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Research Paper

Spatial Interpolation Using Fuzzy Reasoning

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Abstract
Spatial interpolation is an important feature of a Geographic Information System, which is the procedure used to estimate values at unknown locations within the area covered by existing observations. In this paper, we describe a conservative spatial interpolation technique that incorporates the advantages of local interpolation, Euclidean interpolation, and conservative fuzzy reasoning, and a dynamic fuzzy-reasoning-based function estimator with parameters optimised by a genetic algorithm. The main objective of this paper is to formulate a computationally efficient spatial interpolation technique similar to the IDWA technique that can be used in real time application. The main feature of our spatial interpolation technique is a capability for spatial interpolation and extrapolation in a higher-dimensional space. Examples from a rainfall spatial interpolation problem are used to illustrate the applicability of the proposed technique.

1 Introduction

In a Geographic Information System (GIS), spatial interpolation is an essential feature (Burrough and McDonnell 1998). Normally, points with known values are used to estimate values at other points. An application for interpolation is rainfall prediction using spatial interpolation (Lee et al. 1998, Huang et al. 1998, Wong et al. 1997). In this example, interpolation can estimate the amount of rainfall at a location based on the knowledge from the rainfall measurements at nearby locations. All spatial interpolation

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techniques can be grouped into global and local methods (Burrough and McDonnell 1998). In a global method, all the information available is used to estimate an unknown value, while local methods only use a sample of the information for estimation.

In a global method, trend surface analysis is normally performed. The equation that can be used to estimate values at other points using a third-order trend surface is:

$$Z = b_0 + b_1X + b_2Y + b_3X^2 + b_4XY + b_5Y^2 + b_6X^3 + b_7X^2Y + b_8XY^2 + b_9Y^3 \qquad (1)$$

where *b* coefficients are estimated from the available information points.

A more general form of Equation (1) can be written as:

$$Z = f(X, Y) \qquad (2)$$

As for a local method, the spatial interpolation of the value Z_i in the *i- th* surface subregion is:

$$Z_i = f(X_i, Y_i) \qquad (3)$$

This trend surface analysis could be considered as a subset of polynomial regression (Myers 1990). However, in this case in order to construct an unbiased model, a large amount of data is necessary, but this will effectively increase the computational complexity of the system.

In Lee et al. (1998), Huang et al. (1998) and Wong et al. (1997), the authors have discussed the functionality of local methods and found that they provide better results as compared to global methods. Another popular spatial interpolation technique used is Kriging (Matheron 1963). This is normally used to construct a model to describe the variance between points as a function of separation distance. Kriging involves the estimation of a variogram, which is a complex and computationally intensive process.

For computational efficiency, the inverse distance weighted averaging (IDWA) interpolation is preferred over the last few spatial interpolation techniques (Legates and Willmont 1990). IDWA is a deterministic estimation technique that uses values at the known locations to estimate the values at the unknown points by using linear combination. The assumption made by IDWA is that the values from the points closer to the unknown points are more representative of the values to be estimated than those points further away. This method is quite similar to Kriging, but the relationship between the separation distance and the variance is fixed a priori by an arbitrary global function. There is always uncertainty whether the spatial weighting function reflects the spatial structure of the surface. In Tomlin (1990), the author presented another interpolation technique known as Euclidean interpolation. In this technique, he used attribute data to identify the connections that are independent of the physical distance. The Euclidean distance between the sets of attribute data associated with each unknown point are computed. The interpolation is performed based on the minimum Euclidean distance and the geographic distance.

In this paper, we propose a spatial interpolation technique that incorporates the advantages of local interpolation, Euclidean interpolation, and conservative fuzzy reasoning (Gedeon and Kóczy 1997, Kohonen 1995). The main objective of the analysis is to formulate a computationally efficient spatial interpolation technique similar to the IDWA technique that can be used in real time application. Examples extracted from a rainfall spatial interpolation problem (Lee et al. 1998, Huang et al. 1998, Wong et al. 1997) are also used to illustrate the applicability of the proposed technique.

The method presented in this paper has the potential to speed up the spatial interpolation of a region based on large data sets due to its reduction to local homogeneous regions of analysis, which can be very important with the increasing use of interactive spatial analysis tools. The model-free approach also eliminates (or at least greatly reduces) many of the statistical difficulties that appear in spatial interpolation, potentially giving more robust estimation results. The capability of using the evidence of ancillary variables to constrain the interpolation is a definite advantage for process-based analyses.

2 Dynamic Fuzzy-reasoning-based Function Estimator

The dynamic fuzzy-reasoning-based function estimator (DFFE) was proposed by Sun and Davidson (1996). This method starts with the simple concept of interpolation and extrapolation for estimating a function value when certain geometric conditions, "parallel" and "close", are satisfied completely. However, the fuzzy-reasoning component extends the extrapolation and interpolation using non-linear weightings for the neighbouring values based on closeness and the directions of the deviation vectors, as a way to mimic human reasoning. Such fuzzy concepts therefore can tolerate partial satisfaction of the preconditions and take into account the discrepancy in inferring the function values. It is also an assumption-free, model-free and exact interpolator.

Figure 1 shows the architecture of the DFFE. It is composed of a case base of past observations, a dynamic knowledge base creator, a fuzzy reasoning mechanism and an explanation mechanism. When a new input vector is defined, past observations which are similar to the input vector are selected and used to build a knowledge base which consists of a set of fuzzy rules and related truth values. A fuzzy reasoning mechanism is used to infer the response of the system. The explanation mechanism saves the latest rules and truth values so as to be able to answer questions about the response.

A typical DFFE model uses two geometric functions: "close" and "parallel". The functions are usually implemented in the form of fuzzy membership functions.

The "parallel" function can be defined as:

$$\mu_a = \exp(-a \mid CA_{NJ} \mid) \tag{5}$$

where μ_a is the fuzzy membership value of the fuzzy set CA_{NJ} which is the cosine of the spatial angle between the deviation vector of the estimation data point from the reference data point (the closest point to the estimation data point) and the deviation vector

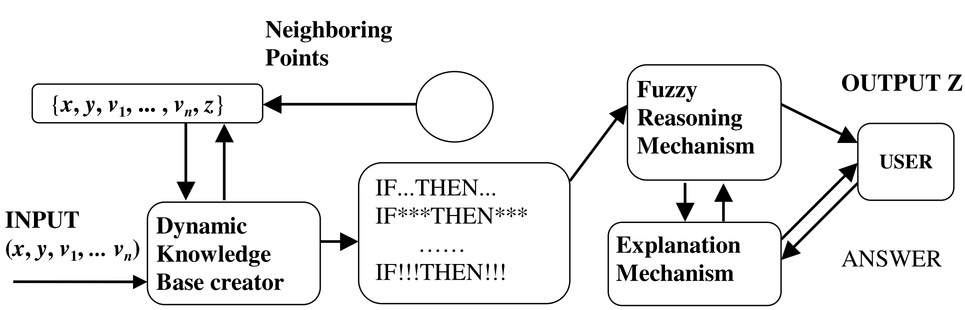

Figure 1 The structure of DFFE

of a neighboring data point from the reference data point. The parameter *a* is the only parameter in the membership function which is usually determined by trial-and-error or cross-validation.

The "close" function can be defined as:

$$\mu_\beta = \exp(-bD_{RN}) \tag{6}$$

where μ_β is the fuzzy membership value of the fuzzy set D_{RN} which is the distance between the estimation data point and the reference point, and *b* is the parameter in the fuzzy membership function. We assume here that the parameters *a* and *b* are optimised, for example by the use of genetic algorithms as in Wong et al. (1997).

2.1 Case Study Results and Discussion

An example interpolation was performed on the rainfall data collected on 8th May 1986 in Switzerland (Wong et al. 1997). One hundred data locations are used as the training data and the data for another 367 locations are then used to verify the prediction accuracy. The input variables used are the 2D coordinate position (x, y); and the output is the rainfall measurements (z). These data are also used in subsequent sections. An extra component of the data set is used here, being the Digital Elevation Model (DEM) data (v). In this section, we use $\{x, y, v\}$ as the input data and z (rainfall) as the output data. Partition of the whole surface is based on the $\{x, y, z\}$ data.

Figure 2 shows a scatter-plot of the estimations versus the true values at the 367 locations. The R^2 was 0.67. The statistics of all the estimated values are tabulated in Table 1. As shown in this table, the statistics of the predictions and the true values are very similar, except for the maximum value being overestimated. This means the overall training of the model was good, but not the performance.

The slope is close to 1, which would make a 'perfect fit'. The intercept is close to 50, indicating an inherent bias in the estimates.

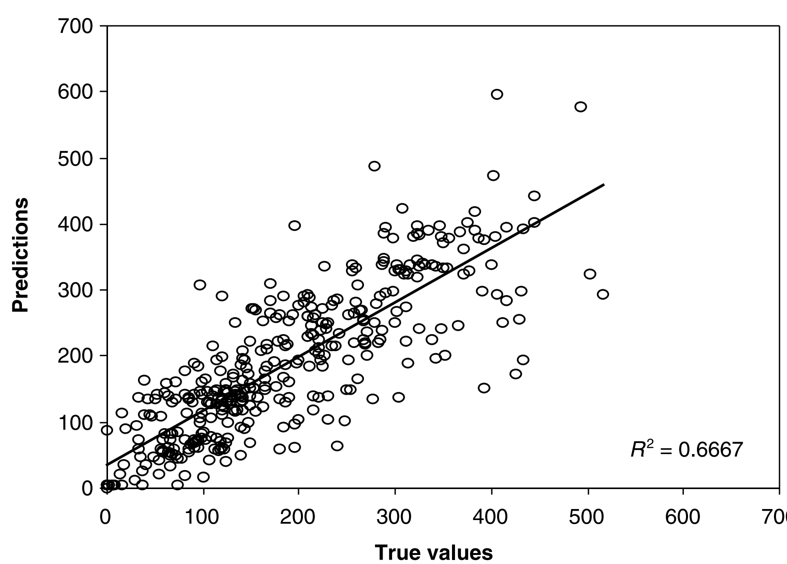

Figure 2 Scatterplot of actual values and DFFE predictions

Table 1 Comparison of the statistics of the 367 data

	True values	Estimates DFFE
Minimum	0	0.9
Maximum	517	595.6
Mean	185.4	188.0
Median	162.0	163.1
Variance	111.0	111.1

Table 2 Comparison of the statistics of the 10 extreme true and estimated values

	Lowest		Highest	
	True values	Estimates	True values	Estimates
Minimum	0.0	0.9	426.0	173.7
Maximum	13.0	86.4	517.0	575.9
Mean	3.3	14.3	455.7	335.3
Median	0.5	5.0	439.0	311.3
Variance	4.3	24.5	21.3	115.1

Table 1 shows that the distribution is quite good, but Figure 2 shows that individual results can be quite poor. Generally, with spatial interpolation, we would like some confidence in the local estimates.

Table 2 provides the statistics of the 10 extreme values. The performance of the 10 lowest estimated values could be acceptable except for the presence of one data point, which had an estimated value of 86 but the true value was 0. This dramatically increases both the mean and variance of the estimation. The performance of the 10 largest estimated values was also not acceptable. The technique overestimated the maximum value and underestimated the minimum value. We will not analyse the subsequent (better) results in such detail.

3 IDWA

The inverse distance weighted averaging (IDWA) interpolation is a deterministic estimation technique that uses values at the known locations to estimate the values at the unknown points by using linear combination, as discussed in the introduction. Here we present the results using IDWA in Figure 3.

The slope is again close to 1, and the intercept is again close to 50. The results are similar, overall, except that clearly the scatter is less, as confirmed by the improved R^2 value.

Table 3 shows that: (1) the DFFE maximum predictions are unacceptably larger, (2) the IDWA estimates substantially compressed the range from the minimum to the maximum, which is consistent with the lower variance, and (3) the mean and median for the IDWA predictions are also both higher.

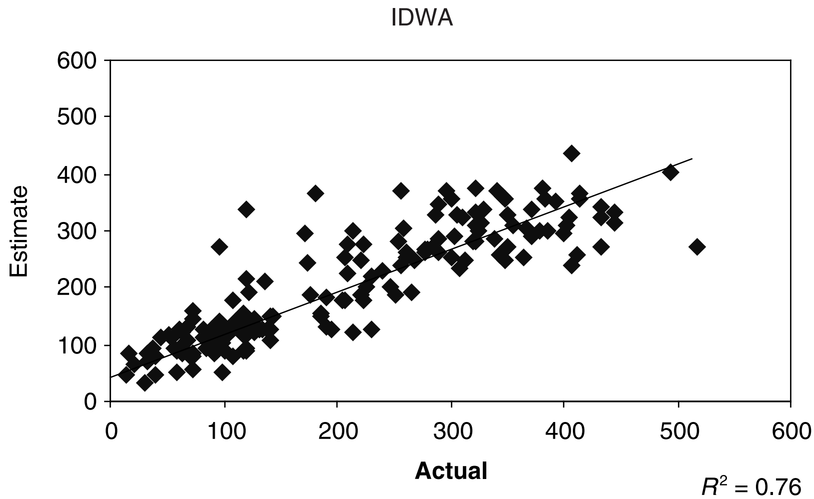

Figure 3 Scatterplot of actual values and IDWA predictions

Table 3 Comparison of the statistics

	True values	Estimates DFFE	Estimates IDW
Minimum	0	0.9	32.9
Maximum	517	595.6	437.1
Mean	185.4	188.0	199.4
Median	162.0	163.1	178.4
Variance	111.0	111.1	99.7

4 Fuzzy Conservative Spatial Interpolation

4.1 Self-organizing Map (SOM)

For local spatial interpolation, the first step is to classify the available data into different classes so that the data are split into homogeneous sub-populations. A SOM is used to divide the data into subpopulations and hopefully reduce the complexity of the whole data space to something more homogeneous. The objective in this step is to make use of an unsupervised learning algorithm to sub-divide the whole population. The SOM is selected for this purpose mainly because it is a fast, easy and reliable unsupervised clustering technique. Beside, the SOM has been successfully applied in spatial interpolation as presented in Wong et al. (1997).

The SOM was designed with the intention to simulate the spatial organizations found in various brain structures and has a close relationship to brain maps (Kohonen 1995). Its main feature is the ability to visualize high dimensional input spaces onto a smaller dimensional display, usually two-dimensional. In this discussion, only two-dimensional arrays will be of interest. Let the input data space $\Re^n$ be mapped by the SOM onto a two-dimensional array with i nodes. The array is usually square, with i chosen as the number of clusters or sub-clusters contained within the data modelled at the desired resolution. Where this is not known, some appropriate heuristic to estimate this must be used in an application dependent manner. Associated with each node i is a parametric reference vector $m_i = [\mu_{i1}, \mu_{i2}, \ldots, \mu_{in}]^T \in \Re^n$, where μ_{ij} is the connection weight between node i and input X_j. Therefore, the input data space $\Re^n$ consisting of input vector $X = [x_1, x_2, \ldots, x_n]^T$, i.e. $X \in \Re^n$, can be visualized as being connected to all nodes in parallel via a set of scalar weights μ_{ij}. The aim of the learning is to map all the n input vectors X_n onto m_i by adjusting weights μ_{ij} such that the SOM gives the best match response locations.

The SOM can also be said to be a nonlinear projection of the probability density function $p(X)$ of the high dimensional input vector space onto the two-dimensional display map. Normally, to find the best matching node i, the input vector X is compared to all reference vectors m_i by searching for the smallest Euclidean distance $\|X - m_i\|$, signified by c.

Therefore,

$$c = \min_i \{\|X - m_i\|\} \tag{6}$$

or

$$c = \|X - m_c\| = \min_i \{\|X - m_i\|\} \tag{7}$$

During the learning process the node that best matches the input vector X is allowed to learn. Those nodes that are close to the node up to a certain distance will also be allowed to learn. The learning process is expressed as:

$$m_i(t+1) = m_i(t) + h_{ci}(t)[X(t) - m_i(t)] \tag{8}$$

where t is the discrete time coordinate and $h_{ci}(t)$ is the neighbourhood function.

After the learning process has converged, the map will display the probability density function $p(X)$ that best describes all the input vectors. At the end of the learning process, an average quantisation error of the map will be generated to indicate how well the map matches the entire input vectors X_n. The average quantisation error is defined as:

$$E = \int \|X - m_c\|^2 p(X)\, dX \tag{9}$$

After the 2-dimensional map has been trained, the reference vectors that were used in the nodes of the map can also be obtained. In spatial interpolation, the reference vector will be the node center and consists of the input variables (x, y) and the output variable (z). As we like the clusters to be formed to facilitate the concept used in Euclidean interpolation, we propose here to construct the clustering boundaries based on the output reference vector of the nodes. This produces a series of overlapping polygons. The rule of thumb for deciding on the clustering boundaries is to examine the distance measure between the neighboring reference values. If the distance measure between the present reference node and the neighboring nodes is high, that suggests another cluster.

4.2 *Conservative Fuzzy Reasoning*

After the whole sample space has been subdivided into local homogeneous sub-populations, a spatial interpolation model based on the concept of conservative fuzzy reasoning is constructed for each sub-population. Before formulating the new approach, the characteristics of conservative fuzziness are discussed.

Conservative fuzzy rule interpolation is used to interpolate those input instances that cannot find any fuzzy rules to fire. This type of fuzzy rule base is known as a sparse rule base (Gedeon and Kóczy 1998, Gedeon and Kóczy 1997, Kóczy and Hirota 1993). By referring to Figure 4, the labels *r* and *s* indicate the spread of the observation and the rule antecedent, which represent their fuzziness. The labels r' and s' indicate the spread of the conclusion and the rule consequent. By observation, B2 is found to be fuzzier than A2, due to the shallower slope, that is, s' is clearly larger than *s*. The A*, A2 and the B*, B2 distances are not normalized, as the values of *u* and u' are used explicitly as appropriate.

The r' value can be calculated by:

$$r' = r\left(1 + \left|\frac{s'}{u'} - \frac{s}{u}\right|\right) \tag{10}$$

The interpolated fuzzy rules produced with this approach can maintain the local change in fuzziness in the rule base.

The fuzzy sets of the input spaces (x, y) and output value (z) are constructed based on the distance between them. We use the nearest four neighboring points of the known values as the spread of the fuzziness. In this case, we can form a trapezoidal membership with these four location points. For the unknown location, it is transformed as the reference (center) point of the membership function, as $(x_0, y_0, \text{ and } z_0)$. Those location points on the left and the right of the reference point for *x* are shown in Figure 5. The

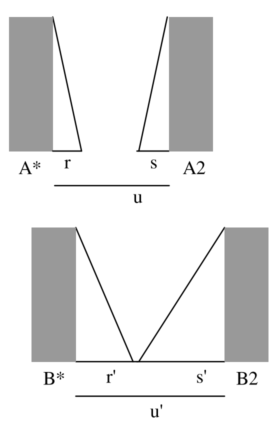

Figure 4 Parameters used in the conservative fuzzy interpolation

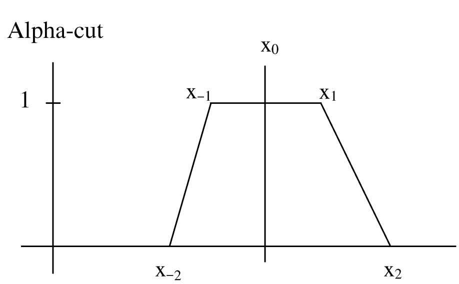

Figure 5 Notations used in the fuzzy sets

notations used for y and z are similar as those used in x. However, the sequence of their orders for y and z may be different.

For the left side of the reference point:

$$d_{(x_{-2},x_{-1})} = x_{-2} - x_{-1} \tag{11}$$

$$d_{(x_{-1},x_0)} = x_{-1} - x_0 \tag{12}$$

$$d_{(x_{-2},x_1)} = x_{-2} - x_1 \tag{13}$$

$$d_{(y_{-2},y_{-1})} = y_{-2} - y_{-1} \tag{14}$$

$$d_{(y_{-1},y_0)} = y_{-1} - y_0 \tag{15}$$

$$d_{(y_{-2},y_1)} = y_{-2} - y_1 \tag{16}$$

$$d_{inL2} = \sqrt{(d_{(x_{-2},x_{-1})})^2 + (d_{(y_{-2},y_{-1})})^2} \tag{17}$$

$$d_{inL1} = \sqrt{(d_{(x_{-1},x_0)})^2 + (d_{(y_{-1},y_0)})^2} \tag{18}$$

$$d_{inL} = \sqrt{(d_{(x_{-2},x_1)})^2 + (d_{(y_{-2},y_1)})^2} \tag{19}$$

$$d_{(z_{-2},z_{-1})} = z_{-2} - z_{-1} \tag{20}$$

$$d_{(z_{-2},z_1)} = z_{-2} - z_1 \tag{21}$$

The reference point can be calculated as:

$$z_{0L} = z_{-1} + d_{inL1}\left(1 + \left|\frac{d_{(z_{-2},z_{-1})}}{d_{(z_{-2},z_1)}} - \frac{d_{inL2}}{d_{inL}}\right|\right) \tag{22}$$

As for the right side of the reference point, similar calculations are used. The reference point can be calculated as:

$$z_{0R} = z_1 - \left[d_{inR1}\left(1 + \left|\frac{d_{(z_2,z_1)}}{d_{(z_{-1},z_2)}} - \frac{d_{inR2}}{d_{inR}}\right|\right)\right] \tag{23}$$

Therefore the interpolated value at the unknown location can be obtained by taking the aggregation of the two:

$$z_0 = \frac{z_{0L} + z_{0R}}{2} \tag{24}$$

The condition for this spatial interpolation is all the data has to be normalized to be within the same range. For example if x is normalized between 0 and 100, y and z will also need to be normalized between 0 and 100. This spatial interpolation technique will only work well with linear functions, therefore it will only work in the local interpolation domain.

4.3 Illustrative Example and Results

The same data set – 467 daily rainfall measurements made in Switzerland on the 8th May 1986 that was introduced in Section 3 – is used here. The 100 training data points are input into the SOM for unsupervised clustering. In that case study, the authors used a

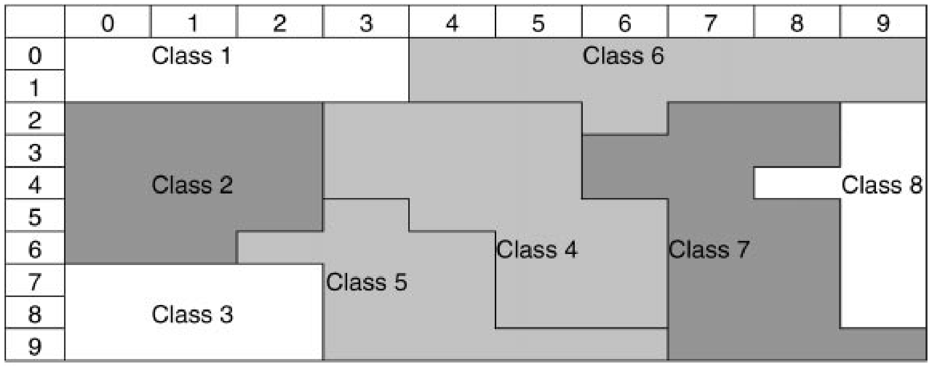

Figure 6 The cluster boundaries for the SOM 2-dimensional map based on output

10 by 10 two-dimensional map. After performing the cluster boundaries determination, the classes were formed, as shown in Figure 5.

To illustrate the conservative fuzzy reasoning technique, a point from that study is presented as follows. The unknown point (all values normalized to be between 0 to 100):

$$
\begin{array}{lll}
x_{-2} = 29.29 & y_{-2} = 89.06 & z_{-2} = 16.22 \\
x_{-1} = 33 & y_{-1} = 78.75 & z_{-1} = 18.11 \\
x_0 = 33.62 & y_0 = 92.54 & z_0 = ?? \\
x_1 = 41 & y_1 = 79.97 & z_1 = 24.59 \\
x_2 = 42.58 & y_2 = 65.05 & z_2 = 27.3
\end{array}
$$

The actual z_0 measured at the location that needs to be interpolated is 23.24. The spatial interpolated z_0 calculated from this conservative spatial interpolation technique is 20.12. The error between the actual and the interpolated z_0 is considered acceptably small, as the difference is 3.12.

The scatterplot of results is shown in Figure 7. Note that for this technique we eliminated points for which interpolation was not possible. That is, when there are only

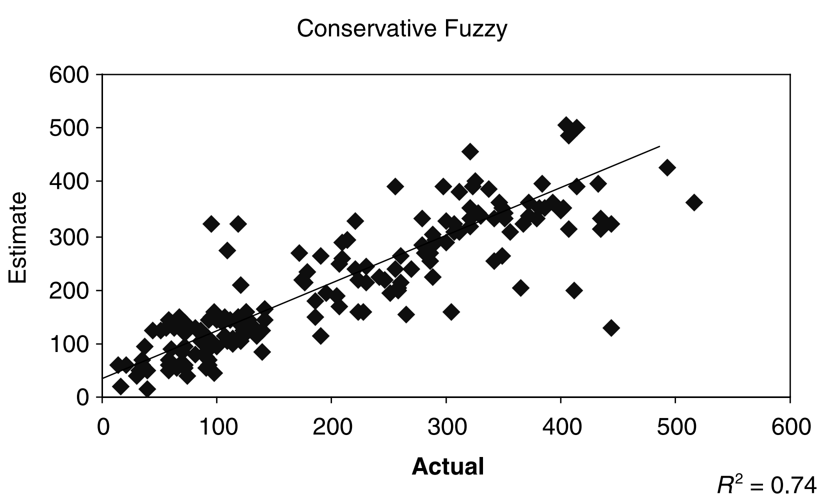

Figure 7 Scatterplot of actual values and Conservative Fuzzy predictions

Table 4 Complete comparison of the statistics

	True values	Estimates DFFE	Estimates IDWA	Estimates Cons. FZ
Minimum	0	0.9	32.9	16.6
Maximum	517	595.6	437.1	506.8
Mean	185.4	188.0	199.4	205.6
Median	162.0	163.1	178.4	163.6
Variance	111.0	111.1	99.7	114.1

left or right points, close to the edge of a region. This is reasonable, as in such regions we would use the best available alternative technique.

The slope is 1, and the intercept is slightly lower than 50. The results are very similar to IDWA, except that clearly the scatter is slightly more, as confirmed by the slightly reduced R^2 value. This is due to some degree of non-linearity in the data.

Clearly from Table 4, the conservative fuzzy method compresses the range from minimum to maximum and, as we expected, we cannot interpolate edge points. The mean value is the highest, yet the median remains acceptably close, unlike IDWA.

The conservative fuzzy spatial interpolation measure is clearly the best based on Table 4 and by comparison of the scatterplots. That is, on the scatterplots it is almost as good as the IDWA measure and thereby providing confidence for local estimates, and better reproduces the general distribution as shown by the statistics.

Our intentions for improvement of our technique are:

1. To incorporate features of nonlinear fuzzy interpolation, as our method at present works with linear interpolation characteristics;
2. To use fuzzy clustering to separate the data so that we may be able to reduce the number of data points that only have one side to interpolate; and
3. A weighted average function to be introduced to incorporate more neighbourhood points for interpolation.

5 Conclusions

This paper has presented a simple and computationally efficient spatial interpolation technique that could be used in real time for interactive data visualisation. The main feature of this spatial interpolation technique is inherited from the concept used in the conservative fuzzy interpolation reasoning for interpolating fuzzy rules in sparse fuzzy rule bases. This paper has also suggested that this technique works well in the local spatial interpolation domain. A self-organising map is used to divide the data into subpopulations and hopefully to reduce the complexity of the whole data space to something more homogeneous. This is the condition for which the conservative spatial interpolation technique could work best. Several examples have also shown that this technique can produce reasonable results.

We also presented the use of a dynamic fuzzy-reasoning-based estimator in an extension of a case study predicting rainfall data from Switzerland. The results of this study show that the dynamic fuzzy-reasoning-based function estimator is suitable for estimating the overall statistics of the predictions, but not the extreme values. The method, however, has many advantages. It is an assumption-free, model-free, and exact estimator and is suitable for handling multi-dimensional inputs and outputs.

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