On a stable and always applicable interpolation method[1]

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Overview

1 Sparse rule bases and fuzzy interpolation

1.1 Sparse rule bases

- Classical methods work on α-covered type rule bases (CRI, Mamdani, Larsen, Takagi-Sugeno)
- Complexity problem: number of rules grows exponentially in terms of the input variables:
$$|R| \leq T^k \qquad \text{where}$$
  - k is the number of variables;
  - $T = \max_{i=1}^{k} T_i$, and T_i is the number of terms α-covering the input space X_i.
- A possible solution: reducing the number of terms T_i in each dimension. Due to the reduction, the α-covering property is often lost. The resulting rule base can be α-sparse or sparse.
- Classical algorithms based on some rule and observation matching for determining the conclusion can not be applied for sparse rule bases. Hence, a conceptually new technique is required.

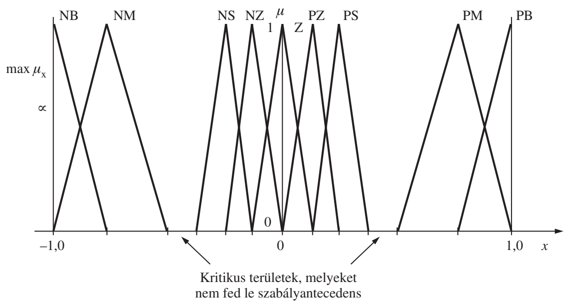

Figure 1: Sparse rule base after tuning

1.2 Fuzzy interpolation

- First results (Kóczy and Hirota, [1, 2, 3])
- Conditions for application
  - Ordering among fuzzy sets in each input dimension X_i [3]
  - The notion of distance among comparable fuzzy sets can be introduced (family of distances):

$$\mathcal{D}(A_1, A_2) = \{d_\alpha(A_1, A_2), \alpha \in (0,1]\} = \{\|A_{1\alpha,L} - A_{2\alpha,L}\|, \|A_{1\alpha,U} - A_{2\alpha,U}\|\}$$

  - The methods are applicable for convex and normal fuzzy (CNF) sets
  - Location of the involved sets (A_1, A_2: antecedents, A^* observation, B_1, B_2 consequents):

$$A_1 \prec A^* \prec A_2 \qquad B_1 \prec B_2 \qquad (\prec \text{ is the ordering})$$

- Conclusion is calculated by the Fundamental Equation of Rule Interpolation (FERI)

$$\mathcal{D}(A^*, A_1) : \mathcal{D}(A^*, A_2) = \mathcal{D}(B^*, B_1) : \mathcal{D}(B^*, B_2)$$

- Simplest version of these method is the linear KH-interpolation (two flanking rules are considered):

$$B^*_{\alpha,C} = \frac{\sum_{i=1}^{2} \frac{1}{d(A^*_{\alpha,C}, A_{i\alpha,C})} B_{i\alpha,C}}{\sum_{i=1}^{2} \frac{1}{d(A^*_{\alpha,C}, A_{i\alpha,C})}} \qquad C \in \{L, U\}$$

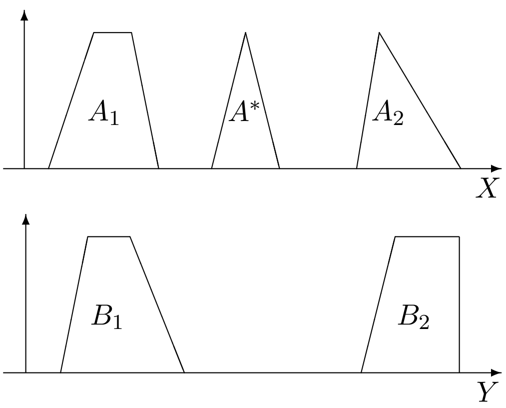

Figure 2: Location of involved sets

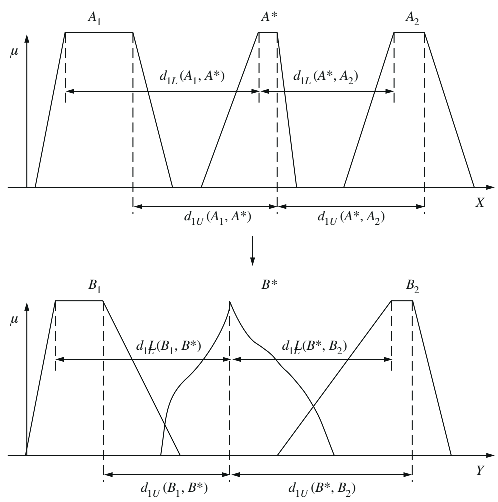

Figure 3: Calculation of the conclusion point-by-point

2 Pros and cons for the linear KH-interpolation

Advantages

- For piecewise linear shaped membership function, the conclusion can be calculate only for the breakpoint (or level) set of the input sets [5]. At the area in between, the slopes of the conclusion can be approximated linearly. Due to this, the time complexity of the method reduces [4].
- Approximation capabilities: the generalized KH interpolation can approximate any continuous function with respect to the supremum and the L_p norm $p \in [1, \infty]$ on a compacta ([6]), see also Section 4.

Disadvantages

- In some location of the input set the conclusion is abnormal (not directly interpretable fuzzy set, see Figures 4 and 5). Several authors contributed in finding
  - either condition for the input sets or conceptually (e.g. Kóczy and Kovács [5]; Shi and Mizumoto [8, 9])
  - or conceptually different methods (e.g., Vass, Kalmár and Kóczy, 1992, [7]; Baranyi and Gedeon, 1996, [10]; Kóczy, Hirota and Gedeon, 1997, [11])

  Common problem: the advantageous computational property could not be preserved.
- It is applicable only for CNF sets.

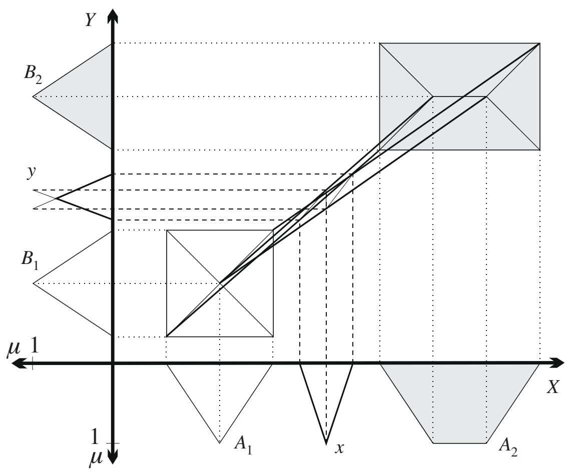

Figure 4: An example for collapsed slopes

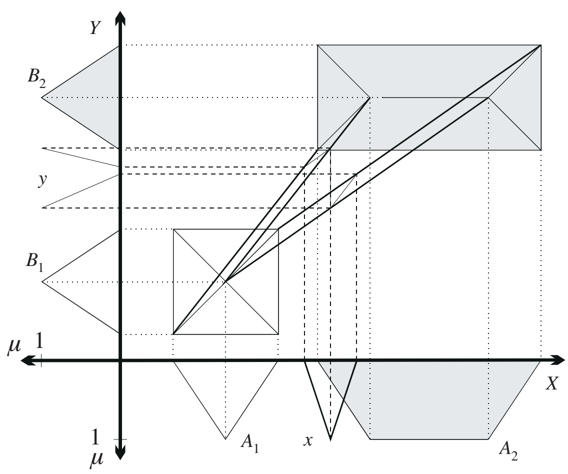

Figure 5: An example when the slopes are even not connected

3 The MACI algorithm

- Termed as Modified Alpha Cut based Interpolation (MACI) algorithm
- Representation: vector description of fuzzy sets due to (Yam and Kóczy [12], see Figure 6)
- Main idea:

  1. Transform the coordinate system in such a way then in the new system CNF conclusions can be guaranteed.

  2. Calculate the conclusion in the new coordinate system.

  3. Then transform the conclusion back to the original coordinate system, i.e., describe the representing vector's coordinate with respect to the original coordinate system.
- Remark: the method can be generalized for arbitrary finite number of characteristic points.

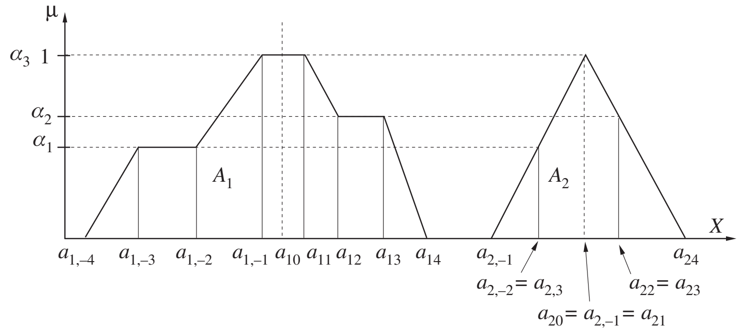

Figure 6: Determination of characteristic points

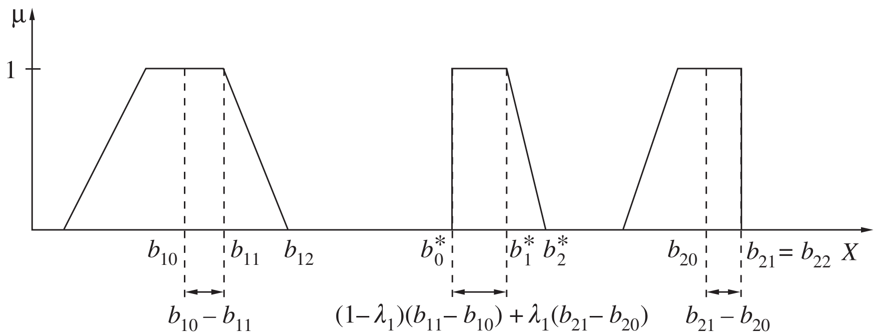

Figure 7: Coordinates according to the MACI method

Theorem 1 *The algorithm outlined so far always gives convex and normal conclusion, i.e., the method is closed in the space of CNF sets.*

Remark: the calculation proceeds separately for the two flanks of the conclusion starting from the reference point. The characteristic points of the right flank monotone increase, the ones of the left flanks monotone decrease.

4 Stability of the KH method

Let us consider the generalized version of the KH interpolation, when not two, but more flanking rule antecedents and their corresponding consequents are taken into consideration in the calculation of the conclusion:

$$B^*_{\alpha,C} = \frac{\sum_{i=1}^m \frac{1}{d(A^*_{\alpha,C}, A_{i\alpha,C})} B_{i\alpha,C}}{\sum_{i=1}^m \frac{1}{d(A^*_{\alpha,C}, A_{i\alpha,C})}}.$$

Definition 1 *Let* $\mathbf{R}^N \supset \Omega = [a_1, b_1] \times \cdots \times [a_N, b_N]$*, further let* $\{\Gamma_n\}_{n=1}^\infty$ *be a sequence of finite subsets of* Ω *with* $\#\Gamma_n = n$*. If*

$$\forall \varepsilon > 0 \ \exists n_0 \ \forall \omega \in \Omega \ \forall n \geq n_0 : \left| \frac{\#(\Gamma_n \cap \omega)}{\#\Gamma_n} - \frac{|\omega|}{|\Omega|} \right| < \varepsilon \qquad (1)$$

then the set Γ_n *are uniformly distributed on the domain* Ω*. Here* $\#(\Gamma_n \cap \omega)$ *denotes the cardinality of the finite set* $(\Gamma_n \cap \omega)$ *and* $|\omega|$ *is the Lebesque-measure of* ω*.*

Theorem 2 *Consider the* L_p *norm* $\| \cdot \|_p$ *with* $p \in [1, \infty]$*, the domain* $\mathbf{R}^N \supset \Omega = [a_1, b_1] \times \cdots \times [a_N, b_N]$ *and a continuous function* $f : \Omega \to \mathbf{R}$*, then for all* $x \in \Omega$ *the expression*

$$\lim_{n\to\infty} K_n(f, x) := \lim_{n\to\infty} \sum_{k=1}^n f\left(x_k^{(n)}\right) \frac{\frac{1}{\|x - x_k^{(n)}\|_p^N}}{\sum_{j=1}^n \frac{1}{\|x - x_j^{(n)}\|_p^N}} \qquad (2)$$

is equal to $f(x)$*, where measurement points* $x_k^{(n)}$ *are uniformly distributed on* Ω *in the sense of* (1)*.*

- Interpretation 1 (fuzzy context): If the antecedents or the observation of a rule base changes slightly it yields not significant change in the conclusion, this is a kind of mathematical stability.
- Interpretation 2 (real function approximation): stabilized KH controllers are universal approximators in the space of continuous function with respect to the L_p norm, $p \in [1, \infty]$.
- Interpretation 3 (fuzzy function approximation): If one approximate a fuzzy-to-fuzzy mappings by KH controllers, the theorem guarantees that for each $\alpha \in (0, 1]$ the approximation converges to the appropriate α-cut of the mapping.

5 Stability of the MACI method

The new method can be generalized by taking more flanking consequents into account for the calculation of the conclusion, as well. Analogously, as in the case of the original KH method, farther an antecedent of a rule is from the observation less significant role plays the corresponding consequent in the determination of the conclusion. Hence, each coordinate values are weighted with the reciprocal of the distance of the appropriate characteristic points

$$b_k^* = \frac{\sum_{i=1}^m \frac{1}{d(A_0^*, A_{i0})} b_{i0}}{\sum_{i=1}^m \frac{1}{d(A_0^*, A_{i0})}} + \sum_{j=1}^k \frac{\sum_{i=1}^m \frac{1}{d(A_j^*, A_{ij})} (b_{ij} - b_{i,j-1})}{\sum_{i=1}^m \frac{1}{d(A_i^*, A_{ij})}}$$

Theorem 3 *The modified α-cut based interpolation method is stable in the sense that for every coordinate slight change in the coordinates involved in the calculation modifies slightly the appropriate characteristic point of the conclusion, i.e., supposing uniform distribution of the antecedents* (1)

$$\lim_{n\to\infty} \left(B^{(k)}\right)_n \left(\sum_{j=1}^k f^j, x\right) = \sum_{j=1}^k \sum_{i=1}^n f^j\left(x_{ij}^{(n)}\right) \frac{\dfrac{1}{\|x - x_{ij}^{(n)}\|_p^m}}{\sum\limits_{\ell=1}^n \dfrac{1}{\|x - x_{\ell j}^{(n)}\|_p^m}}$$

where f^k $(k = 0, \ldots, N)$ *are continuous functions on domain* $\mathbf{R}^N \supset \Omega = [a_1, b_1] \times \cdots \times [a_m, b_m]$ *and* $x_{ij}^{(n)}$ *are the* j*th characteristic point of the* i*th antecedent when, in total,* n *rule are taken into consideration.*

For proving this theorem we use the following

Lemma 1 *If* F *and* G *are stable interpolatory operator in the sense of* Theorem 2 *with uniform distribution* (1) *then* $F \pm G$ *is also stable in the same sense.*

Hence, the MACI method is also stable in mathematical sense, and can be considered as universal approximator in the space of continuous function on a compacta with respect to the L_p $(p \in [1, \infty])$ norm.

6 Conclusion

We have shown that the stable behaviour of the original KH interpolation can be carried over for the modified interpolation technique. It means that the modified method shares the advantageous properties of the original method, such as low computational complexity and stability, while it solves the problem of abnormal conclusion, as well.

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