

A Generalized Concept for Fuzzy Rule Interpolation

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Abstract—The concept of *fuzzy rule interpolation* in sparse rule bases was introduced in 1993. It has become a widely researched topic in recent years because of its unique merits in the topic of *fuzzy rule base complexity reduction*. The first implemented technique of fuzzy rule interpolation was termed as α -cut distance based fuzzy rule base interpolation. Despite its advantageous properties in various approximation aspects and in complexity reduction, it was shown that it has some essential deficiencies, for instance, it does not always result in immediately interpretable fuzzy membership functions. This fact inspired researchers to develop various kinds of fuzzy rule interpolation techniques in order to alleviate these deficiencies. This paper is an attempt into this direction. It proposes an interpolation methodology, whose key idea is based on the interpolation of relations instead of interpolating α -cut distances, and which offers a way to derive a family of interpolation methods capable of eliminating some typical deficiencies of fuzzy rule interpolation techniques. The proposed concept of interpolating relations is elaborated here using fuzzy- and semantic-relations. This paper presents numerical examples, in comparison with former approaches, to show the effectiveness of the proposed interpolation methodology.

Index Terms—Fuzzy rule interpolation, sparse fuzzy rule-base.

I. INTRODUCTION

THE concept of interpolating in sparse rule bases, termed as *fuzzy rule interpolation*, and its first implementation, termed as α -cut distance based fuzzy rule base interpolation (α -cut interpolation shortly), were introduced in 1990/1991 [26]–[30]. Despite the advantages of *fuzzy rule interpolation* in different issues of fuzzy theory shown in a number of articles, it was also proved that the α -cut interpolation does not accommodate some elementary conditions of fuzzy concept in cases. In this regard, conditions were investigated by Kovács [31]–[34], Kawase and Chen [25], and Shi and Mizumoto [47]. As a result, *three typical problems of interpolation* have come into the focus of the related literature, which are addressed as *abnormal conclusion*, *nonpreserving linearity*, *restriction to convex normalized fuzzy (CNF) sets*. *Abnormal conclusion*

means that the interpolation yields at least two membership values over at least one element of the output universe, or the resulted membership values are not bounded by $[0, 1]$. *Nonpreserving linearity* is addressed when not piece-wise linear conclusion is inferred from piece-wise linear rules and observations. *Restriction to CNF sets* means the interpolation does not function with arbitrary type of fuzzy sets, but with CNF sets. Inspired by the purpose of eliminating these typical deficiencies in certain cases, various interpolation methods were developed in the 1990s.

A. Brief Overview of Fuzzy Rule Interpolation Techniques

The explicit form of α -cut interpolation, called the *fundamental equation* of fuzzy rule interpolation, is actually a fuzzy extension of the classical linear interpolation of given points. The α -cut interpolation method infers a conclusion based on carrying the proportion of the fuzzy distances [30] between the observation and the rule antecedents over the corresponding consequents and the conclusion [26]–[29]. The fuzzy distance utilized in the α -cut interpolation reflects to some extent of the ideas of *approximate analogical reasoning* proposed by Turksen in 1988 [53], and is actually a family of α -cut distances. This is the reason why we call the first interpolation method α -cut distance based fuzzy rule interpolation [48], [61]. According to the definition of the fuzzy distance, a class of methods can be derived in the fashion of α -cut interpolation. For instance, different definitions of distance were proposed by Vass *et al.* in [58] and [18]. It is remarkable that the method proposed in [18] eliminated the problem of *abnormal conclusion*, however, it did not function with certain crisp fuzzy sets. Its improved version was proposed in 1997. These techniques assume that the fuzzy premises and consequents are CNF sets. Let us group these approaches as α -cut based methods. As a matter of fact, in the case of arbitrary shaped CNF sets theoretically an infinite number of α -levels should be taken into account, in order, to yield a proper conclusion. To achieve an acceptable computational requirement for practical cases one may restrict the computation to a finite number of α -levels (usually three or four), after assuming that the fuzzy sets applied are piecewise linear, for instance triangular or trapezoidal shaped. Unfortunately, the aforementioned methods do not preserve linearity, which simply means that calculating the piecewise linear sets only at certain α -levels and connecting the resulting points of the conclusion by linear pieces yields an approximation of the accurate conclusion. The deviation of the piecewise linear approximation from the accurate conclusion is, however, dispensable in the case of α -cut interpolation, as pointed out in [48] and [31]–[34]. Another method proposed by Dubois and Prade in [15]–[17] operated with all possible distances among the elements of fuzzy sets at each α -level and computed all corresponding elements

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of the conclusion for the same α -level. The membership function of the conclusion is obtained here by bounding the resulting elements at each α -level. In contrast with the aforementioned methods, Dubois and Prade's method was the first one, which could be applied to rule-bases, which were not restricted to CNF sets, as stated in [19]. As opposed to this advantage, Dubois and Prade's method might yield *abnormal conclusion* in certain cases [19]. In view of its essence, this approach is also included in the group of α -cut based methods. In order to keep the simplicity of α -cut interpolation, but to eliminate the problem of *abnormal conclusion*, papers [7] and [8] propose the transformation of the fundamental equation of α -cut interpolation to the space of normal conclusions. This method was termed *modified α -cut interpolation*. Shortly afterwards, Tikk *et al.* analyzed its various properties in [48]. It was shown that it also did not preserve linearity, but the deviation of the piecewise linear conclusion from the accurate one was less than in the case of α -cut interpolation. Tikk *et al.* also showed that the modified method inherited the approximation stability of the α -cut interpolation [49], [50]. The modified α -cut method was extended to nonconvex fuzzy sets by Tikk *et al.* in [52]. One of the recent methods of this narrow topic uses the combination of different interpolation techniques proposed by Wong *et al.* [60], [61].

In 1995, a conceptually different method was introduced by the authors [1]–[6]. It was termed “*solid cutting method*” and “*generalized interpolation method*.” Its essential difference from the former approaches is that this method infers the conclusion based on the interpolation of relations instead of α -cut distances. It has two main steps. In the first step a rule is interpolated from the rule-base as “close” to the observation as possible, based on a spatial solid cutting technique. The term “close” means here that at least partial overlapping is ensured between the observation and the interpolated rule, which implies the firing of the interpolated rule. In the second step, the conclusion is inferred from the consequent of the fired rule according to the similarity between the observation and the interpolated antecedent. The advantage of this method is that it is applicable to arbitrary shaped sets and does not yield abnormal conclusion. This method can readily be extended to fuzzy rule extrapolation as detailed in [5]. The drawback of this method is its high computational complexity. Some practical simplifications of this method have been done for piece-wise linear fuzzy sets [40]–[42]. These simplified methods preserve linearity. In 1997, the authors replaced the fuzzy interpolation in the first step by the interpolation algorithm of semantic relations [6]. Simultaneously, Kawaguchi *et al.* proposed a B-spline technique based fuzzy interpolation method in [21]–[24], which could be viewed as a kind of generalization of the first step. Kawaguchi's method functions with fuzzy sets given by a finite number of characteristic points. Let these algorithms be included in a group named *generalized methods*.

In 1997, Yam introduced a vector based approach to represent membership functions as points in high-dimensional Cartesian space [54]–[57]. This method transforms the ideas of fuzzy rule interpolation to the interpolation of vector mapping. A recent variation was proposed in [54], which we include in the group of *generalized methods* since it is constructed by the two steps of the generalized concept in terms of matrix operations. However,

it is restricted to sets given by a finite number of characteristic points.

Various further techniques have been proposed in the last years. For instance Bouchon-Meunier introduced the graduality based interpolative reasoning [9]–[11]. Kóczy and Hirota and others published results about the use of α -cut interpolation in hierarchically structured rule-bases [35], [44]. Mizik *et al.* compared various interpolation techniques in a uniform description [40]–[42]. In 1996, Kovács *et al.* proposed an interpolation technique based on the approximation of the vague environment of fuzzy rules and applied it in the control of an automatic guided vehicle system [36]–[38]. Jenei introduced an axiomatic treatment of linear interpolation and extrapolation as a new way of interpolation of compact fuzzy quantities and proposed its multi-dimensional extension in [19], [20]. He also investigated various properties of interpolation techniques. Bouchon-Meunier proposed a comparative view of fuzzy interpolation methods in [12].

B. Aim of this Paper

The aim of this paper is to introduce a methodology for fuzzy rule interpolation. This methodology has already been partially initialized by the authors in [1]–[6]. Further, this interpolation methodology is capable of eliminating typical deficiencies of fuzzy interpolation methods. By the help of the proposed methodology a class of linear and nonlinear fuzzy interpolation methods can be developed. The key idea of this interpolation methodology is based on the interpolation of relations. As an implementation, two groups of algorithms are developed in this paper. One is based on the interpolation of *fuzzy relation*. The other is based on the interpolation of *semantic relation*. The comparison of the resulting interpolation methods to the former techniques is given in this paper. Various further comparison have already been published. Detailed comparisons, by Mizik [40]–[42], Tikk *et al.* [48] and by the authors, analyze the α -cut, the modified α -cut interpolation and solid cutting method in a uniform coordinate system. Further, [19] compares the α -cut based interpolation, Dubois and Prade's technique and the *solid cutting method*. Consequently, all the results of these comparisons, done for the preliminary works such as *solid cutting method* and the *revision principle based technique*, can straightforwardly be carried over the interpolation methodology which will be discussed in this paper. Some, numerical examples are investigated both in this paper and in [1] and [2].

II. DEFINITIONS AND NOTATION

This section introduces some elementary definitions and concepts utilized in the further developments. Before starting with the definitions, some comments are enumerated on the notation. To facilitate the distinction between the types of given quantities, they will be reflected by their representation: scalar values are denoted by lower-case letters $\{x, y, \dots\}$; fuzzy sets by capital letters as $\{A, B, \dots\}$; and letter R is reserved to denote fuzzy rule IF A , THEN B , briefly $R : A \mapsto B$. Letters X, Y , and S are, respectively, reserved to input–output universes and to the third dimension of geometrical representation, see later. In order to enhance the overall readability characters $i, j, k \dots$

are in the meaning of subscripts (counters), and I, J, K, \dots are reserved to denote the respective upper bounds of the subscripts, unless stated otherwise. Further, K is assigned to the index of fuzzy rules. The membership functions of the fuzzy sets used through this paper are continuous with bounded support. Antecedent fuzzy sets are denoted by $A \in F(X)$; consequent sets by $B \in F(Y)$. Notations A^* and B^* are for the observation and conclusion, respectively. Superscript i indicates that the given quantity being “interpolated”. Notations \underline{x}, \bar{x} simply mean the lower and upper bound of x . $\lambda \in [0, 1]$ is an interpolation parameter.

Definition 1 (Lower and Upper Bound of Fuzzy Set A): Given fuzzy set $A \in F(X)$. The lower and upper bounds of A in X are given by $\underline{A} = \text{support}(A)$ and $\bar{A} = \text{support}(A)$.

Definition 2 (Centre Point of Fuzzy Set A , $\text{cp}(A)$): The center point of a given fuzzy set $A \in F(X)$ is: $\text{cp}(A) = (\underline{A}_\alpha + \bar{A}_\alpha)/2$, where $\alpha = \text{height}(A)$. A_α denotes the α -cut of A .

Definition 3 (Normal Fuzzy Set): A fuzzy set A is normal if it has at least one element whose membership value is one.

Definition 4 (Subnormal Fuzzy Set): A fuzzy set A is subnormal if it does not have any elements whose membership value is one.

The next Definition 5 is used to describe the relation between the elements of two universes. Its idea is actually taken from the so called *revision principle*, where the relation is defined as the *revision function* or *interrelation function* (see [13], [14], [43], [45], and [46]). In this paper, a piecewise linear variant is utilized.

Definition 5 (Piecewise Linear Interrelation Function): $y = \Lambda(x, \mathbf{p}_1, \mathbf{p}_2)$, where $x \in [\underline{x}, \bar{x}]$, $y \in [\underline{y}, \bar{y}]$, $\mathbf{p}_1 = [p_{1,1} \ p_{1,2} \ \dots \ p_{1,M}] \in \mathfrak{R}^M$, where $p_{1,1} = a$ and $p_{1,M} = b$, and $\mathbf{p}_2 = [p_{2,1} \ p_{2,2} \ \dots \ p_{2,M}] \in \mathfrak{R}^M$, where $p_{2,1} = c$ and $p_{2,M} = d$, subject to $p_{i,m} \leq p_{i,m+1}$, $i = 1, 2$. The interrelation function is a piecewise linear function where the linear pieces are defined by point-pairs $(p_{1,m}, p_{2,m})$. Fig. 1 depicts an interrelation function, where $M = 4$, namely, the vector \mathbf{p}_i consists of four elements.

The interrelation function is used to assign the nonzero membership valued elements of sets A and B of a rule $R : A \mapsto B$.

Definition 6 (Interrelation Area): The interrelation area of interrelation function is a rectangular area defined by points which the interrelation function ends in, namely, by points (a, c) and (b, d) ; see Figs. 1, 15, and 16. This can also be implemented as the area defined by the support of antecedent A and consequent B of a rule $R : A \mapsto B$.

Definition 7 (Linear Interpolation of Two Points): Function $x^i = \Gamma(x_1, x_2, \lambda) = (1 - \lambda)x_1 + \lambda x_2$, $\lambda \in [0, 1]$, is the linear interpolation between given x_1 and x_2 (superscript i denotes “interpolated”).

III. KEY IDEA OF THE GENERAL FUZZY INTERPOLATION METHODOLOGY

This section is intended to introduce the fundamental concept of the generalized method. To capture the main idea, first interpolation in a one variable rule-base is discussed in

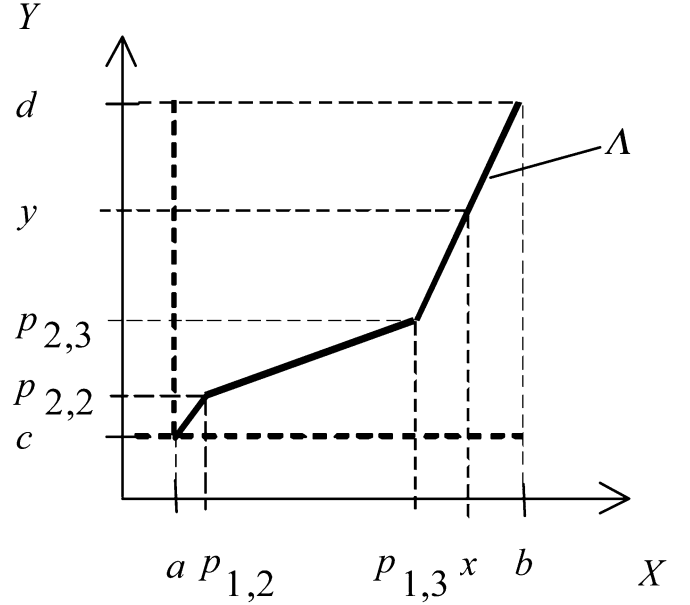


Fig. 1. Linear interrelation function $y = \Lambda(x, \mathbf{p}_1, \mathbf{p}_2)$.

Sections IV–VII. Multidimensional extension is treated in Sections VIII and IX. The discussion of this paper is restricted to the elementary step of interpolation, namely, to the interpolation between two rules selected from the rule-base. In order to avoid overlapping with preliminary papers, the way of selecting two rules is not in focus here. For the sake of simplicity, let us assume that the rule selection is done by the selection technique proposed for the α -cut interpolation. In order to initialize further discussion, let the following assumptions and statements be recalled from [29], [30], [48], and [54]. The variables, including input universe X and output universe Y are bounded and gradual in the sense of [15]. So, a linear ordering in each of them exists. In this case, a partial ordering between the elements of X and Y . Having this partial ordering between $A_1, A_2 \in F(X)$, denoted by \prec , if A_1 and A_2 are comparable, i.e., $A_1 \prec A_2$, it is possible to define a distance between these two fuzzy sets that will be denoted by $d(A_1, A_2)$. Assume that an observation $A^* : \mu_{A^*}(x) \in F(X)$ is given, to which the conclusion $B^* : \mu_{B^*}(y) \in F(Y)$ is searched for. Further, assume that two fuzzy rules are selected so that

$$\begin{aligned} R_k : A_k \mapsto B_k, \quad k = 1, 2 \quad \text{and} \\ A_1 \prec A^* \prec A_2 \quad \text{and} \quad B_1 \prec B_2 \end{aligned}$$

where $A_k : \langle x, \mu_{A_k}(x) \rangle$ and $B_k : \langle y, \mu_{B_k}(y) \rangle$.

In the following, the main steps of the proposed method are presented.

A. Generalized Method for Fuzzy Rule Interpolation

1) *Generation of an Interpolated Firing Rule:* In this step, an interpolated rule $R^i : A^i \mapsto B^i$ is generated, which is located between R_1 and R_2 , in such a way that A^i is as “close” to A^* as possible, but in any case it has at least partial overlapping with A^* . For brevity, let this step be denoted by

$$R^i = f^{\text{Interpolation}}(R_1, R_2) \quad (1)$$

where $f^{\text{Interpolation}}$ is a mapping from pairs of X - Y rules into the set of possible X - Y rules. $f^{\text{Interpolation}} : \mathfrak{R}^2 \rightarrow \mathfrak{R}, \mathfrak{R}\{R_i | A_i(x) \rightarrow B_i(y)\}$. As a matter of fact, this definition of “closeness” and the degree overlapping can be selected in a suitable way rather freely, but shall, be used later on consistently.

2) *Inference of the Conclusion*: Let the newly generated interpolated rule be considered temporarily as if it were one of the existing rules of the rule-base. The overlapping of the observation and the interpolated antecedent implies the firing of the interpolated rule. Let this step be denoted as

$$B^* = f^{\text{Inference}}(R^i, A^*).$$

Remark 1: Various algorithms introduced in the related literature can be substituted into the above steps and analyzed in regards to the three typical deficiencies of the fundamental versions of interpolation discussed in the first paragraph of the Introduction. For instance, Kawaguchi’s B-spline based rule interpolation is directly substitutable into the first step. Its immediate consequence is that the method obtained in this way inherits restriction to piecewise fuzzy sets given by a finite number of pieces. In the case of the second step, *single rule reasoning techniques* have prominent roles. For instance, the use of the *revision principle* introduced by Shen *et al.* are detailed in this paper.

B. Further Characterization

This section introduces further characterizations for the overall view of fuzzy interpolation. In [7], [8], [40]–[42], and [48], a technique is presented that is capable of comparing different interpolation methods in a uniform coordinate system. This technique implicitly determines a generable usable reference point for the fuzzy sets in the rules. Let a reference point of a fuzzy set be defined by Definition 8 as follows.

Definition 8 (Reference Point $\text{rp}(A)$): $\text{rp}(A)$ is a point of X assigned to fuzzy set A , so that $\text{rp}(A)$ expresses the “most typical” location of fuzzy set A . $\text{rp}(A) \in \text{supp}(A)$ in every case. It could be, for instance, some kind of defuzzified value of A .

The use of the idea of reference point helps with examining the global feature of the interpolations via simplified explicit forms. A global feature of the interpolation can, hence, be described by the function of the reference points of the inferred conclusions with respect to the reference points of the observations; see Figs. 9 and 10. Let this function be termed as follows.

Definition 9 (Interpolation Generatrix): Let the interpolation generatrix G be the function of the $\text{rp}(\text{observation})$ in respect to $\text{rp}(\text{conclusion})$, such that $G : X \rightarrow Y$ and $G(x_0) = y_0$, whenever $\text{rp}(A^*) = x_0, \text{rp}(B^*) = y_0$ and $R(A^*) = B^*, R$ being the rule base interpolation function.

For example, if fuzzy numbers are used in α -cut interpolation and $\text{rp}(A) = \text{cp}(A)$ is fixed for all sets, then the interpolation generatrix between two neighboring rules is a straight line showing the linear feature of the α -cut interpolation. In this case, the set of points defined by the $\alpha = 1$ cut of all the possible observations and conclusions equals the interpolation generatrix.

One of the aims of this paper is to propose various algorithms for the implementation of the interpolation method. Before

dealing with the algorithms in detail a brief digression needs to be taken here to define a concept of “closeness” by introducing a simple distance between two fuzzy sets that will be used later on in this paper. Let

$$d(A, B) = |\text{rp}(A) - \text{rp}(B)|$$

which is a crisp distance in contrast with α -cut distance-based methods. Without the loss of generality let in this paper the reference point be fixed to the center point, so that

$$\text{rp}(A) = \text{cp}(A) \quad (2)$$

is used from now on. Therefore, if $\text{cp}(A_1) \leq \text{cp}(A_2)$ then A_1 and A_2 will be comparable, i.e., we write $A_1 \prec A_2$. Let the interpolated rule be determined subject to

$$d(A^*, A^i) = |\text{cp}(A^*) - \text{cp}(A^i)| = 0 \quad (3)$$

which ensures an overlapping between A^* and A^i . Again the restrictions in (2) and (3), are not necessary for the interpolation method in general, it can be set in various ways rather freely. Equations (2) and (3) have been chosen for the implementations, detailed in the next sections, of the interpolation method.

IV. RULE INTERPOLATION

Two groups of algorithms are introduced in this section as possible implementations of the first step of the proposed interpolation method. The first group is based on *fuzzy relation interpolation*, the second one is based on *semantic relation interpolation*.

A. Fuzzy Relation Interpolation

The first algorithm in this group is the detailed version of the *solid cutting method* proposed by the authors in [1]–[5]. The second and the third algorithms respectively apply the *fixed point law* (FPL) and the *fixed value law* (FVL) theory, introduced by Shen *et al.* [13], [14], [43], [45], [46], to fuzzy set interpolation. The essential difference between FPL and FVL algorithms is that the FPL method considers the membership values, while the FVL operates with the “fuzziness” of the sets, see later. Fuzzy relation interpolation is computed here via interpolating the fuzzy sets

$$A^i = f^{\text{Interpolation}}(A_1, A_2, \lambda_a) \quad \text{and} \\ B^i = f^{\text{Interpolation}}(B_1, B_2, \lambda_b)$$

where $R^i : A^i \mapsto B^i$

where $\lambda_a, \lambda_b \in [0, 1]$. First, the fuzzy set interpolation techniques will be proposed and then the fuzzy relation interpolation algorithm will be introduced.

Algorithm 1 (Solid Cutting (SC) Fuzzy Set Interpolation):

$$A^i = f_{\text{SC}}^{\text{Interpolation}}(A_1, A_2, \lambda).$$

Let $g_k(s, x_k), s \in S, S \times X$ (dimension S is orthogonal on $\mu \times X$ on Figs. 2 and 3), $x_k = \text{cp}(A_k)$ be the function (see Figs. 2 and 3) that is obtained by rotating the membership function $A_k : \mu_{A_k}(x)$ by 90° around the axis t_k that is positioned at $\text{rp}(A_k) = \text{cp}(A_k) : g_k(x - x_k, x_k) = \mu_{A_k}(x)$. Let a solid

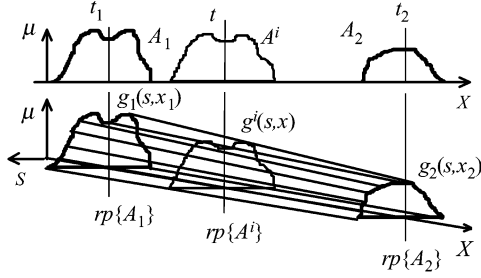


Fig. 2. Interpolating fuzzy sets by solid cutting.

be constructed by fitting a surface on generatrices $g_k(s, x_k)$. Let $g^i(s, x)$ be the cross-section of this imagined solid at position $x = \text{cp}(A^i)$, where $\text{cp}(A^i) = \Gamma(\text{cp}(A_1), \text{cp}(A_2), \lambda)$. Turning back $g^i(s, x)$ into its original position the interpolated fuzzy set $A^i : \mu_{A^i}(x) = g^i(x - \text{cp}(A^i), \text{cp}(A^i))$ is obtained.

Great variety of algorithms capable of fitting a surface to the generatrices $g_k(s, x_k)$ can be defined according to specific desired properties. Regarding the length of this paper only one algorithm is discussed here as a possible solution. Let such an algorithm be developed here which holds the following properties.

Property 1 (Compatibility With the Rule Base): The intersection of the solid at points $\text{cp}(A_1)$ and $\text{cp}(A_2)$ must, respectively, be equivalent to $g_1(s, \text{cp}(A_1))$ and $g_2(s, \text{cp}(A_2))$. This means that if $\lambda = 0$, then $A^i = A_1$ and if $\lambda = 1$, then $A^i = A_2$.

Property 2 (Avoiding Abnormal Fuzzy Set): The intersection of the solid at any points is a function and bounded by $[0, 1]$. This simply means that all intersections are interpretable as fuzzy sets.

Property 3 (Normalization): If A_1 and A_2 are normalized fuzzy sets (Definition 3) then the interpolated fuzzy set A^i is a normalized fuzzy set.

Property 4 (Preserving Linearity): If A_1 and A_2 are given by the same number of linear pieces then the interpolated set is also a piecewise linear set.

The surface of the imagined solid is created by simple conical and cylindrical line surfaces in the next part of this section. In order to facilitate further discussion first the key steps are illustrated on a simple example.

Step 1) Let the generatrices $g_k(s, x_k)$ be divided into pieces by characteristic points. These pieces will determine the bound of the conical and the cylindrical line surfaces. As an example, Fig. 3 shows that function $g_1(s, \text{cp}(A_1))$ and $g_2(s, \text{cp}(A_2))$ are divided by five and three points, respectively.

Step 2) Let the characteristic points be assigned between generatrices $g_k(s, x_k)$. Following the example on Fig. 3 let the characteristic points be assigned as $(p_{1,1}, p_{2,1}), (p_{1,2}, p_{2,2}), (p_{1,3}, p_{2,2}), (p_{1,4}, p_{2,2})$ and $(p_{1,5}, p_{2,3})$.

Step 3) Those characteristic points, which are assigned to one, determine a conical surface and other pairs of points determine the bound of a cylindrical surface. For example points $p_{1,2}, p_{1,3}$ and $p_{2,2}$ surround a conical surface. Similarly, points $p_{1,3}, p_{1,4}$ and $p_{2,2}$

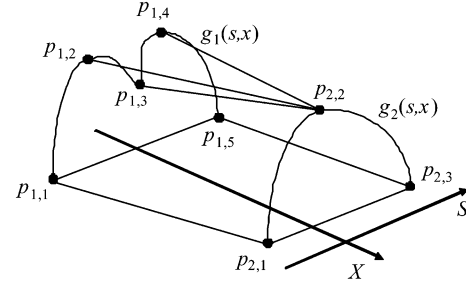


Fig. 3. Assigning the characteristic points.

also determine a conical surface. Cylindrical surfaces are bounded by points $p_{1,1}, p_{2,1}, p_{1,2}, p_{2,2}$ and $p_{1,4}, p_{2,2}, p_{1,5}, p_{2,3}$.

The following proposes possible solutions for the above three steps. Let the characteristic points, discussed in Step 1), be defined by the following conditions

- i) Let $p_{k,1}$ and p_{k,M_k} , namely, the first and the last characteristic points be those ones, which are corresponding to the lower and the upper bound of A_k . Therefore, $p_{k,1} = g_k(\underline{A}_k - \text{cp}(A_k), \text{cp}(A_k))$ and $p_{k,M_k} = g_k(\overline{A}_k - \text{cp}(A_k), \text{cp}(A_k))$.
- ii) Those elements of the generatrices, which correspond to the minimum and the maximum elements of $(A_k)_\alpha$, where $\alpha = \text{height}(A_k)$, are chosen to be characteristic points as: $p_{k,a} = g_k((\underline{A}_k)_\alpha - \text{cp}(A_k), \text{cp}(A_k))$ and $p_{k,b} = g_k((\overline{A}_k)_\alpha - \text{cp}(A_k), \text{cp}(A_k))$.
- iii) Let $\text{cp}(A_k)$ also be included among the characteristic points.
- iv) Let p_{k,m_k} and p_{k,m_k+1} the end points of the linear pieces in the rotated membership function also be selected for characteristic points.
- v) Those points where the function $g_k(s, \text{cp}(A_k))$ has break points (where function $(dg_k(s, \text{cp}(A_k))/ds)$ is not continuous) are also defined as characteristic points.
- vi) Local minimum and maximum points of $g_k(s, \text{cp}(A_k))$, like on Fig. 3, can also be considered as characteristic points.
- vii) Inflexion points of the generatrix are also taken into account as characteristic points.

Among the characteristic points of a generatrix there are four distinguished ones, defined under points i) and ii), such as $p_{k,1}, p_{k,a}, p_{k,b}$ and p_{k,M_k} . According to these points let the characteristic points of the k th generatrix be divided into three groups. Group $G_{k,1}$ consists of points $G_{k,1} = \{p_{k,m}\}$, where $p_{k,1} \leq p_{k,m} \leq p_{k,a}$. Similarly: let $G_{k,2} = \{p_{k,i}\}$, where $p_{k,a} \leq p_{k,m} \leq p_{k,b}$ and $G_{k,3} = \{p_{k,m}\}$, where $p_{k,b} \leq p_{k,m} \leq p_{k,M_k}$ (note that points $p_{k,a}$ and $p_{k,b}$ are included in two groups). Let the process of Step 2) be started by assigning the distinguished characteristic points as $(p_{1,1}, p_{2,1}), (p_{1,a}, p_{2,a}), (p_{1,b}, p_{2,b})$ and (p_{1,M_1}, p_{2,M_2}) , which ensures Property 3, because the assigned points will be connected by straight lines contained in the surface of the solid, see later. Therefore, if both rotated antecedents are normalized then at least one of these lines is parallel with the plane $S \times X$

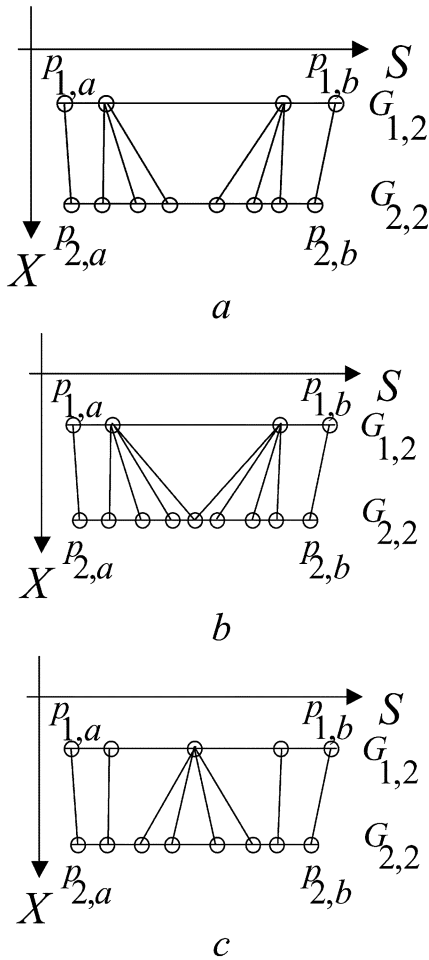


Fig. 4. Topology of the characteristic points.

and lies at level $\mu = 1$. This implies that any intersection of the solid has at least one point which coordinate μ equals one, the interpolated set, hence, is normalized. The remaining characteristic points are assigned between the same numbered groups, namely the points of $G_{1,i}$ are assigned to the points of $G_{2,i}$. The point pairs between the groups are defined in the same way for all $i = 1 \dots 3$. Fig. 4 shows an example where group $G_{k,2}$ of the generatrices are depicted. Assume that the number of points in $G_{1,2}$ is less than in $G_{2,2}$. The point pairs are simultaneously determined from the left and from the right side (see Fig. 4(a)) where points $(p_{1,a}, p_{2,a})$ and $(p_{1,b}, p_{2,b})$ are assigned first. Then the next two points are assigned from left and right, see points $(p_{1,a+1}, p_{2,a+1})$ and $(p_{1,b-1}, p_{2,b-1})$. This is repeated until there is no more point or only one point remains in $G_{1,2}$ [Fig. 4(b)]. If there is no more point in $G_{1,2}$ then the points connected last in the right and the left side are connected to the remaining points in $G_{2,2}$. Namely, simultaneously one remaining point from right side in $G_{2,2}$ connected to right point in $G_{1,2}$ and the left remaining point in $G_{2,2}$ is connected to the left point in $G_{1,2}$, see Fig. 4(a) and (b). If the number of points in $G_{2,2}$ is odd then the last point in $G_{2,2}$ is connected with the both points connected last in $G_{1,2}$ as shown in Fig. 4(c). If the number of point in $G_{1,2}$ is odd, namely, one point is remained then this point is connected with all remaining point in $G_{2,2}$ as depicted

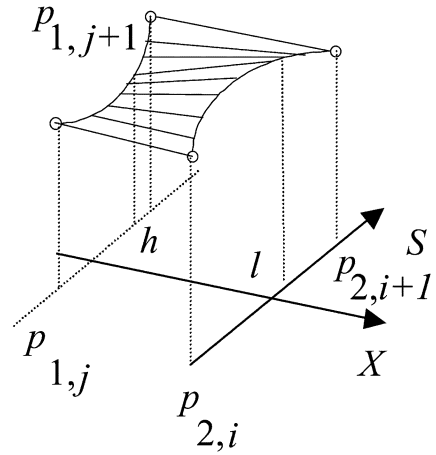


Fig. 5. Cylindrical surface.

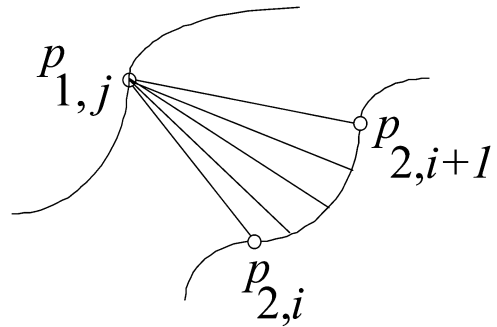


Fig. 6. Conical surface.

on Fig. 4(c). Fig. 4 shows that the topology of the connections yields triangular and quadrangular forms. The triangular forms are covered by simple conical surfaces and the quadrangular forms are covered by cylindrical surfaces. The cylindrical surface is a line surface fit to two generatrices, where all points of the generatrices are connected by lines. Fig. 5 shows an example, where point $h \in [p_{1,j}, p_{1,j+1}]$ and $l \in [p_{2,i}, p_{2,i+1}]$ are connected. Let the relation between h and l be defined as: $d(p_{1,j}, h) : d(h, p_{1,j+1}) = d(p_{2,i}, l) : d(l, p_{2,i+1})$. In the case of a conical surface, all points of the generatrix are connected to one point; see Fig. 6.

The previously outlined technique holds Properties 1–4. Additionally, Property 5 can also be observed.

Property 5 (Continuity): For $\varepsilon > 0$ there exists $\delta > 0$ such that if $d(\text{cp}(A^i), \text{cp}(C^i)) \leq \delta$, then for the intersections of the solid at $\text{cp}(A^i)$ and $\text{cp}(C^i)$ we have $\forall s : \max_s (|g(s, \text{cp}(A^i)) - g(s, \text{cp}(C^i))|) \leq \varepsilon$. Therefore, if $|\lambda - \lambda'| \leq \delta$ then $\forall x : \max_x (|\mu_{A^i}(x) - \mu_{C^i}(x)|) \leq \varepsilon$, where $A^i = f_{\text{SC}}^{\text{Interpolation}}(A_1, A_2, \lambda)$ and $C^i = f_{\text{SC}}^{\text{Interpolation}}(A_1, A_2, \lambda')$.

Property 5 comes from the fact that the solid is constructed by continuous line surfaces. As a matter of fact, the characteristic points can be assigned in various ways. All assigning may result in different solids. All solids have restrictions and advantages on their own. Our future work is to investigate the features of various kinds of techniques capable of generating a solid based on the given generatrices.

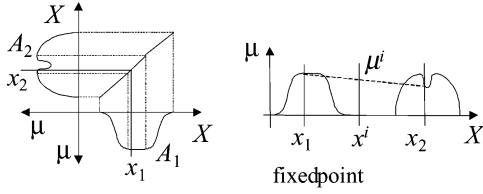


Fig. 7. Fixed point law set interpolation.

Algorithm 2 (Fixed Point Law Fuzzy Set Interpolation (FPLI)):

$$A^i = f_{\text{FPL}}^{\text{Interpolation}}(A_1, A_2, \lambda).$$

According to the theory of FPL technique, this algorithm gives the convex combination of the membership functions. Assume that the elements x_k of fuzzy sets A_k are assigned by a piecewise linear interrelation function $x_2 = \Lambda(x_1, \mathbf{p}_1, \mathbf{p}_2)$, where $\mathbf{p}_k = [\underline{A}_k \ p_{k,2} \ \cdots \ \bar{A}_k]$ then the membership values of the fixed elements x_1 and x_2 are interpolated as

$$\mu_{A^i}(x) = \Gamma(\mu_{A_1}(x_1), \mu_{A_2}(x_2), \lambda) \quad (4)$$

(for Γ see Definition 7) which is the membership value over the interpolated element $x = \Gamma(x_1, x_2, \lambda)$.

A substantial point of this algorithm is the definition of the interrelation function. Vectors \mathbf{p}_1 and \mathbf{p}_2 , respectively, contain the characteristic points of the fuzzy sets A_1 and A_2 . The length of the vectors must agree, see Definition 3. This implies, that the fuzzy sets must be divided by the same number of characteristic points. These points can, for instance, be defined by the above proposed conditions i)–vi). As an example, a particular algorithm is discussed in the following, which is proposed to frequent practical cases, when the shapes of the antecedents are “rather nice” like on Fig. 7. Let the membership function of A be divided into pieces by characteristic points $\underline{A}, \bar{A}, g \in [\underline{A}, \text{cp}(A)]$ and $h \in [\text{cp}(A), \bar{A}]$ (notation \bar{A} is defined in Definition 1). g and h are the maximum and minimum elements of A_α , where $\alpha = \text{height}(A)$, like the characteristic points in the case of trapezoidal sets. Therefore, the interrelation is $x_2 = \Lambda(x_1, \mathbf{p}_1, \mathbf{p}_2)$, (this function Λ is defined in Definition 5) $\mathbf{p}_k = [\underline{A}_k \ g_k \ \text{cp}(A_k) \ h_k \ \bar{A}_k]$ and the interpolated membership values are determined by (4).

Note that the proposed FPLI algorithm can be viewed as a special case of the previously outlined SC technique.

Algorithm 3 (Fixed Value Law Fuzzy Set Interpolation (FVLI)):

$$A^i = f_{\text{FVL}}^{\text{Interpolation}}(A_1, A_2, \lambda).$$

According to the theory of FVL technique, the membership value is fixed first, then the fuzziness is interpolated at the fixed α -level, see Fig. 8. In order to be applicable to subnormal sets as well let the first step be the normalization of the given sets. Let the normalization be understood as

$$A' : \mu_{A'}(x) = \frac{\mu_A(x)}{\text{height}(A)}.$$

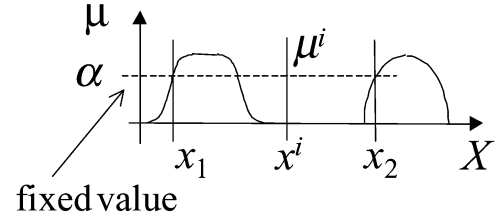


Fig. 8. Fixed value law set interpolation.

For further notation, let fuzzy set A' denote the normalized fuzzy set A . Interpolation of the fuzziness at α level

$$\underline{A}'_\alpha = \Gamma(\underline{A}'_{1,\alpha}, \underline{A}'_{2,\alpha}, \lambda); \quad \bar{A}'_\alpha = \Gamma(\bar{A}'_{1,\alpha}, \bar{A}'_{2,\alpha}, \lambda) \\ \alpha \in [0, 1].$$

Finally, the interpolated A^i is renormalized from A^i in such a way that the renormalized height of the interpolated set is

$$\text{height}(A^i) = \Gamma(\text{height}(A_1), \text{height}(A_2), \lambda).$$

Note that this algorithm is restricted to convex fuzzy (CF) sets.

As mentioned at the beginning of this section, the rule interpolation is done via fuzzy set interpolation executed on both the input X and output universe Y .

Algorithm 4 (Fuzzy Relation Based Rule Interpolation): This step summarizes the aforementioned algorithms

$$R^i = f_\circ^{\text{Interpolation}}(R_1, R_2) \quad \text{where } \circ \in \{\text{SC}, \text{FPL}, \text{FVL}\}. \quad (5)$$

Formula (5) is performed as

$$A^i = f_\circ^{\text{Interpolation}}(A_1, A_2, \lambda_a) \quad \text{and} \\ B^i = f_\circ^{\text{Interpolation}}(B_1, B_2, \lambda_b). \quad (6)$$

Remark 2: An important issue of choosing λ_a and $\lambda_b = f(\lambda_a)$ should be addressed here. When defining λ_a , (3) should be taken into account, that leads to

$$\lambda_a = \frac{d(A_1, A^*)}{d(A_1, A_2)}.$$

Remark 3: In order to master the use of the previous algorithms in the proposed general concept let the simplest choice $\lambda_a = \lambda_b$ be briefly discussed.

Lemma (Linear Interpolation Generatrix): The set of points $(\text{cp}(A^i), \text{cp}(B^i))$ defined by (6) where $\lambda_a = \lambda_b \in [0, 1]$ determine a straight line which consists of points $(\text{cp}(A_1), \text{cp}(B_1))$ and $(\text{cp}(A_2), \text{cp}(B_2))$. With (2) the interpolation generatrix exhibits the linear feature of the rule interpolation.

As an example, see Fig. 9. Defining different proper functions for $\lambda_b = f(\lambda_a)$ a family of nonlinear interpolations can be derived. For instance, generatrix $y = f(x)$ depicted on Fig. 10 is achieved by defining a weighted combination of the rules such as

$$\lambda_b = f(\lambda_a) = \frac{(1 - \lambda_a)\text{cp}(B_1)w_1 + \lambda_a\text{cp}(B_2)w_2 - \text{cp}(B_1)}{\text{cp}(B_2) - \text{cp}(B_1)}$$

where w_1 and w_2 are the weights of the consequent sets.

Remark 4: Algorithms SC, FPLI, and FVLI hold Properties 1–5, which implies that the rule interpolation defined in (1) inherits these properties.

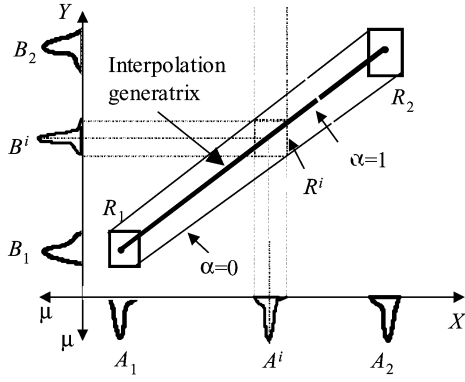


Fig. 9. Linear interpolation.

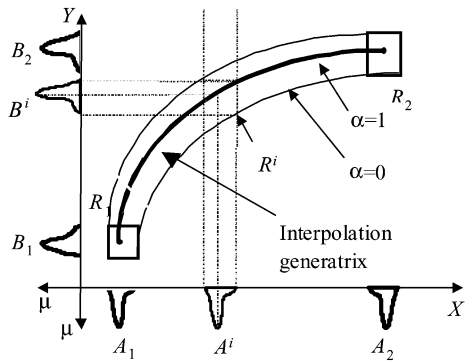


Fig. 10. Nonlinear interpolation.

B. Semantic Relation Interpolation

This section uses the ideas of *semantic revision principle* techniques to describe the relation between antecedents and consequent sets via semantic interpretation. Analogously to the revision principle methods FPL and FVL, two kinds of *semantic revision methods* (SRM) have been introduced by Shen *et al.* [13], [14], [43], [45], [46]. They are termed as SRM-I and SRM-II. Both techniques define an *interrelation* between the elements of fuzzy sets and a *semantic relation* function to capture the similarities of the membership values. In the case of SRM-I the element pairs, whose membership values are recorded into a semantic function, are predefined by interrelation function, which implies that the idea of the FPL is followed. As opposed to this in the case of the SRM-II, first the membership values are assigned by semantic function and then, according to this assignment, the interrelation function records the fuzziness of the sets. This idea emerges in the FVL methods. Consequently, it can be concluded that the SRM methods are the extensions of the FPL and the FVL in this sense. For more details, see [13], [14], [43], [45], and [46]. In order to facilitate the undersatnding of the semantic relation based interpolation, first the basics of SRM methods is recalled.

Definition 10 (Semantics and Interrelation for SRM-I):

a) *Interrelation:*

$$\mathbf{IR}_{A,B}^{(I)} = \{(x, y) \mid x \in [\underline{A}, \bar{A}], y = \Lambda(x, \mathbf{p}_1, \mathbf{p}_2)\}$$

$$\mathbf{p}_1 = [\underline{A} \text{ cp}(A) \bar{A}] \text{ and } \mathbf{p}_2 = [\underline{B} \text{ cp}(B) \bar{B}]$$

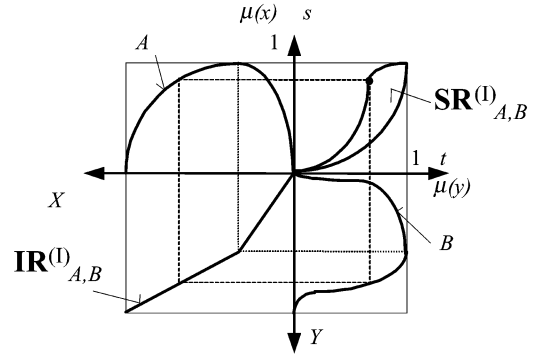


Fig. 11. Interrelation and semantic relation of SMR-I.

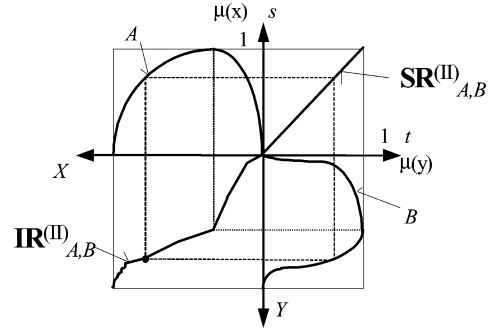


Fig. 12. Interrelation and semantic relation of SMR-II.

b) *Semantic relation:*

$$\mathbf{SR}_{A,B}^{(I)} = \{(s, t) \mid s = \mu_A(x), t = \mu_B(y), (x, y) \in \mathbf{IR}_{A,B}^{(I)}\}$$

For an illustration, see Fig. 11.

Definition 11 (Semantics and Interrelation for SRM-II):

a) *Semantic relation:*

$$\mathbf{SR}_{A,B}^{(II)} = \{(s, t) \mid t = \frac{\text{height}(B)}{\text{height}(A)}s, s \in [0, \text{height}(A)]\}. \tag{7}$$

b) *Interrelation:*

$$\mathbf{IR}_{A,B}^{(II)} = \left\{ (x, y) \mid \mu_A(x) = s, \mu_B(y) = t, (s, t) \in \mathbf{SR}_{A,B}^{(II)}, \left(\frac{ds}{dx} \cdot \frac{dt}{dy} \psi > 0 \text{ or } \frac{ds}{dx} = \frac{dt}{dy} = 0 \right) \right\}.$$

For an illustration, see Fig. 12.

In the next part, one more interpolation technique is proposed. Its aim is to compute the interpolated semantic and interrelation functions to the observation as

$$\mathbf{IR}^i = f^{\text{Interpolation}}(\mathbf{IR}_1, \mathbf{IR}_2);$$

$$\mathbf{SR}^i = f^{\text{Interpolation}}(\mathbf{SR}_1, \mathbf{SR}_2). \tag{8}$$

Notations “-I” and “-II” utilized in the next algorithms are reserved to indicate that the concept of SRM-I or SRM-II is applied.

Algorithm 5 (Interpolation of Semantic Relation, IS-I):

a) Interrelation:

$$\begin{aligned} \mathbf{IR}_{A^i, B^i}^{(I)} &= \left\{ (x, y) \mid x = \Gamma(x_1, x_2, \lambda_a), y = \Gamma(y_1, y_2, \lambda_b), x_2 \right. \\ &= \Lambda(x_1, \mathbf{p}_1, \mathbf{p}_2), (x_k, y_k) \in \mathbf{IR}_{A_k, B_k}^{(I)} \left. \right\} \\ \mathbf{p}_k &= [\underline{A}_k \quad \text{cp}(A_k) \quad \bar{A}_k]. \end{aligned}$$

b) Semantic relation:

$$\begin{aligned} \mathbf{SR}_{A^i, B^i}^{(I)} &= \left\{ (s, t) \mid s = \Gamma(s_1, s_2, \lambda_a); t = \Gamma(t_1, t_2, \lambda_b); s_k \right. \\ &= \mu_{A_k}(x_k), t_k = \mu_{B_k}(y_k); x_2 \\ &= \Lambda(x_1, \mathbf{p}_1, \mathbf{p}_2); (x_k, y_k) \in \mathbf{IR}_{A_k, B_k}^{(I)} \left. \right\}. \end{aligned}$$

Algorithm 6 (Interpolation of Semantic Relation, IS-II): Along the same line as in the case of FVL interpolation let the semantic relations be normalized first. Therefore, let

$$\begin{aligned} \mathbf{SR}_{A, B}^{(I/II)} &= \left\{ (s', t') \mid s' = \frac{s}{\text{height}(A)}; t' \right. \\ &= \frac{t}{\text{height}(B)}; (s, t) \in \mathbf{SR}_{A, B}^{(I/II)} \left. \right\} \quad (9) \end{aligned}$$

where (I/II) means that the equation applicable for both IS-I and IS-II, and $\mathbf{SR}_{A, B}^{(I/II)}$ denotes normalized $\mathbf{SR}_{A, B}^{(I/II)}$, which in fact means the same as the $\mathbf{SR}_{A', B'}^{(I/II)}$ of the normalized sets A' and B' . Note that, if the sets are normalized, then their semantic relations equal the set of points defined by $s = t$, see (7) and (9). This implies that the semantic relations become equivalent for all the rules in the case of IS-II.

a) Semantic relation

$$\begin{aligned} \mathbf{SR}^{(II)} &= \mathbf{SR}_{A_k, B_k}^{(II)} = \{(s'_k, t'_k) \mid s'_k = t'_k; t'_k \in [0, 1]\} \\ &= \mathbf{SR}_{A^i, B^i}^{(II)} = \{(s'_k, t'_k) \mid s'_k = t'_k; t'_k \in [0, 1]\}. \end{aligned}$$

b) Interrelation

In order to give graphical interpretation of the interpolation let the interpretation of SRM-II be particularly modified here. Let the points of $\mathbf{IR}_{A^i, B^i}^{(II)}$ and $\mathbf{IR}_{A_k, B_k}^{(II)}$ be defined in three dimensional space spanned by X, Y , and S as

$$\begin{aligned} \mathbf{IR}_{A_k, B_k}^{(II)} &= \left\{ (x, y, s) \mid s = \mu_{A_k}(x); t = \mu_{B_k}(y); (s, t) \right. \\ &\in \mathbf{SR}^{(II)}; \frac{ds}{dx} \cdot \frac{dt}{dy} > 0 \text{ or } \frac{ds}{dx} = \frac{dt}{dy} = 0 \left. \right\} \end{aligned}$$

An illustration is given in Fig. 13.

The interpolation can easily be defined in the three-dimensional space as

$$\begin{aligned} \mathbf{IR}_{A^i, B^i}^{(II)} &= \left\{ (x, y, s) \mid x = \Gamma(x_1, x_2, \lambda_a); y \right. \\ &= \Gamma(y_1, y_2, \lambda_b); (x_k, y_k, s) \in \mathbf{IR}_{A_k, B_k}^{(II)} \left. \right\}. \end{aligned}$$

Fig. 14 shows the interpolation of the interrelation functions.

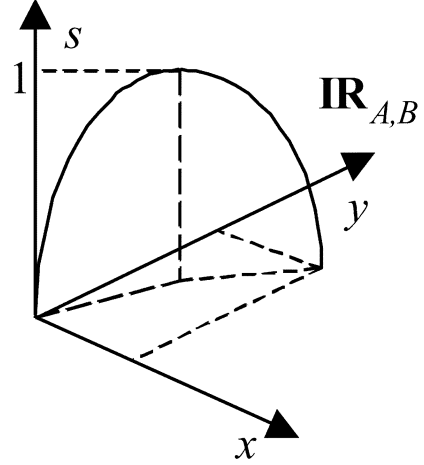


Fig. 13. Illustration of three dimensional $\mathbf{IR}_{A_k, B_k}^{(II)}$.

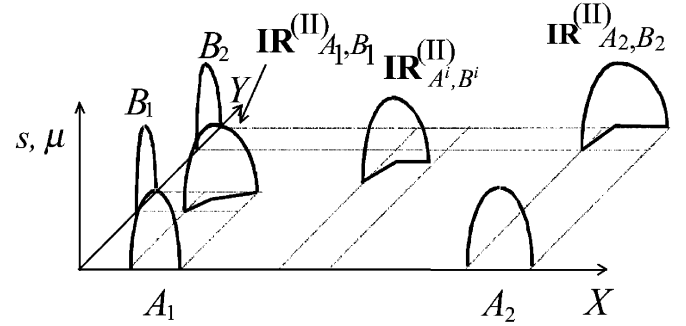


Fig. 14. Interpolation of the interrelation function.

Similar to the case of FVLI algorithm, the final step is to renormalize the interpolated sets, namely, to renormalize the semantic relation function based on (8)

$$\begin{aligned} \mathbf{SR}_{A^i, B^i}^{(II)} &= \left\{ (s, t) \mid s = s' \Gamma(\text{height}(A_1), \text{height}(A_2), \lambda_a); t \right. \\ &= t' \Gamma(\text{height}(B_1), \text{height}(B_2), \lambda_b); (s', t') \in \mathbf{SR}^{(II)} \left. \right\}. \end{aligned}$$

In order to use the semantic interpolation techniques in the proposed general concept the determination of λ_a and λ_b will be addressed. The same conclusions can be drawn here as at the discussion of fuzzy relation based interpolation algorithms. Let $\lambda_a = (d(A_1, A^*)/d(A_1, A_2))$ according to (3). In the case of globally linear featured interpolation, $\lambda_b = \lambda_a$ is chosen. Again, like in the case of fuzzy relation-based interpolation, the equation

$$\lambda_a = f(\lambda_b) \quad (10)$$

determines the global feature of the interpolation, where (10) is defined according to a desired interpolation generatrix.

V. SINGLE RULE INFERENCE

The objective of this section is to propose three kinds of techniques capable of firing the interpolated rule with the observation and inferring the conclusion. These algorithms are for the second step of the generalized interpolation method proposed in Section III. Single rule reasoning approaches, hence,

have prominent roles in this section. The proposed inference algorithms of this section are originated from the FPL, SRM-I, and SRM-II single rule inference methods [13], [14], [43], [45], [46], which have been developed for such cases when the support of the observation consists of all elements and only those elements, which are contained in the interrelation function of the fired rule. Namely, the interrelation area (Definition 6) of the interpolated rule should agree with the support of the observation. The interpolated relation (fuzzy or semantic), however, may not fulfill this condition. Therefore, this section introduces how to expand the interpolated relation, in order, to match the required area defined by the observation. This fitting ensures the proper matching of the interpolated relation with the observation. First, an algorithm is proposed capable of transforming the interrelation area of the interpolated relation to the observation. Then two algorithms, a fuzzy and a semantic relation-based, are discussed which transform the interpolated relation to the transformed interrelation area. The use of these transformations means that the interpolated relation is expanded in the “near” neighborhood of the observation. This is based on the assumption that the resulting relation is an acceptable approximation of the relation of this area.

Definition 12 [*Spanning the Interrelation Area* (c, d) = $\Phi(A, B, a, b)$]: Assume that a fuzzy rule $R : A \mapsto B$ is given. Its rectangular interrelated area is defined by intervals $[\underline{A} \ \bar{A}]$ and $[\underline{B} \ \bar{B}]$. The new area, which is proportionally spanned to a given interval $[a, b]$ is defined by intervals $[a, b]$ and $[c, d]$ as

$$\begin{aligned} c &= \Lambda(a, \mathbf{p}_1, \mathbf{p}_2) \quad \text{and} \quad d = \Lambda(b, \mathbf{p}_1, \mathbf{p}_2); \\ \mathbf{p}_1 &= [\underline{x} \ \underline{A} \ \text{cp}(A) \ \bar{A} \ \bar{x}] \\ \mathbf{p}_2 &= [\underline{y} \ \underline{B} \ \text{cp}(B) \ \bar{B} \ \bar{y}]. \end{aligned}$$

Fig. 15 gives an illustration how the interrelation area is transformed.

The fuzzy and the semantic relation defined over the interrelation area can be proportionally transformed accordingly to the spanning of the interrelation area. First, a fuzzy relation then a semantic relation based transformation are proposed.

A. Transformation of Fuzzy Relation

The fuzzy relation is transformed via set transformations on both the input and the output universe.

Transformation 1 [*Transformation of the Fuzzy Relation to a Given Interrelation Area* $R^t = T(R, a, b, c, d)$]: Assume fuzzy rule $R : A \mapsto B$. Let fuzzy rule $R^t : A^t \mapsto B^t$ be a transformed fuzzy rule whose interrelation area is defined by $[a, b]$ and $[c, d]$. Superscript t means that “transformed.” The transformed antecedent set is determined as $A^t : \mu_{A^t}(x) = \mu_A(\Lambda(x, \mathbf{p}_1, \mathbf{p}_2))$, where $\mathbf{p}_1 = [a \ \text{cp}(A) \ b]$ and $\mathbf{p}_2 = [\underline{A} \ \text{cp}(A) \ \bar{A}]$. As a result $\underline{A}^t = a$ and $\bar{A}^t = b$. The consequent set is calculated in the same way as: $B^t : \mu_{B^t}(y) = \mu_B(\Lambda(y, \mathbf{p}_3, \mathbf{p}_4))$, where $\mathbf{p}_3 = [c \ \text{cp}(B) \ d]$ and $\mathbf{p}_4 = [\underline{B} \ \text{cp}(B) \ \bar{B}]$, which leads to $\underline{B}^t = c$ and $\bar{B}^t = d$.

An illustration of the transformation is given in Fig. 16.

B. Transformation of Semantic Relation

The following transformation technique is applicable to the semantic relation and results in a proportionally enlarged se-

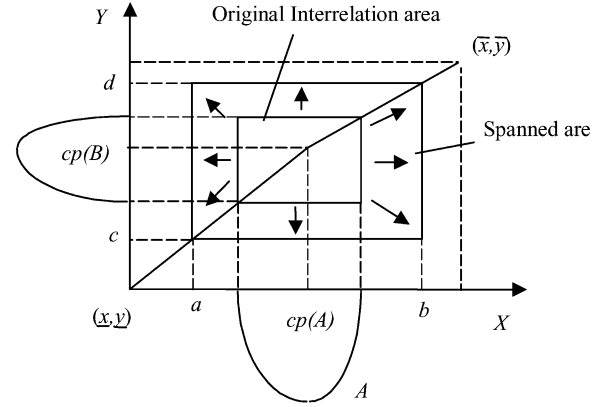


Fig. 15. Spanned interrelation area.

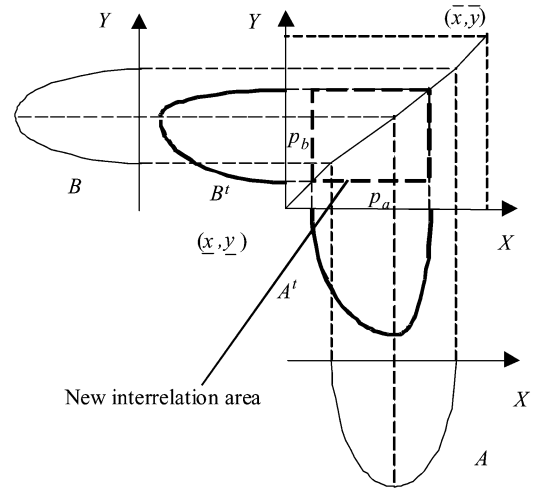


Fig. 16. How A^t and B^t are defined according to the new interrelation area.

semantic relation which fits the new interrelation area. In this case only the transformation of the interrelation is of interest, since the semantic relation is independent on the size of the interrelation area. This also implies that, the same transformation can be proposed for methods I and II.

Transformation 2 (*Transformation of the Semantic Relation to a Given Interrelation Area*): Assume that $\mathbf{IR}_{A,B}^{(I/II)}$ and $\mathbf{SR}_{A,B}^{(I/II)}$ (let $\mathbf{IR}_{A,B}$ and $\mathbf{SR}_{A,B}$ respectively be used for brevity) are given. Let $\mathbf{IR}_{A,B}^t$ be the transformed interrelation function to a given interrelation area defined by $[a, b]$ and $[c, d]$ as $\mathbf{IR}_{A,B}^t = \{(x, y) \mid x = \Lambda(x', \mathbf{p}_1, \mathbf{p}_2); y = \Lambda(y', \mathbf{p}_3, \mathbf{p}_4); (x', y') \in \mathbf{IR}_{A,B}\}$, where $\mathbf{p}_1 = [\underline{A} \ \text{cp}(A) \ \bar{A}]$; $\mathbf{p}_2 = [a \ \text{cp}(A) \ b]$; $\mathbf{p}_3 = [\underline{B} \ \text{cp}(B) \ \bar{B}]$ and $\mathbf{p}_4 = [c \ \text{cp}(B) \ d]$. The semantic relation is not changed by the transformation, so, $\mathbf{SR}_{A,B}^t = \mathbf{SR}_{A,B}$.

Note that both Transformations 1 and 2 hold Properties 1–5. For instance, if the interpolated antecedent equals to the observation, then the transformed set equals the interpolated set. The transformations conserve piecewise linearity and normality. The transformation is continuous.

According to the previous section, the relations are interpolated to the observation in such a way that (3) is held. Again, in the second step the interrelation area and the interpolated relation is transformed to the observation. Finally, the conclu-

sion is generated from the transformed relation and the observation after the transformed relation is assumed to be a good approximation of the interpolated relation. Having the transformed fuzzy and semantic relation immediately leads to the use of the revision principle methods, namely the FPL, SRM-I, and II single rule inference techniques. These methods are slightly specialized here according to (3). For further discussion let us assume that R^i is transformed to the support of A^* , namely, $R^t : A^t \mapsto B^t$ has already been calculated from $R^i : A^i \mapsto B^i$. In order to facilitate the notation “ t ” (transformed) and “ i ” (interpolated) is not used in the next part, the interpolated and transformed single rule to be fired is simply denoted as $R : A \mapsto B$. The conclusion is generated by the following methods.

C. Inference by Fuzzy Relation

Algorithm (Inference of the Conclusion by FPL, $B^ = f_{\text{FPL}}^{\text{Inference}}(R, A^*)$):* This algorithm fires the transformed fuzzy relation. Performing the ideas of FPL, the membership functions are compared over each interrelated element of the sets. The deviation between the transformed antecedent and the observation over a fixed element is carried to the interrelated element on the output universe to yield the deviation of the conclusion from the consequent. Considering all elements of the sets results in the membership function of the conclusion

$$\mu_{B^*}(y) = \Lambda(\mu_{A^*}(x), \mathbf{p}_1, \mathbf{p}_2);$$

$$\mathbf{p}_1 = [0 \quad \mu_A(x) \quad 1] \quad \text{and} \quad \mathbf{p}_2 = [0 \quad \mu_B(y) \quad 1].$$

Elements x and y are assigned by the transformed interrelation function.

D. Inference by Semantic Relation

Two semantic revision based methods are discussed in the next part, namely, the concepts of SRM-I and II-based inference techniques, which are capable of concluding B^* in respect to A^* based on the transformed semantic relation. These algorithms have originally been developed for CNF sets and are slightly specialised here to $\text{cp}(A^*) = \text{cp}(A)$. One more important condition of the SRM methods should be taken into account here. The inference by SRM methods assumes that $\text{height}(A^*) = \text{height}(A)$ which is, as a matter of fact, not ensured by any of the previous steps for all cases. Because of this let the semantic relations be normalized in the same way as in the case of IS-II [see (9)].

Algorithm 7 (Inference by SRM-I, $B^ = f_{\text{SRM-I}}^{\text{Inference}}(\mathbf{IR}_{A,B}^{(I)}, \mathbf{SR}_{A,B}^{(I)}, A^*)$, from [45]):* The essential point is to carry the semantic and interrelation of the fired rule over the observation and the conclusion.

$$\text{a) } \mathbf{SR}_{A^*,B^*}^{(I)} \equiv \mathbf{SR}_{A,B}^{(I)} \quad \text{and} \quad \mathbf{IR}_{A^*,B^*}^{(I)} \equiv \mathbf{IR}_{A,B}^{(I)}$$

Then, for all $(x, y) \in \mathbf{IR}_{A,B}^{(I)}$, the following holds.

$$\text{b) The conclusion is found by simply solving the following equation for } \mu_{B^*}(y):$$

$$(\mu_{A^*}(x), \mu_{B^*}(y)) = (\mu_A(x'), \mu_B(y')) \in \mathbf{SR}_{A^*,B^*}^{(I)}$$

$$\text{where } (x, y) \in \mathbf{IR}_{A,B} \quad \text{and} \quad (d\mu_A(x')/dx')(d\mu_B(y')/dy') > 0 \quad \text{or} \quad (d\mu_A(x')/dx') = (d\mu_B(y')/dy') = 0.$$

Algorithm 8 (Inference by SRM-II, $B^ = f_{\text{SRM-II}}^{\text{Inference}}(\mathbf{IR}_{A,B}^t, \mathbf{SR}_{A,B}^{(II)}, A^*)$, from [45]):* Following the same idea as before, let

$$\text{a) } \mathbf{SR}_{A^*,B^*}^{(II)} \equiv \mathbf{SR}_{A,B}^{(II)}, \quad \mathbf{IR}_{A^*,B^*}^{(II)} \equiv \mathbf{IR}_{A,B}^{(II)}$$

The next two steps are solved for all t .

$$\text{b) } (x, y) \in \mathbf{IR}_{A^*,B^*}^{(II)}, \quad \text{and} \quad (\mu_{A^*}(x), t) \in \mathbf{SR}_{A^*,B^*}^{(II)}$$

$$\text{c) } \mu_{B^*}(y) = t.$$

If the fired relation is normalized, then the above obtained conclusion should be renormalized to the height of

$$\text{height}(B^*) = \Lambda(\text{height}(A^*), \mathbf{p}_1, \mathbf{p}_2)$$

$$\text{where } \mathbf{p}_1 = [0 \quad \text{height}(A) \quad 1] \quad \text{and} \\ \mathbf{p}_2 = [0 \quad \text{height}(B) \quad 1].$$

Note that these inference techniques keep Properties 1–3 and 5. Property 4 is guaranteed in the case of triangular shaped interpolated rules.

VI. DISCUSSION OF THE PROPOSED ALGORITHMS

The previous sections proposed a generalized fuzzy rule interpolation method, and few techniques as examples for possible implementation. Let the generalized fuzzy rule interpolation be denoted as

$$B^* = I_{\circ_1, \circ_2}(R_1, R_2, A^*)$$

where $\circ_1 \in \{\text{SC}, \text{FPL}, \text{FVL}, \text{IS-I}, \text{IS-II}\}$ indicates which interpolation technique is used and $\circ_2 \in \{\text{FPL}, \text{SRM-I}, \text{SRM-II}\}$ defines the single rule inference technique applied in the second step of the method. This section is intended to investigate some properties of the interpolation methods implemented by the proposed algorithms. Special attention is paid on the three typical deficiencies of interpolation methods discussed in the first paragraph of the Introduction. First of all, let the main steps be summarized and simply demonstrated via the SC interpolation method, namely, $B^* = I_{\text{SC}, \text{FPL}}(R_1, R_2, A^*)$. Fig. 17 demonstrates the essential points of the concept. Assume that fuzzy rules $R_k : A_k \mapsto B_k, k = 1, 2$, and observation A^* are given. In the first step a fuzzy rule $R^i : A^i \mapsto B^i$ is interpolated in such a way that $d(A^*, A^i) = 0$, namely, $d(\text{rp}(A^*), \text{rp}(A^i)) = 0$ (see (3), and note that $\text{rp}(A) = \text{cp}(A)$ is set in (2)). In other words, the rule-base is interpolated at the observation. The SC interpolation algorithm is actually executed on the fuzzy sets. Both the interpolated antecedent and consequent sets are generated by the *solid cutting* technique, see Fig. 17. The interpolated rule overlapping with A^* is considered temporarily like any rules of the rule base and is fired by A^* in the second step of the proposed generalized concept (see Fig. 17). The interpolated rule is depicted by dotted line in the column of STEP II. The key idea of the inference is to keep the interpolated fuzzy relation of R^i between A^* and B^* . Then the fuzzy relation of R^i is transformed to A^* , see the sets with superscript t and drawn by thin line in the column of STEP II. Finally, the conclusion is inferred by the FPL method.

Let some properties of the fuzzy interpolation techniques implemented by the proposed algorithms be investigated in the following.

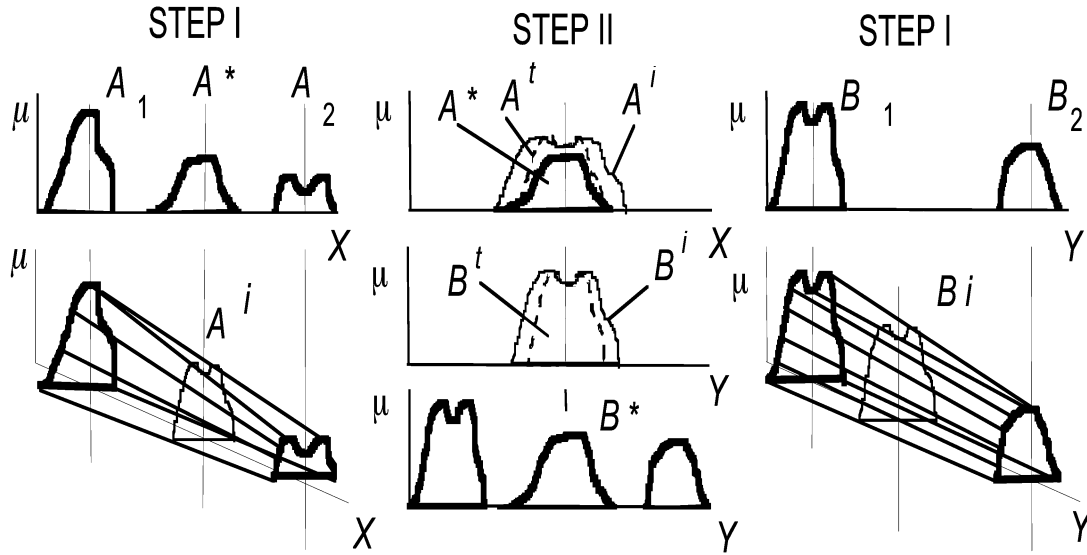


Fig. 17. Main steps of the proposed method.

Property 6 (Avoiding Abnormal Fuzzy Conclusion): Interpolation algorithms $B^* = I_{o_1, o_2}(R_1, R_2, A^*)$, $o_1 \in \{SC, FPL, FVL, IS-I, IS-II\}$, $o_2 \in \{FPL, SRM-I, SRM-II\}$ always result in a normal conclusion (Property 2 is held in all steps).

Property 7 (No Restriction to CNF Sets): Interpolations $I_{o_1, o_2}(R_1, R_2, A^*)$, $o_1 \in \{SC, FPL\}$ and $o_2 \in \{FPL\}$ are not restricted to CNF sets, but are applicable to arbitrary shaped sets. Interpolations $I_{o_1, o_2}(R_1, R_2, A^*)$, $o_1 \in \{FVL\}$ and $o_2 \in \{SRM-I, SRM-II\}$ are not restricted to normal, but to arbitrary convex fuzzy sets.

Property 8 (Preserving Linearity): Interpolations $I_{o_1, o_2}(R_1, R_2, A^*)$, $o_1 \in \{SC, FPL, FVL, IS-I, IS-II\}$ and $o_2 \in \{FPL, SRM-I, SRM-II\}$ conserve the piece-wise linearity in the case of triangular fuzzy sets (Property 4 is held for triangular sets in all steps).

Property 9 (Compatibility With the Rule Base): This property is the *modus ponens* in logic. For $\forall k$ and $\forall A^*$ it follows from $A^* = A_k$ that $B^* = B_k$. All techniques proposed in this paper fulfill this condition (Property 1 is held in all steps) since if $A^* = A_k$ then $A^i = A_k$ and $B^i = B_k$, which simply implies that $A^* = A^i$. The discussed single rule reasoning techniques keep the nondeviation property, namely, they infer $B^* = B^i$ if $A^* = A^i$. This immediately leads to $B^* = B_k$ if $A^* = A_k$.

Property 10 (Preserving Normality): The resulting conclusion by $I_{o_1, o_2}(R_1, R_2, A^*)$, $o_1 \in \{SC, FPL, FVL, IS-I, IS-II\}$ and $o_2 \in \{FPL, SRM-I, SRM-II\}$ is normal if the sets in the rules and the observation are normal (Property 3 is held in all steps).

Property 11 (Monotonicity): If $A^* \in F(X)$ is more specific than $C^* \in F(X)$, then $I_{o_1, o_1}(A^*)$ is more specific than $I_{o_1, o_1}(C^*)$, i.e., for all $A^*, C^* \in F(X)$ inequality $A^* \subseteq C^*$ implies $I_{o_1, o_1}(A^*) \subseteq I_{o_1, o_1}(C^*)$.

In addition to the basic properties listed previously, the “smoothness” Property 12 (see later) of the mapping I is of high interest as well, because in many applications “similar” observations are expected to induce “similar” conclusions. In order to make this property more precise, adequate concepts like “continuity” for mappings, which map fuzzy subsets to

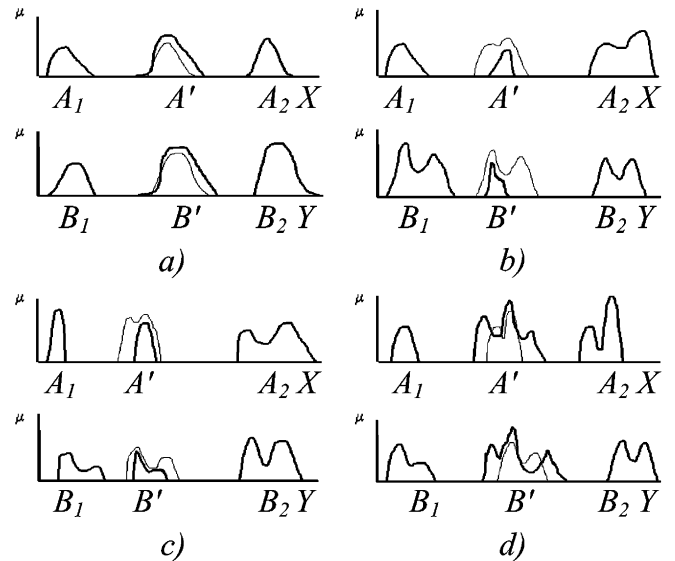


Fig. 18. Numerical example.

fuzzy subsets, are required. Let d_x and d_y be a deviation metrics on $F(X)$ and $F(Y)$, respectively.

Property 12 (Continuity): In the case of the all proposed techniques, for arbitrary $\varepsilon > 0$ there exists $\delta > 0$ such that if $A^*, C^* \in F(X)$, and $d(A^*, C^*) \leq \delta$ then for the corresponding conclusions $d(I_{o_1, o_2}(A^*), I_{o_1, o_2}(C^*)) \leq \varepsilon$ hold.

It is worth noticing here that interpolation $I_{SC, FLP}(R_1, R_2, A^*)$ eliminates three typical deficiencies (while Property 8 is held only for triangular sets) of interpolation techniques discussed in the first paragraph of Section I.

VII. NUMERICAL EXAMPLES

This section is intended to show some examples computed by $I_{SC, FLP}(R_1, R_2, A^*)$. The fuzzy sets used in the examples are not restricted to CNF sets. The results of the first example are depicted in Fig. 18(a)–(d). Each figure contains two coordinate systems. The observations are depicted in the upper ones, while the conclusions can be seen in the lower ones. The sets

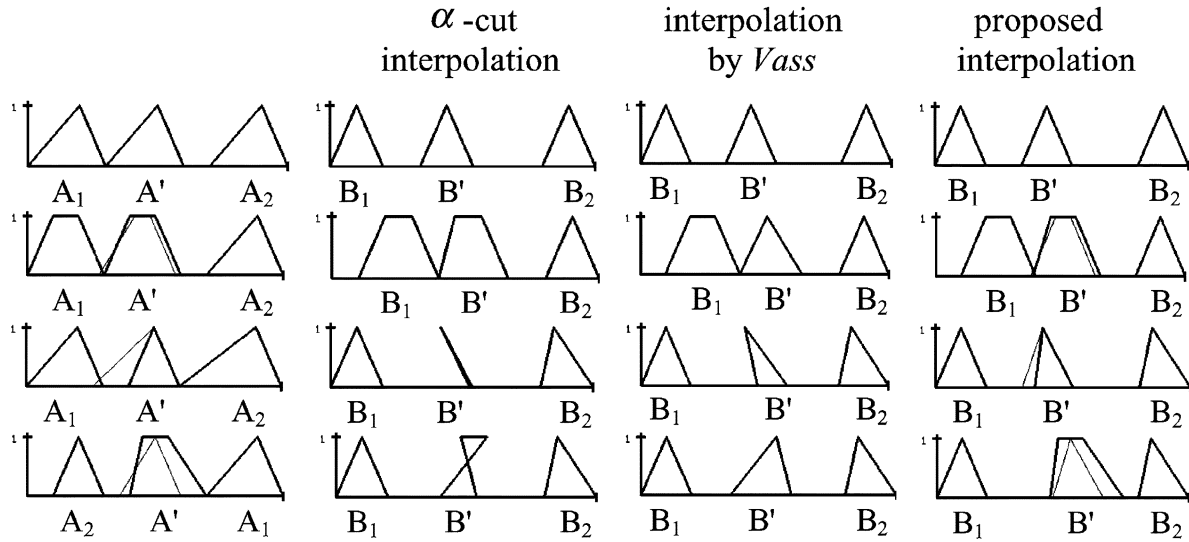


Fig. 19. Numerical comparison.

drawn by thin line are the sets of the interpolated rule. Fig. (a) shows a simple case, where one can observe the similarity of the conclusion to the observation. In Fig. (b), it can be seen that the elements belonging mostly to the conclusion and the interpolated consequent are the same, similarly to elements of the observation and the interpolated antecedent. Fig. (c) presents a similar case. Fig. (d) shows a case, where the functions of the conclusion can be hardly determined by using only human comprehension.

The next example focuses on practical cases when simple triangular or trapezoidal shaped sets are used. In Fig. 19, several examples of sets A_1 , A_2 and observation A' defined on X can be seen in the first column. The fuzzy sets, depicted by thin line, in the first and last columns are contained in the interpolated rules. Figures of columns 2–4 compare three different interpolation methods. The second column treats the results obtained by the first technique, namely, by the α -cut interpolation, and the third column contains results obtained by its improved version proposed by Vass *et al.* [58]. The last column shows the result computed by $I_{SC,FPL}(R_1, R_2, A^*)$. In the first row, the fuzzy terms are rather “nice,” and so every method results in a normal fuzzy set conclusion. It can be observed that, if all methods infer normal fuzzy set conclusion and only triangular fuzzy sets are used, the results are almost identical. The second line shows an example, with trapezoidal fuzzy sets, where the results in the second and third columns are significantly different. The results in the second and fourth columns are more in accordance with the features of the observation. Comparing these three different methods it can be said that the α -cut interpolation usually gives an almost identical conclusion (if it is a normal fuzzy set) with the $I_{SC,FPL}(R_1, R_2, A^*)$. In order to see one of the advantages of the proposed generalized interpolation method, the third and fourth rows present examples where the specialized method results in normal fuzzy set conclusions while the others do not. It is worth mentioning here that various nonlinear interpolation techniques can be defined easily via the proposed generalized interpolation method which always infer a normal conclusion fuzzy set.

interpolation by Vass proposed interpolation

VIII. EXTENSION TO MULTIVARIABLE RULE BASE

In order to propose some implementation algorithms to the generalized method on multi variable rule bases, let all fuzzy and semantic relation based techniques discussed in the previous sections be extended to multivariable rules. Assume that an N -variable rule-base is given which consists of rules

$$R_k : A_{1,k} \& A_{2,k} \& \dots \& A_{N,k} \mapsto B_k$$

where $A_{n,k} : \mu_{A_{n,k}}(x_n) \in F(X_n)$ is the k th fuzzy set on input universe X_n . Assume that two rules are selected ($k = 1, 2$) to observations A^*_{n} subject to

$$A_{n,1} \prec A^*_{n} \prec A_{n,2} \quad B_1 \prec B_2.$$

The conclusion B^* is searched subject to $B_1 \prec B^* \prec B_2$. The first step of the generalized concept is to find

$$R^i = f^{\text{Interpolation}}(R_1, R_2).$$

subject to (3) which could look in multidimensional case as

$$\forall n : d(A^*_{n}, A_n^i) = 0. \quad (11)$$

Since all proposed techniques use the interrelation function, let us define its multidimensional extension.

Definition 13 (Multidimensional Piecewise Linear Interrelation Function): $\mathbf{x} = \Lambda(y, \mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_N, \mathbf{p}_b)$, where vector $\mathbf{x} \in \mathbb{R}^N$ consists of elements $x_n, x_n \in [\underline{x}_n, \bar{x}_n]$, further, $y \in [\underline{y}, \bar{y}]$, $\mathbf{p}_n = [\underline{x}_n = p_{n,1} \ p_{n,2} \ \dots \ \bar{x}_n = p_{n,M}] \in \mathbb{R}^M$ and $\mathbf{p}_b = [\underline{y} = p_{b,1} \ p_{b,2} \ \dots \ \bar{y} = p_{b,M}] \in \mathbb{R}^M$ subject to $p_{n,m} \leq p_{n,m+1}$, and $p_{b,m} \leq p_{b,m+1}$. The interrelation function assigns one element of each universe X_n to one element of the output universe Y . Consequently, it can be defined by N number of univariable interrelation function as the elements of \mathbf{x} are determined by $x_n = \Lambda(y, \mathbf{p}_n, \mathbf{p}_b)$.

In the next section, the algorithms of the previous sections are extended to multi-variable rules.

A. Interpolation of Fuzzy Relation

The fuzzy relation based rule interpolation algorithms, introduced in Section IV, can easily be extended to multivariable cases, since the interpolation is done via set interpolation executed on the input and output universes separately. In the multidimensional case the interpolation can be done in the same fashion. Consequently, with the help of the set interpolation techniques SC, FPL, and FVL, a fuzzy rule $R^i : A_1^i \& A_2^i \& \dots \& A_N^i \mapsto B^i$ is computed as: $A_n^i = f_{\circ}^{\text{Interpolation}}(A_{n,1}, A_{n,2}, \lambda_n)$, and $B^i = f_{\circ}^{\text{Interpolation}}(B_1, B_2, \lambda_b)$, $\circ \in \{\text{SC}, \text{FPL}, \text{FVL}\}$, where $\lambda_n = (d(A_{n,1}, A^*)/d(A_{n,1}, A_{n,2}))$ according to (11). λ_b is a function of λ_n as

$$\lambda_b = f(\lambda_1, \lambda_2, \dots, \lambda_N).$$

depending on a desired interpolation generatrix.

For example, a globally linear interpolation, namely, a linear interpolation generatrix can be achieved if the following is applied: $\lambda_b = (1/N) \sum_n \lambda_n$.

B. Interpolation of Semantic Relation

In order to extend the interpolation technique of semantic and interrelation functions let multidimensional inter- and semantic relation functions be defined in such a way that each antecedent function has its own inter and semantic relation function to the consequent.

Definition 14 (Multivariable Semantic and Interrelation for SRM-I): Assume fuzzy rule $R : A_1 \& A_2 \& \dots \& A_N \mapsto B$.

a) *Interrelation:*

$$\begin{aligned} \mathbf{IR}_{A_n, B}^{(I)} &= \{(x_n, y) \mid x_n \in [\bar{A}_n, \underline{A}_n], y = \Lambda(x_n, \mathbf{p}_n, \mathbf{p}_b)\} \\ \mathbf{p}_n &= [\underline{A}_n \quad \text{cp}(A_n) \quad \bar{A}_n] \quad \text{and} \\ \mathbf{p}_b &= [\underline{B} \quad \text{cp}(B) \quad \bar{B}]. \end{aligned}$$

b) *Semantic relation:*

$$\begin{aligned} \mathbf{SR}_{A_n, B}^{(I)} &= \left\{ (s, t) \mid s = \mu_{A_n}(x_n), t = \mu_B(y), (x_n, y) \in \mathbf{IR}_{A_n, B}^{(I)} \right\}. \end{aligned}$$

Definition 15 (Multivariable Semantic and Interrelation for SRM-II):

a) *Semantic relation:*

$$\mathbf{SR}_{A_n, B}^{(II)} = \left\{ (s, t) \mid t = \frac{\text{height}(B)}{\text{height}(A_n)} s, s = [0, \text{height}(A_n)] \right\}.$$

b) *Interrelation:*

$$\begin{aligned} \mathbf{IR}_{A_n, B}^{(II)} &= \left\{ (x_n, y) \mid \mu_{A_n}(x_n) = s, \mu_B(y) = t, (s, t) \right. \\ &\left. \in \mathbf{SR}_{A_n, B}^{(II)}, \left(\frac{ds}{dx_n} \cdot \frac{dt}{dy} \psi > 0 \text{ or } \frac{ds}{dx_n} = \frac{dt}{dy} = 0 \right) \right\}. \end{aligned}$$

The interpolated relations are determined by

$$\begin{aligned} \mathbf{IR}_{A_n^i, B^i} &= f^{\text{Interpolation}}(\mathbf{IR}_{A_{n,1}, B_1}, \mathbf{IR}_{A_{n,2}, B_2}); \\ \mathbf{SR}_{A_n^i, B^i} &= f^{\text{Interpolation}}(\mathbf{SR}_{A_{n,1}, B_1}, \mathbf{SR}_{A_{n,2}, B_2}) \end{aligned}$$

as introduced in Section IV. Again, the definition of the interpolation parameters λ_n and λ_b determines the global interpolation feature, namely the interpolation generatrix.

The interpolation in the first step is responsible to determine the location of the conclusion by the interpolated rule, but the inference technique, in the second step, yields the shape, namely, the fuzziness of the conclusion that is originated from the interpolated consequent and is modified according to the differences between the observations and the interpolated antecedents. The fundamental idea in the following multi dimensional inference techniques is that the conclusion is modified according to the average of the differences between the observations and the interpolated antecedents after assuming that the inputs are equally considered e.g., they have the same contribution to the output. Of course, in special cases the averaging operator could be replaced by any kinds of convex combinations emphasizing the different contribution weights of the inputs to the output.

C. Inference of the Conclusion

In the univariable case, the interrelation space is defined by the support of the interpolated antecedents. The interpolated interrelation function is transformed to the interrelation space of the observations. The same technique is applied in multivariable case.

Definition 16 (Spanning the Interrelation Space, $(c, d) = \Phi(A_1, A_2, \dots, A_N, B, \mathbf{a}, \mathbf{b})$): Vector \mathbf{a} and \mathbf{b} , respectively, consist of elements a_n and b_n . Assume that a fuzzy rule $R : A_1 \& A_2 \& \dots \& A_N \mapsto B$ is given. Its interrelation space is defined by intervals $[\underline{A}_n \quad \bar{A}_n]$ and $[\underline{B} \quad \bar{B}]$. The new space, which is spanned to a given interval $[a_n, b_n]$ on each input universe is defined by intervals $[a_n, b_n]$ and $[c \quad d]$ as

$$c = \frac{1}{N} \sum_n \Lambda(a_n, \mathbf{p}_n, \mathbf{p}_b) \quad \text{and} \quad d = \frac{1}{N} \sum_n \Lambda(b_n, \mathbf{p}_n, \mathbf{p}_b)$$

$$\begin{aligned} \mathbf{p}_n &= [\underline{x}_n \quad \underline{A}_n \quad \text{cp}(A_n) \quad \bar{A}_n \quad \bar{x}_n]; \\ \mathbf{p}_2 &= [\underline{y} \quad \underline{B} \quad \text{cp}(B) \quad \bar{B} \quad \bar{y}]. \end{aligned}$$

The difference from the univariable case is that the interval on the output universe is determined based on the average of the intervals in full accordance with the assumption that each input has equal contribution to the output. Along the same line the transformation of the fuzzy relation to the new space is done by fuzzy set transformation executed on all universes, like in the univariable case. In the input universes, the antecedents are transformed to intervals $[a_n, b_n]$ and the consequent is transformed to interval $[c, d]$.

Transformation 3 (Transformation of Multivariable Fuzzy Relation to a Given Interrelation Space, $R^t = T(R^i, \mathbf{a}, \mathbf{b}, c, d)$): Assume fuzzy rule $R : A_1 \& A_2 \& \dots \& A_N \mapsto B$. Let fuzzy rule $R^t : A_1^t \& A_2^t \& \dots \& A_N^t \mapsto B^t$ be a transformed fuzzy rule whose interrelation space is defined by $[a_n, b_n]$ and $[c, d]$. The transformed antecedent sets are determined as $A_n^t : \mu_{A_n^t}(x) = \mu_{A_n^t}(\Lambda(x_n, \mathbf{p}_1, \mathbf{p}_2))$, where $\mathbf{p}_1 = [a_n \quad \text{cp}(A_n) \quad b_n]$ and $\mathbf{p}_2 = [\underline{A}_n \quad \text{cp}(A_n) \quad \bar{A}_n]$. As a result, $\underline{A}_n^t = a_n$ and $\bar{A}_n^t = b_n$. The consequent set is calculated in the same way as: $B^t : \mu_{B^t}(y) = \mu_{A_n}(\Lambda(y, \mathbf{p}_3, \mathbf{p}_4))$, where $\mathbf{p}_3 = [c \quad \text{cp}(B) \quad d]$ and $\mathbf{p}_4 = [\underline{B} \quad \text{cp}(B) \quad \bar{B}]$, which leads to $\underline{B}^t = c$ and $\bar{B}^t = d$.

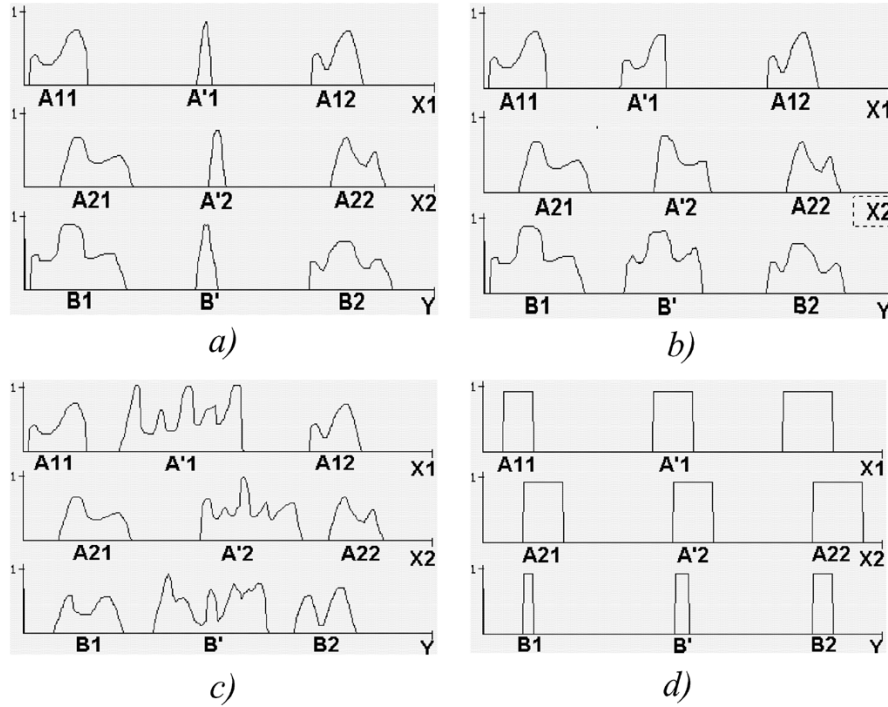


Fig. 20. Results of the proposed method.

Let the relation transformation used in methods SRM-I and II be extended in the same way. The semantic relation is not changing by expanding the interrelation area like in the univariable case.

Transformation 4 (Transformation of the Multivariable Semantic Relation to a Given Interrelation Space): Assume that $\mathbf{SR}_{A_n, B}$ and $\mathbf{IR}_{A_n, B}$ are given. Let $\mathbf{IR}_{A_n, B}^t$ be the transformed interrelation function to a given interrelation area defined by $[a_n, b_n]$ and $[c, d]$ as

$$\mathbf{IR}_{A_n, B}^t = \{(x_n, y) \mid x_n = \Lambda(x'_n, \mathbf{p}_1, \mathbf{p}_2); y = \Lambda(y', \mathbf{p}_3, \mathbf{p}_4); (x'_n, y') \in \mathbf{IR}_{A_n, B}\}$$

where $\mathbf{p}_1 = [\underline{A}_n \text{ cp}(A_n) \bar{A}_n]$; $\mathbf{p}_2 = [a_n \text{ cp}(A_n) b_n]$; $\mathbf{p}_3 = [\underline{B} \text{ cp}(B) \bar{B}]$ and $\mathbf{p}_4 = [c \text{ cp}(B) d]$. Finally, let $\mathbf{SR}_{A_n, B}^t = \mathbf{SR}_{A_n, B}$.

Having the transformed relations techniques FPL, SRM-I, and SRM-II methods can be executed to generate the conclusion B^* . These methods can be executed to each input universes like to an univariable rule as $R_n : A_n^t \mapsto B^t$ in respect to A_n^* and yield conclusion B_n^* on the output universe. The obtained sets B_n^* share a common support ($\forall n : \underline{B}_n^* = c; \bar{B}_n^* = d$, see Transformation 4) ensured by the transformation of the interrelation space. Taking the previous assumption into account, namely, if the shape of all observations and antecedents are equally considered in each input universe, the membership function of the final conclusion B^* is the average of the membership functions of B_n^* as

$$\mu_{B^*}(y) = \frac{1}{N} \sum_n \mu_{B_n^*}(y).$$

As a matter of fact, depending on the actual purposes in mind the averaging operator can be replaced by any convex combination techniques as outlined above.

The determination of the conclusion by the multidimensional interpolation can essentially be viewed as computing the average of the conclusions resulted by the univariable interpolation executed to each input–output pair. Consequently, all the properties investigated in Section VI can be stated for the multivariable algorithms introduced in this section as well.

IX. NUMERICAL EXAMPLES FOR MULTI-VARIABLE CASE

In Fig. 20, the results obtained by $I_{SC, FPL}(R_1, R_2, A^*)$ are tested for two-dimensional cases. There are three coordinate systems in Fig. 20(a)–(d). The first and second coordinate systems represent the input universes X_1 and X_2 , while the third one shows the output Y . The computer simulation allowed the use of fuzzy sets drawn by hand, permitting observation of all the particularities in the process. Fig. (c) shows a case, where the conclusion set can be determined difficultly if only human comprehension is used. Fig. (d) shows an example with crisp sets. The consequents of the rules are defined by the AND operation of the antecedents: B_k equals A_{1k} AND A_{2k} , $k = 1, 2$. One can observe that the resulting conclusion B' also equals A'_1 AND A'_2 .

Fig. 21 shows examples when triangular or trapezoidal sets are used in a two variable fuzzy rule base. This example compares the results of α -cut, *Vass*, and the proposed $I_{SC, FPL}(R_1, R_2, A^*)$ interpolation techniques. In Fig. 21(a), every method results in a normal fuzzy set conclusion. Comparing the results it can be said that the α -cut interpolation method usually gives similar conclusion with the proposed

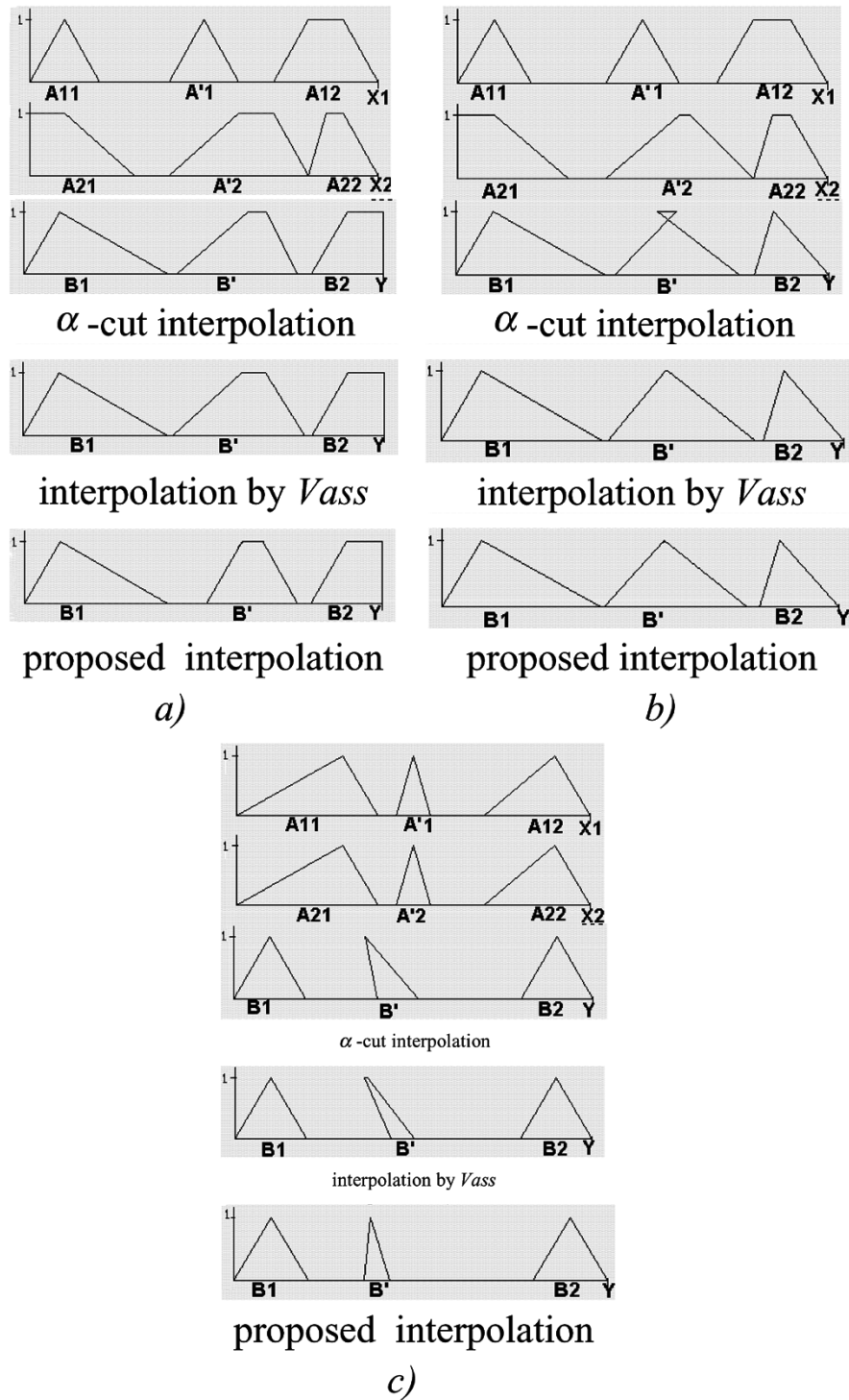


Fig. 21. Comparison of three different methods.

method. Fig. 21 (b) and (c) present examples, where the proposed interpolation method yields normal fuzzy sets while the others do not.

X. CONCLUSION

In this paper, a family of interpolation methods is proposed. These methods offer a structure to derive a family of fuzzy

rule interpolation techniques capable of avoiding the three typical deficiencies of interpolation techniques addressed as *abnormal conclusion*, *preserving linearity*, *restriction to CNF sets*. Some of the derived interpolation techniques are not restricted to convex fuzzy sets and always result in a fuzzy set unlike former interpolation techniques. The method is introduced as a relation interpolation in general sense in Section III, and is performed to fuzzy and semantic relations via some algorithms as possible implementations in Sections IV and V. Another contribution of

this paper to the topic of fuzzy rule interpolation is that the linear or nonlinear type of the interpolation can easily be changed by the help of interpolation generatrix without loosing the above advantages. Based on the concept of α -cut distance based techniques, developing a nonlinear interpolation which always results in a normal fuzzy set leads to a rather hard problem.

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