α-cut Interpolation Technique in the Space of Regular Conclusion

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Abstract –**The first published method of fuzzy rule interpolation is the α–cut distance based fuzzy interpolation. Its stable-approximation property has been deemed important not only in fuzzy theory, but in numerical analysis as well. However, its applicability is restricted by its interpretability in fuzzy theory. Many other fuzzy interpolation methods have been proposed to alleviate these restrictions, but they are rather complicated from a practical point of view. As a result the simplicity of α-cut fuzzy interpolation requiring small computational effort is lost. In this paper, a modified α-cut based fuzzy interpolation method is proposed which eliminates the problem of abnormal conclusion while maintains low computational capacity.**

I. INTRODUCTION

The classical approaches of fuzzy control deal with dense rule bases where the universe of discourse is fully covered by the antecedent fuzzy sets of the rule base in each dimension. On the other hand, fuzzy interpolation is a completely new technique of reasoning and control capable of handling rule bases with „gaps" in the input space. Since there is no information provided by the *if - then* rules in these gaps, hence, no consequent can be constructed by means of the classical methods should the observation fall in a gap.

The use of such sparse rule base and the accompanying fuzzy rule base interpolation technique has its importance in different aspects of fuzzy theory. First, the recently emerged research topic of fuzzy rule base reduction [1,2] was the main motivation of fuzzy rule interpolation [2]. The use of sparse rule base allow removal of redundant rules with proper technique even if the resulted rule set contains „gaps". Second, sparse rule base can be generated by tuning algorithms (*Burkhardt*, 1992, [9]). The starting term set is an α-cover type (every point in the input space covered by the rule antecedents is at least at level $\alpha \in (0,1]$). By tuning the rules, the antecedents are partially shifted and shrunk. The tuned model may therefore contain gaps [9]. Third, one has to deal with sparse rule base if only partial information is available about the modeled system modeled. Fourth, „Gaps" can be defined between rule bases. Hence, fuzzy interpolation technique has important role in hierarchically structured systems [14].

Besides the above, there is also a great interest among researchers in the field of mathematical analysis to finding a stable interpolating method. The inherent requirement of an interpolation technique is that it should be stable independently from the measurement points, e.g., it should converge to the approximated function regardless of the measurement points as long as their numbers converges to infinity. The classical methods in numerical analysis generally do not fulfil this condition. They require that the approximated function be sampled at some specially defined points, which in practice is hardly possible as the sampling process has certain randomness in itself.

It is a remarkable result that the α-cut distance based fuzzy interpolation technique of [2], (termed as KH-interpolation) actually has the stability property. The method maintains stability property not only in fuzzy theory, but also as a function approximator converging to the approximated function independent of the sampling points (*Joó et al.*, 1997, [10]). Furthermore, a class of stable interpolating function can be derived from the function of α-cut interpolation technique [10]. Despite its many advantages, the α-cut interpolation, however, is strongly restricted by the fact that in many cases it yields a conclusion that is not interpretable as a fuzzy set (abnormal conclusion). In order to alleviate this problem conditions are imposed by *Kovács* (1994, [11]), *Kawase, Chen* (1996, [19]) *Shi* and *Mizumoto* (1996, [8]) so as to yield a normal fuzzy set.

The deficiency of the α-cut interpolation technique inspired various conceptually different fuzzy interpolation approaches to be proposed. *Vas* and *Kalmár* have proposed an algorithm in 1992 [2] that reduces the problem of abnormal conclusion. *Gedeon* has published a method in 1996 [21] and its modification in 1997 [18] that is based on the conservation of relative fuzziness. A vague environment based interpolation technique has been proposed by *Kovács* in 1996 [7]. A general fuzzy interpolation and extrapolation techniques based on the interpolation of fuzzy- (1996 [12]), semantic- and interrelation (1998 [13]) have also been proposed. The technique ensures normal conclusion even in case when arbitrary type fuzzy sets are used. Normal conclusion is also ensured by the „vector mapping" based fuzzy interpolation proposed in 1997 [20]. In addition, *Kawaguchi* has proposed techniques based on triangular shaped rules in 1997 [16] and has published the fuzzy spline interpolation technique in 1998 [17].

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Though some of the new algorithms are able to solve the problem of abnormal conclusion even for the case of general fuzzy set, they are, however, conceptually different and much more complicated algorithms. In the frequent practical cases of using piece-wise linear sets with only three or four characteristic points (triangular or trapezoidal) the new methods maintain their relatively high computation complexity. This implies that the advantageous of the original technique, mainly, convenient practical applicability, simplicity, and the established stability property are lost. Without these advantageous the calculation complexity reduction property is destroyed.

The main objective of this paper is to modify the α-cut interpolation to ensure normal conclusion in the practical case when the fuzzy set are of a finite number of characteristic points while at the same time maintaining the advantageous of the original method. From theoretical point of view the general form is given for the use of sets with given analytically expressed membership functions.

II. α-CUT FUZZY DISTANCE BASED INTERPOLATION

The basic concepts of α-cut based interpolation, which include the definitions of convex normal fuzzy (CNF) sets, partial ordering of CNF sets and α-cut distance of CNF sets, are defined in the fuzzy literature (*Kóczy-Hirota*, 1992, [3]). Similar ideas can be found in the theory of gradual rules (*Dubois* and *Prade,* 1992, [64,5]). The sets involved in α-cut interpolation have to be CNF, and the state variables (including the input and the output universes as well) must be bounded and gradual which guarantees the existence of full ordering in each dimension. In this case a partial ordering can be introduced among the elements of X (i.e. among CNF sets) with the help of their α-cuts. If $\forall\alpha\in[0,1]: \inf\{A_\alpha\}\le\inf\{B_a\}$ and $\sup\{A_\alpha\}\le\sup\{B_a\}$ then A and B are comparable, i.e. $A \prec B$. Distance can be defined among comparable fuzzy sets. Two extremes of the α-cuts are considered for $\forall\alpha\in[0,1]$ and this pair-wise distances are named such as „lower" and „upper fuzzy" distance:

$$d_{\alpha L/U}(A_\alpha, B_\alpha) = \inf/\sup\{A_\alpha\} - \inf/\sup\{B_\alpha\}\,. \qquad (1)$$

For multi-dimensional case the *Minkowski* -distance is used.

$$\mathrm{d}_{\alpha L/U}(A_1, A_2) = \left(\sum_{i=1}^{n}\left(\mathrm{d}_{i\alpha L/U}(A_{i,1}, A_{i,2})\right)^W\right)^{\frac{1}{w}} \qquad (2)$$

For more details see [3].

If $\prec$ and $\mathrm{d}_{\alpha L/U}$ are interpretable on X and on Y, then given an observation $A^*(x)$ and the rules R_k: if A_k then B_k, k=1,2, where

$$A_1 \prec A^* \prec A_2 \text{ and } B_1 \prec B_2 \qquad (3)$$

conclusion $B^*(y)$ can be generated by the *fundamental equation of interpolation*:

$$\mathrm{d}(A^*, A_1) : \mathrm{d}(A_2, A^*) = \mathrm{d}(B^*, B_1) : \mathrm{d}(B_2, B^*)\,. \qquad (4)$$

(4) is an extension of the classical linear interpolation to CNF sets based rules, in fully accordance with the semantic interpretation of rules, as proposed by *Dubois* and *Prade* in 1992 [4]: "the more similarity we have between the observation and an antecedent, the more similar conclusion must be concluded to the corresponding consequent set". This semantic interpretation is the extended version of the analogical inference, proposed by *Turksen* in 1988 [6], and the meaning of revision principle, proposed by *Ding*, *Shen* and *Mukaidono* in 1993 [15].

The solution of the fundamental equation (4) by replacing (1) is:

$$\inf/\sup\{B^*_{\alpha i}\} = \frac{\dfrac{1}{\mathrm{d}_{\alpha L}(A^*_i, A_{i,1})}\inf/\sup\{B_{1\alpha}\} + \dfrac{1}{\mathrm{d}_{\alpha L}(A^*_i, A_{i,2})}\inf/\sup\{B_{2\alpha}\}}{\dfrac{1}{\mathrm{d}_{\alpha L}(A^*_i, A_{i,1})} + \dfrac{1}{\mathrm{d}_{\alpha L}(A^*_i, A_{i,2})}}$$

The KH-interpolation has been extended to more than two rules [2].

III. THE MODIFIED α–CUT BASED INTERPOLATION

This section proposes a modified α–cut based interpolation method that always results in normal conclusion. We first introduce the simplified vector description of [20], which represents the membership functions as points in high dimensional Cartesian spaces using the positions of the finite number of characteristic points as coordinates. With this representation, a fuzzy rule base then becomes a mapping between the antecedent and consequent Cartesian spaces. The region of normal and abnormal conclusion can also be readily characterised.

Suppose that two rules are given as a result of rule-selecting process as: *If* A_1 *then* B_1 and *If* A_2 *then* B_2, where A_1, A^*, A_2 are defined on X and B_1, B^*, B_2 are defined on Y. The selection is done on the same way as in the α–cut interpolation using the definition of ordering and ensuring (3).

The abnormality problem of the original α–cut based interpolation under the vector description of [20] is presented. Suppose that piece-wise linear fuzzy sets A_j are described by characteristic points as $\{(\alpha_i, a_{j,i}) \mid \alpha_i = \mu_{A_j}(a_{j,i})\}$. Suppose that two vectors contain values α_i as: $\underline{\alpha}_L = [\alpha_{L,i}]$ and $\underline{\alpha}_U = [\alpha_{U,i}]$.

Let us form two vectors for each CNF set A_j that contain values $a_{j,i}$ as: $\underline{\mathbf{a}}_{j,L/U} = [a_{L/U,j,i}]^{\mathrm{T}}$ (T means transpose), where L denotes the lower and U denotes the upper slope of the sets, namely

$\forall i : a_{L/U,j,i} \in [\inf/\sup\{A_{j,\alpha=0}\}, \inf/\sup\{A_{j,\alpha=1}\}]$.

The above is similar to the distance definition of the original α–cut based interpolation, where lower and upper α–cut distance is used considering the „lower" and „upper" slope of the sets separately (see (1)). Let us consider only the upper slope as the method is the same for the lower slope (for simplicity the notation L and U are dropped from now on). Suppose that the elements in $\underline{\alpha}$ are ordered as: $\alpha_k \geq \alpha_{k+1}$, $k = n_u - 1$, and $\alpha_1 = 1$ (n_u is the number of characteristic point of the upper slope). This implies in the case of CNF sets, that the conditions of normal CNF set A is:

$$\underline{\mathbf{a}} = [a_k]^{\mathrm{T}} : a_k \leq a_{k+1},\ k = n_u - 1 \tag{5}$$

Let us describe the original α–cut based interpolation in this form. The conclusion is calculated as:

$$\underline{\mathbf{b}}^* = \left(\underline{\underline{\mathbf{I}}} - \underline{\underline{\mathbf{I}}}\underline{\Lambda}\right)\underline{\mathbf{b}}_1 + \underline{\underline{\mathbf{I}}}\underline{\Lambda}\underline{\mathbf{b}}_2 \tag{6}$$

where $\underline{\underline{\mathbf{I}}}$ is the identity matrix

$$\underline{\Lambda} = [\lambda_i] : \lambda_i = \frac{a^*_i - a_{1,i}}{a_{2,i} - a_{1,i}}; \qquad i = n_u \tag{7}$$

Figure 1 shows a two dimensional subspace $V_k \times V_{k+1}$ for the k-th and (k+1)-th element of the vectors $\underline{\mathbf{a}}_1$, $\underline{\mathbf{a}}_2$ and $\underline{\mathbf{a}}^*$ as denoted by sub vector A_1, A_2, and A^*. The location of A^* satisfies (3), thus point A^* must be in the rectangle drown by dotted line in figure 1. The corresponding two dimensional consequent subspace $Z_k \times Z_{k+1}$ for the k-th and (k+1)-th element of the vectors $\underline{\mathbf{b}}_1$, $\underline{\mathbf{b}}_2$ and $\underline{\mathbf{b}}^*$, as denoted by B_1, B_2, and B^*, is depicted in figure 2. Vector $\underline{\mathbf{b}}^*$ would represent abnormal conclusion if point B^* is under line l: $z_{k+1} = z_k$, $z_v \in Z_v$ (see (5)). The possible area for conclusion is over line l.

Note that (3) and (7) ensures that $\forall i : \lambda_i \in [0,1]$, which implies that (6) is a convex combination. Therefore point B^* is always obtained in the rectangle drawn by dotted line in figure 2. If this rectangular is crossed by line l then there is always a chance for abnormal conclusion.

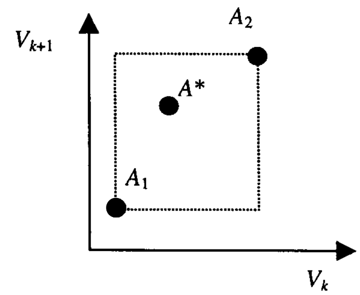

Figure 1

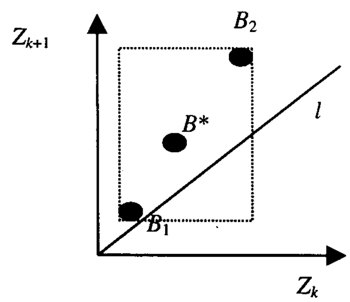

Figure 2

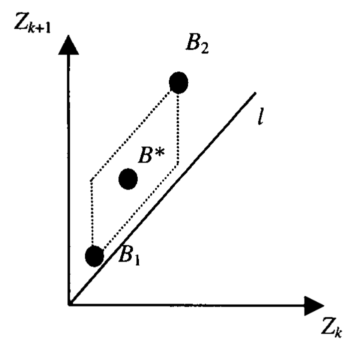

Figure 3

Eliminating the abnormal conclusion

We propose to map the possible region of conclusion to be above the line l, as in the area bordered by dotted line in figure 3. To do so, we transform vectors $\underline{\mathbf{b}}_j$, $j = 1..2$ to vector $\underline{\mathbf{b}}'_j$, which is the representation of the same point using the bases vectors along directions Z_{k+1} and line l. Then applying α–cut based interpolation (hkc) to the

transformed $\underline{\mathbf{b}}'_j$ vectors yield a „conclusion vector„ $\underline{\mathbf{b}}'^*$ that lies always in the parallelogram in figure 3. Hence normal conclusion is always obtained. For simplicity let us define the base vectors $\underline{z}_i = [z_{i,j}]$ of the new coordinate system as $j \ge i : z_{i,j} = 1$ and $j < i : z_{i,j} = 0$. The original coordinates b_i of all points defined by not negative coordinates in the new system hold $b_{i+1} \ge b_i$.
Let us summarize the proposed method.

1) Transformation of vectors $\underline{\mathbf{b}}_j$ to the new coordinate system.

$$\underline{\mathbf{b}}'_j = \underline{\underline{\mathbf{T}}}\underline{\mathbf{b}}_j$$

where $\underline{\underline{\mathbf{T}}} = [t_{r,s}]$, $t_{r,s} = 1$ if $r = s$ and $t_{r,s} = -1$ if $s = r-1$, otherwise $t_{r,s} = 0$.

2) Applying (6) as:

$$\underline{\mathbf{b}}'^* = \left(\underline{\underline{\mathbf{I}}} - \underline{\underline{\mathbf{I}}}\underline{\underline{\Lambda}}\right)\underline{\mathbf{b}}'_1 + \underline{\underline{\mathbf{I}}}\underline{\underline{\Lambda}}\underline{\mathbf{b}}'_2 \tag{8}$$

where $\underline{\underline{\mathbf{I}}}$ is identity matrix.

3) Transformation to the original coordinates:

$$\underline{\mathbf{b}}^* = \underline{\underline{\mathbf{T}}}^{-1}\underline{\mathbf{b}}'^*$$

namely $b^*_1 = b'^*_1$ and $b^*_k = b'^*_k + b'^*_{k-1}$, for $k = 2..n_c$.

Multi-variable antecedent case can be conducted similarly as the transformation applied only to the consequent part in (4).

Using sets given by membership functions

The original α–cut based interpolation for the upper (and lower) slope of the sets can in this case be written as: for $\lambda \in [0,1]$

$$f_{B^*}(\alpha) = (1-\lambda(\alpha)) f_{B_1}(\alpha) + \lambda(\alpha) f_{B_2}(\alpha), \tag{9}$$

where $f_B(\alpha) = \mu_B^{-1}(\alpha)$ and $\lambda(\alpha) = \dfrac{\mu_{A^*}^{-1}(\alpha) - \mu_{A_1}^{-1}(\alpha)}{\mu_{A_2}^{-1}(\alpha) - \mu_{A_1}^{-1}(\alpha)}$

Let the interval $\alpha \in [0,1]$ be divided into $n_u - 1$ equidistant intervals of $\Delta\alpha = \dfrac{1}{n_u - 1}$. Subintervals ($\alpha_k = 1-(k-1)\Delta\alpha$).

In this case the elements of vector $\underline{\mathbf{b}}'_j$ is the difference between the membership values as: $b'_{j,1} = \mu_{B_j}^{-1}(1)$; and

$$b'_{j,k} = \mu_{B_j}^{-1}(\alpha_k) - \mu_{B_j}^{-1}(a_k + \Delta\alpha);\ k = 2..n_u.$$

Dividing the elements in the vectors by $\Delta\alpha$ and increasing n_u to infinite leads to the key idea of transformation. Let the result of transformation of $\mu_{B_j}^{-1}(\alpha)$ be: $f_{B_j}(\alpha)$ and constant C_{B_j}, where

$$\mu_{B_j}^{-1}(\alpha) = \int f_{B_j}(\alpha) d\alpha + C_{B_j}. \tag{10}$$

The condition (5) becomes:

If $\forall\alpha : f_{B_j}(\alpha) \le 0$ then B_j is CNF set.

(for the lower slope: $\forall\alpha : f_{B_j}(\alpha) \ge 0$).

Using the original α–cut based interpolation (9) $f_{B^*}(\alpha)$ is obtained, that is always ≤ 0 as (9) is the convex combination of $f_{B_j}(\alpha) \le 0$. Using (10), where the C_{B^*} is obtained from:

$$\mu_{B^*}^{-1}(1) = (1-\lambda(1))\mu_{B_1}^{-1}(1) + \lambda(1)\mu_{B_2}^{-1}(1)$$

fully in accordance with (8), when the first elements of the vectors is calculated as $b^*_1 = \mu_{B^*}^{-1}(1)$.

IV. CONCLUSION

In this paper, a fuzzy rule interpolation technique is presented and compared to former methods. The proposed method is a modification of the original proposed α-cut interpolation function. The modified version eliminates the problem regarding abnormal conclusion, while maintaining the simplicity and the advantages of the original method.

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