dimension, where $d \geq 2$. Let $w^{(d)}_{mesh}(n;k)$ or $w^{(d)}_{torus}(n;k)$ be the number of wavelengths needed for realizing all-to-all routing in a d-dimensional mesh or a d-dimensional torus with n nodes in each dimension using k-hop routing. For a d-dimensional torus ($d \geq 3$) of $N = n^d$ nodes, it has been shown that $w^{(d)}_{torus}(n;1) = n^{d+1}/8$[2], [27] when n is even, $w^{(d)}_{torus}(n;1) \leq (n+1)^{d+1}/8$ otherwise [2]. For a d-dimensional mesh either $w^{(d)}_{mesh}(n;1) = n^{d+1}/4$ when n is even [27], or $w^{(d)}_{mesh}(n;1) \leq (n+1)^{d+1}/4$ by the result for torus in [2].

A. *The 2-D Square Mesh or Torus on the 2-Hop Model*

The $n \times n$ torus or mesh is partitioned into a number of node-disjoint square blocks (square sub-meshes) such that the subgraph induced by the nodes in each block is connected. Each block is treated as a *supernode*, and there is a *super link* between two supernodes if there is any link between the nodes in them. A *super square torus* or a *super square mesh* consists of the supernodes and super links.

The basic idea of the proposed algorithm consists of two phases. In the first phase it realizes all-to-all routing in the super square torus or the super square mesh on the single hop model. In the second phase it realizes all-to-all routing in each square block (square sub-mesh) on a single hop model. Let W_i be the set of wavelengths used in phase i, $i = 1, 2$, $W_1 \cap W_2 = \emptyset$ due to the fact that a link within a sub-mesh can be contained by the routing paths in both phases. The total number of wavelengths for 2-hop routing is $|W_1| + |W_2|$.

Given the partition parameter q, let $n = ql + r$, $0 \leq r < q$. We consider the block partition in the square mesh or torus as follows.

The nodes in each column are partitioned into q groups such that there are r groups with each containing $l+1$ nodes and $(q-r)$ groups with each containing l nodes. As a result, the nodes in each column are partitioned into q groups $C(1), C(2), \ldots, C(q)$, and use the exact partition, the nodes in each row can be partitioned into q groups $R(1), R(2), \ldots, R(q)$, where $C(i)$ and $R(i)$ are the sets of the row and column coordinates of the nodes. Thus, there is a block partition for a square torus or mesh, which partitions the torus or mesh into q^2 blocks $\{G(i,j) \mid 1 \leq i \leq q, 1 \leq j \leq q\}$, where $G(i,j) = \{(x,y) \mid x \in C(i), y \in R(j)\}$. Consequently, the size of a block in this partition is one of the values, $l \times l$, $l \times (l+1)$, $(l+1) \times l$, or $(l+1) \times (l+1)$.

(i) If $r = 0$, then we have a $q \times q$ super square torus or mesh and q^2 $l \times l$ square sub-meshes. The numbers of wavelengths for all-to-all routing in a super square torus and mesh are $(1/8)l^2 \cdot q^3$ and $(1/4)l^2 \cdot q^3$ respectively. The number of wavelengths for all-to-all routing in a square sub-mesh is $(1/4)q^2 \cdot l^3$, because each node in the sub-mesh now contains q^2 messages to be sent after the first phase. Thus, we have

$$w^{(2)}_{torus}(n;2) \leq \frac{1}{8}l^2 \cdot q^3 + \frac{1}{4}q^2 \cdot l^3 \tag{4}$$

and

$$w^{(2)}_{mesh}(n;2) \leq \frac{1}{4}l^2 \cdot q^3 + \frac{1}{4}q^2 \cdot l^3 \tag{5}$$

$w^{(2)}_{torus}(n;2) = (1/2^{3/2})n^{5/2} = (1/2^{3/2})N^{5/4}$ is minimum when $l = \sqrt{n/2}$, and $w^{(2)}_{mesh}(n;2) = (1/2)n^{5/2} = (1/2)N^{5/4}$ is minimum when $l = \sqrt{n}$. So far we have assumed that both $\sqrt{n/2}$ and $\sqrt{n}$ are integers and $r = 0$.

(ii) If $r \neq 0$, then we add dummy columns and rows to the blocks such that each of them becomes a square block. The detailed description is as follows.

For the mesh, we assume that $K^2 < n < (K+1)^2 = K^2 + 2K + 1$. Let $n = K^2 + r$, then $1 < r \leq 2K$. Let $m = (2K+1) - r$. We evenly insert m dummy columns and m dummy rows in the network such that $n + m = (K+1)^2$. In the worst case a block may have 2 dummy columns and 2 dummy rows. As we argued before, we have

$$\begin{aligned}
w^{(2)}_{mesh}(n;2) &\leq w^{(2)}_{mesh}(n+m;2)\\
&\leq \frac{1}{4}(K+1)^2 \cdot (K+1)^3 + \frac{1}{4}(K+1)^2 \cdot (K+1)^3\\
&= \frac{1}{2}K^5 + \frac{5}{2}K^4 + 5K^3 + 5K^2 + \frac{5}{2}K + \frac{1}{2}\\
&= \frac{1}{2}K^5 + O(K^4)\\
&= \frac{1}{2}K^5 + o(K^5)\\
&= \frac{1}{2}N^{5/4} + o(N^{5/4}).
\end{aligned}$$

For the torus, we assume that $K^2 < n/2 < (K+1)^2$. Let $n = 2K^2 + r$, then $1 < r \leq 4K + 1$. Let $m = 4K + 1 - r$. We evenly insert m dummy columns and m dummy rows such that $n + m = 2(K+1)^2$. Using arguments similar to those we used for the mesh case, we have

$$\begin{aligned}
w^{(2)}_{torus}(n;2) &\leq w^{(2)}_{torus}(n+m;2)\\
&\leq \frac{1}{8}(K+1)^2 \cdot (2K+1)^3 + \frac{1}{4}(2K+1)^2 \cdot (K+1)^3\\
&= 2K^5 + \frac{15}{2}K^4 + 11K^3 + \frac{59}{8}K^2 + \frac{11}{4}K + \frac{3}{8}\\
&= 2K^5 + O(K^4)\\
&= 2K^5 + o(K^5)\\
&= \frac{1}{2^{3/2}}N^{5/4} + o(N^{5/4}).
\end{aligned}$$

We therefore have the following theorem.

Theorem 4: Given a 2-D $n \times n$ square torus or mesh of N $(= n^2)$ nodes, there is an algorithm for realizing all-to-all routing with 2-hop routing. The numbers of wavelengths is no more than $(1/2^{3/2})N^{5/4} + o(N^{5/4})$ for the square torus, and $(1/2)N^{5/4} + o(N^{5/4})$ for the square mesh.

B. The 2-D Square Mesh or Torus on the k-Hop Model

In this section we generalize the algorithm for a 2-D square mesh or torus on the 2-hop model to the k-hop model as follows. The algorithm consists of two phases. In the first phase it realizes all-to-all routing in the super square torus or the super square mesh on a single hop model. In the second phase it realizes all-to-all routing in each square sub-mesh on a $(k-1)$-hop model, recursively. We then have the following theorem.

Theorem 5: Given a 2-D $n \times n$ square torus or mesh of N nodes with $N = n^2$, there is an algorithm for realizing all-to-all routing on a k-hop model. The number of wavelengths for the square mesh is no more than $(1/4)kN^{1+1/2k} + o(N^{1+1/2k})$, and the number of wavelengths for the square torus is no more than $(1/2^{2+1/k})kN^{1+1/2k} + o(N^{1+1/2k})$. Moreover, if $n = p^k$ for mesh and $n = 2p^k$ for torus, then the bounds can be improved to $(1/4)kN^{1+1/2k}$ and $(1/2^{2+1/k})kN^{1+1/2k}$, respectively.

Proof: The theorem has been proved for $k = 2$ in Section V.A. We now prove this theorem by induction on k.

We first consider a $n \times n$ square mesh. We assume that $n = p^k$. We divide n nodes in a row or column into p groups with each containing p^{k-1} nodes exactly. Thus, there is a $p \times p$ super square mesh in which there are $p^{k-1} \times p^{k-1}$ networks, and there are p^2 $p^{k-1} \times p^{k-1}$ square sub-meshes. The number of wavelengths needed in the first phase is $(p^{k-1} \cdot p^{k-1}) \cdot (1/4)p^3$, and the number of wavelengths needed in the second phase is $p^2 \cdot (1/4)(k-1) \cdot p^{2(k-1)+1}$ by induction. Therefore, the total number of wavelengths will be

$$\begin{aligned} w_{mesh}^{(2)}(n;k) &\le \frac{1}{4}p^{2k+1} + \frac{1}{4}(k-1)p^{2k+1} \\ &= \frac{1}{4}kp^{2k+1} \\ &= \frac{1}{4}kn^{2+1/k} \\ &= \frac{1}{4}kN^{1+1/2k}. \end{aligned}$$

Now we consider the case where $n \neq p^k$. For this case we have an integer p such that $p^k < n < (p+1)^k$. By adding dummy nodes at each recursion, we get a new line (row or column) that has $n_1 = (p+1)^k$ nodes. Because the work done by a dummy node is less than the work required for each real node, we have $w_{mesh}^{(2)}(n;k) \le w_{mesh}^{(2)}(n_1;k) \le (1/4)kn_1^{2+1/k}$. Since k is a fixed constant, we have

$$\begin{aligned} n_1^{2+1/k} &\le (p+1)^{2k+1} \\ &= p^{2k+1} + O(p^{2k}) \\ &= p^{2k+1} + o(p^{2k+1}). \end{aligned}$$

Therefore,

$$\begin{aligned} w_{mesh}^{(2)}(n;k) &\le \left(\frac{1}{4}\right)kp^{2k+1} + o(p^{2k+1}) \\ &\le \left(\frac{1}{4}\right)kN^{1+1/2k} + o(N^{1+1/2k}). \end{aligned}$$

We then consider a $n \times n$ square torus. We first assume that $n = 2p^k$. We divide n nodes in a row or column into $2p$ groups with each containing p^{k-1} nodes exactly. Thus, there is a $2p \times 2p$ super square torus in which there are $p^{k-1} \times p^{k-1}$ networks, and there are $4p^2$ $p^{k-1} \times p^{k-1}$ square sub-meshes. The number of wavelengths needed in the first phase is $(p^{k-1} \cdot p^{k-1}) \cdot (1/8)(2p)^3$, and the number of wavelengths needed in the second phase is $(2p)^2 \cdot (1/4)(k-1) \cdot p^{2(k-1)+1}$ by the above result for a 2-D square mesh. Therefore, the total number of wavelengths will be

$$\begin{aligned} w_{torus}^{(2)}(n;k) &\le p^{2k+1} + (k-1)p^{2k+1} \\ &= kp^{2k+1} \\ &= \frac{1}{2^{2+1/k}}kn^{2+1/k} \\ &= \frac{1}{2^{2+1/k}}kN^{1+1/2k}. \end{aligned}$$

We now assume that $n \neq 2p^k$. For this case we have an integer p such that $2p^k < n < 2(p+1)^k$. By adding dummy nodes at each recursion, we get a new ring (row or column) that has $n_1 = 2(p+1)^k$ nodes. Since the work done by a dummy node is less than the work required for each real node, we have $w_{torus}^{(2)}(n;k) \le w_{torus}^{(2)}(n_1;k) \le (1/2^{2+1/k})kn_1^{2+1/k}$. k is a fixed constant, so we have

$$\begin{aligned} n_1^{2+1/k} &\le (2(p+1)^k)^{2+1/k} \\ &= 2^{2+1/k}(p+1)^{2k+1} \\ &= 2^{2+1/k}p^{2k+1} + O(p^{2k}) \\ &= 2^{2+1/k}p^{2k+1} + o(p^{2k+1}). \end{aligned}$$

Therefore,

$$\begin{aligned} w_{torus}^{(2)}(n;k) &\le \frac{1}{2^{2+1/k}}k2^{2+1/k}p^{2k+1} + o(p^{2k+1}) \\ &= kp^{2k+1} + o(p^{2k+1}) \\ &< \frac{1}{2^{2+1/k}}kn^{2+1/k} + o(n^{2+1/k}) \\ &= \frac{1}{2^{2+1/k}}kN^{1+1/2k} + o(N^{1+1/2k}). \end{aligned}$$

The theorem thus follows. ■

C. The d-Dimensional Mesh and Torus on the k-Hop Model

We now generalize the approach for 2-D square torus or mesh to the d-dimensional torus or mesh of n nodes in each dimension with $d \ge 3$. Following the similar discussion for a 2-D square mesh or torus on the k-hop model, we partition the d-dimensional torus or mesh into a number of blocks (d-dimensional sub-meshes), and each of the blocks as a super node. The algorithm proceeds in two phases. In the first phase it implements all-to-all routing in the super d-dimensional torus or mesh on a single hop model, and in the second phase it realizes all-to-all routing in each of the blocks (d-dimensional sub-meshes) on the $(k-1)$-hop model recursively. We thus have the following theorem.

Theorem 6: Given a d-dimensional torus or mesh of $N = n^d$ nodes with n nodes in each dimension, there is an algorithm for realizing all-to-all routing in it with k-hop routing. The number of wavelengths needed for the torus is no more than $(1/2^{2+1/k})kN^{1+1/kd} + o(N^{1+1/kd})$, and the number of wavelengths needed for the mesh is no more than $(1/4)kN^{1+1/kd} + o(N^{1+1/kd})$.

Proof: The proof is similar to Theorem 5. In the following we only prove the results for a d-dimensional mesh. The results for a d-dimensional torus can be shown similarly.

The special case where $d = 2$ has been proved in Theorem 5. We now prove this theorem for $d \geq 3$ by induction on k. We first assume that $n = p^k$ and p is even. It is obvious that the theorem is true when $k = 1$ for the single hop model. Suppose the theorem is true for the $(k-1)$-hop model. We prove that it is true for the k-hop model. We divide n nodes in each dimension into p groups with each containing p^{k-1} nodes exactly. Thus, there is a super d-dimensional mesh with p nodes in each dimension, and in this super d-dimensional mesh, there are p^d d-dimensional sub-meshes with each containing $p^{(k-1)d}$ nodes. The number of wavelengths needed in the first phase is $p^{(k-1)d} \cdot (1/4)p^{d+1}$, and the number of wavelengths needed in the second phase is $p^d \cdot (1/4)(k-1) \cdot p^{d(k-1)+1}$ by induction. Therefore, the total number of wavelengths is

$$\begin{aligned} w_{mesh}^{(d)}(n;k) &\leq \frac{1}{4}p^{kd+1} + \frac{1}{4}(k-1)p^{dk+1} \\ &= \frac{1}{4}kp^{dk+1} \\ &= \frac{1}{4}kn^{d+1/k} \\ &= \frac{1}{4}kN^{1+1/kd}. \end{aligned}$$

Now we consider the case where $n \neq p^k$ or p is odd. For this case we have an integer p such that either $p^k < n < (p+1)^k$ or $p^k = n < (p+1)^k$. By adding dummy nodes at each recursion, we get a new line (row or column) of $n_1 = (p+1)^k$ nodes. Because the work done by a dummy node is less than the work required for each real node, we have $w_{mesh}^{(d)}(n;k) \leq w_{mesh}^{(d)}(n_1;k) \leq (1/4)kn_1^{d+1/k}$. Since k is a fixed constant, we have

$$\begin{aligned} n_1^{d+1/k} &\leq (p+1)^{kd+1} \\ &= p(p+1)^{kd} + (p+1)^{kd} \\ &= p^{kd+1} + O(p^{kd}) \\ &= p^{kd+1} + o(p^{kd+1}). \end{aligned}$$

Therefore,

$$\begin{aligned} w_{mesh}^{(d)}(n;k) &\leq \left(\frac{1}{4}\right) kp^{kd+1} + o(p^{kd+1}) \\ &= \left(\frac{1}{4}\right) kN^{1+1/kd} + o(N^{1+1/kd}). \end{aligned}$$

Using the similar argument, it can be shown that $w_{torus}^{(d)}(n;k) \leq (1/2^{2+1/k})kN^{1+1/kd} + o(N^{1+1/kd})$. The theorem follows. ■

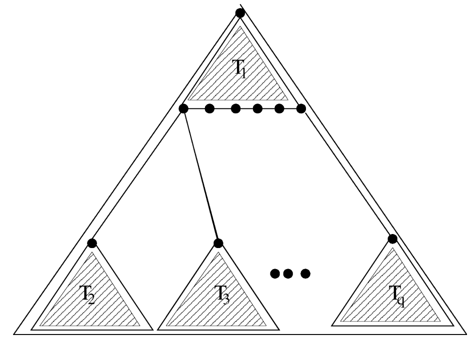

Fig. 3. A complete binary tree is partitioned into q complete binary subtrees with equal size.

Note that our generic method provides a neat and clear recursive structure for the routing paths without introducing tedious dimension-specific routing details and designs as did in [17]. The hard-obtained bounds in [17] become special cases ($d = 2$, 3) in our solution.

VI. All-to-All Routing in a Complete Binary Tree

A *complete binary tree* is a full binary tree in which all leaf nodes are at the bottom level, and there are 2^{h-1} leaf nodes if the height of the tree is h $(h \geq 1)$. Given a complete binary tree T of N nodes with height h, then $h = \log(N+1)$. We partition T into q subtrees such that each of the subtrees is still a complete binary tree with roughly equal size. The q subtrees are constructed as follows. The first subtree T_1 is a tree rooted at the original root with height h_1, and it contains $(2^{h_1}-1)$ nodes. For each leaf node in T_1, the two children of the leaf node will be the roots of the other two complete binary subtrees. There are 2^{h_1} such subtrees with each having height $(h - h_1)$. To make the size of each of these $(q-1)$ subtrees be roughly identical to that of T_1, we require

$$2^{h_1} - 1 = 2^{h-h_1} - 1. \tag{6}$$

Thus, $h_1 = \log(N+1)/2$ and each subtree contains $2^{h_1} - 1 = \sqrt{N+1} - 1$ nodes. Fig. 3 illustrates the partition of the q subtrees.

Assume that each subtree is a *supernode*. There is a *super link* between two supernodes if there is any link in the original complete binary tree between the nodes in the two supernodes. A graph is constructed, which consists of supernodes and super links. Obviously, the graph is a *super star*, and its center is the supernode corresponding to T_1. It is not difficult to show that $w_{star}(N;1) = N-1$ for a star of N nodes and $w_{tree}(N;1) = (N^2-1)/4$ for a complete binary tree of N nodes under the single hop model.

The algorithm for realizing all-to-all routing in a complete binary tree on the k-hop model is as follows. It realizes all-to-all routing in the super star on the single hop model, followed by realizing all-to-all routing in each subtree under the $(k-1)$-hop model recursively. The number of wavelengths for realizing all-to-all routing in the first phase is $(2^{h_1}-1) \cdot (q-1)$ because the super star contains q supernodes, and $q = 2^{h_1} + 1 = \sqrt{N+1} + 1$. In the second phase another set of wavelengths will be used for realizing all-to-all routing in each subtree of $(2^{h_1}-1)$ $(=\sqrt{N+1}-1)$ nodes under the $(k-1)$-hop model. We thus have the following theorem.

Theorem 7: There is an algorithm for realizing all-to-all routing in a complete binary tree of N nodes on the k-hop model. The number of wavelengths used is no more than $2^{(k-1/2^{k-1})}N^{1+1/2^{k-1}}+o(N^{1+1/2^{k-1}})$.

Proof: We show the theorem by induction. When $k=1$, it is known that $w_{tree}(N;1)=(N^2-1)/4$.

If the height h of T is even, following the partitioning approach above, the tree T is partitioned into q_1 subtrees, and the number of nodes in each subtree is $N_1=\sqrt{N+1}-1$, where $q_1=\sqrt{N+1}+1$. Obviously $N=q_1N_1$. Otherwise (h is odd), we add dummy nodes in the bottom of T to form a new complete binary tree T' with height $h+1$. The number of nodes in T' is $N'=2^{h+1}-1=2(2^h-1)+1=2N+1$. Then, the tree T' is partitioned into q_1 subtrees, and the number of nodes in each subtree is $N_1=\sqrt{2N+2}-1$, where $q_1=\sqrt{2N+2}+1$. Obviously $2N+1=q_1N_1$.

At the first phase we have a super star of q_1 supernodes, and each supernode is a complete binary tree of N_1 nodes. Thus, the number of wavelengths needed for the first phase is $N_1(q_1-1)=N-N_1$. After the first phase we apply the $(k-1)$-hop routing on each subtree recursively. Therefore, the total number of wavelengths needed is

$$w_{tree}(N;k)=(N-N_1)+q_1\cdot w_{tree}(N_1;k-1).$$

We assume that $w_{tree}(N';k')\le 2^{(k'-1/2^{k'-1})}N'^{1+1/2^{k'-1}}+o(N'^{1+1/2^{k'-1}})$ holds when $N'<N$ and $k'<k$.

It is obvious that the assumption holds when $k'=1$ or $N'=1$. In the following we show $w_{tree}(N;k)\le 2^{(k-1/2^{k-1})}N^{1+1/2^{k-1}}+o(N^{1+1/2^{k-1}})$ holds for any general integers N and $k>1$.

When h is even, we have

$$\begin{aligned}
&w_{tree}(N;k)\\
&\quad=q_1\cdot w_{tree}(N_1;k-1)+(N-N_1)\\
&\quad\le(\sqrt{N+1}+1)\cdot 2^{((k-1)-1/2^{k-2})}(\sqrt{N+1}-1)^{1+1/2^{k-2}}\\
&\qquad+o((\sqrt{N+1}-1)^{1+1/2^{k-2}})+(N-\sqrt{N+1}+1)\\
&\quad=2^{(k-1/2^{k-1})}N^{1+1/2^{k-1}}+o(N^{1+1/2^{k-1}}).
\end{aligned}$$

When h is odd, we have

$$\begin{aligned}
&w_{tree}(N;k)\\
&\quad\le q_1\cdot w_{tree}(N_1;k-1)+(N-N_1)\\
&\quad\le(\sqrt{2N+2}+1)\cdot 2^{((k-1)-1/2^{k-2})}(\sqrt{2N+2}-1)^{1+1/2^{k-2}}\\
&\qquad+(N-\sqrt{2N+2}+1)+o((\sqrt{2N+2}-1)^{1+1/2^{k-2}})\\
&\quad=2^{(k-1/2^{k-1})}N^{1+1/2^{k-1}}+o(N^{1+1/2^{k-1}}).
\end{aligned}$$

Therefore, $w_{tree}(N;k)\le 2^{(k-1/2^{k-1})}N^{1+1/2^{k-1}}+o(N^{1+1/2^{k-1}})$. ■

VII. Conclusions

In this paper we have dealt with the all-to-all routing problem in several standard directed symmetric WDM optical networks including lines, rings, d-dimensional tori or meshes with n nodes in each dimension ($d\ge 2$), and complete binary trees, on a k-hop model for $k\ge 2$. For each of these networks we have proposed an algorithm for realizing all-to-all routing such that the routing is fault tolerant and both node load and link load are well balanced. The technique adopted is to partition the network into a number of subnetworks with roughly equal size. All-to-all routing in a network on a k-hop model is implemented through first realizing all-to-all routing in the same type of super network as the original network on a single hop model, and then realizing all-to-all routing in each subnetwork, which is either the same or of a similar type to the original, on a $(k-1)$-hop model, recursively. As a future work, we conjecture that all bounds obtained here are tight.

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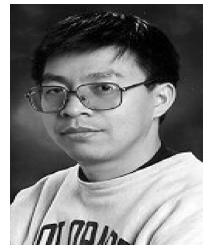

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