Integrator Backstepping using Barrier Functions for Systems with Multiple State Constraints

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Motivating problem

Consider the idealized model of the longitudinal dynamics of an aircraft:

- Control margin provided by elevators is sufficient for all required manoeuvres.
- Magnitude constraints on:
  - Climb rate due to stall.
  - Pitch attitude and pitch rate due to “passenger comfort” factor.
- Control design must guarantee that the constrained states always remain below their limits.
Some flight control design techniques

Gain scheduled linear control design:
Does not explicitly include constraint limits. Constrained designs tend to be based on low gain arguments.

Feedback linearization:
Demands accurate an dynamic model. Does not directly incorporate constraints.

Model Predictive Control:
Requires a good model and is computationally intensive.

Forwarding:
Incorporates input constraints and can be extended to state constraints. Low gain design.
Backstepping

➤ Affords control designers great freedom in selecting final control law.

➤ Able to accommodate large nonlinearities and uncertainties in system’s model, ignored dynamics, input/measurement disturbances.

➤ Aircraft longitudinal dynamics can be transformed into “lower-triangular feedback” form.

➤ The lack of ‘gain-limiting’ feedforward interconnection terms means no gain restriction on intermediate control signals.

➤ High gains are required to impose state constraints.
Motion systems dynamics

\[ \dot{\xi}_1 = \xi_2 \] Translation position (unconstrained)
\[ \dot{\xi}_2 = \xi_3 \] Translation velocity
\[ \dot{\xi}_3 = \xi_4 \] Attitude
\[ \dot{\xi}_4 = b(\xi) + a(\xi)u \] Angular velocity

Magnitude constraints on system states

\[ |\xi_i(t)| \leq \Xi_i, \quad i = 2, 3, 4 \]

The proposed approach can be extended to:

➤ Arbitrary lower triangular systems.
➤ Systems of arbitrary order.
Step I: Growth condition on first state for cLf

Consider the first scalar system

\[ \dot{\xi}_1 = \xi_2 \]

Since \( \xi_1 \) is unbounded the cLf design must incorporate a growth bound on its dependence on \( \xi_1 \).

\[ V_1(\xi_1) = O(\xi_1), \quad \text{as} \quad |\xi_1| \to \infty \]

A candidate cLf is

\[ V_1(\xi_1) = k_1 \xi_1 \arctan(\xi_1), \quad k_1 > 0 \]

The time differential of \( V_1 \) is bounded in \( \xi_1 \)

\[ \dot{V}_1 = k_1 \xi_2 \left[ \arctan(\xi_1) + \frac{\xi_1}{1 + \xi_1^2} \right] \]
Step I: Bounded reference trajectory

Define the error variable $z_1$

$$z_1 = \xi_2 - \xi_{2_{ref}}$$

The reference trajectory $\xi_{2_{ref}}$ must be bounded.

$$\xi_{2_{ref}} = -c_1 \arctan(\xi_1), \quad c_1 > 0$$

By choosing bounded growth in $V_1$ and bounded reference trajectory, $\xi_{2_{ref}}$, this ensures bounded propagation of virtual error in the back-stepping procedure.
Step I: Error dynamics of $\xi_1$-subsystem

\[
\begin{align*}
\dot{\xi}_1 &= -c_1 \arctan(\xi_1) + z_1 \\
\dot{V}_1 &= -k_1 c_1 \arctan(\xi_1) \left[ \arctan(\xi_1) + \frac{\xi_1}{1 + \xi_1^2} \right] \\
&\quad + k_1 z_1 \left[ \arctan(\xi_1) + \frac{\xi_1}{1 + \xi_1^2} \right] \\
&\leq -W(\xi_1) + k_1 z_1 \left[ \arctan(\xi_1) + \frac{\xi_1}{1 + \xi_1^2} \right]
\end{align*}
\]

where $W_1(\xi_1)$ is positive-definite in $\xi_1$, 

Multi-state constrained backstepping.  
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Step II: Barrier function on $z_1$

Need to choose an augmented cLf $V_2$ that takes $V_1$ and adds a term for the stability of the $z_1$ state. We propose to incorporate a barrier condition on the cLf $V_2$ with respect to $z_1$

$$|z_1| \to k_2 \implies V_2(z_1) \to \infty$$

A candidate cLf is

$$V_2(\xi_1, z_1) = V_1 + \frac{1}{2} k_3 \log \left( \frac{k_2^2}{k_2^2 - z_1^2} \right)$$

- Local quadratic structure of $V_2$

  $$V_2 \approx k_1 \xi_1^2 + k_3 z_1^2, \quad \text{for } |\xi_1|, |z_1| \text{ small}$$

- Linear growth in $\xi_1$.

- Unbounded growth in $V_2$ as $z_1 \to k_2$. 

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Step II: Choosing stabilising function $\xi_{3_{\text{ref}}}$

The time-derivative of $V_2$ is

$$\dot{V}_2 = -W_1(\xi_1) + z_1 \left[ k_1 \arctan(\xi_1) + \frac{k_1 \xi_1}{1 + \xi_1^2} + \frac{k_3 \dot{z}_1}{k_2^2 - z_1^2} \right]$$

The stabilizing function $\xi_{3_{\text{ref}}}$ is chosen as

$$\xi_{3_{\text{ref}}} = -c_2 z_1 - \frac{c_1 \xi_2}{1 + \xi_1^2} - \frac{(k_2^2 - z_2^2)}{k_3} \left[ k_1 \arctan(\xi_1) + \frac{k_1 \xi_1}{1 + \xi_1^2} \right],$$

Thus,

$$\dot{z}_1 = -c_2 z_1 - \frac{(k_2^2 - z_2^2)}{k_3} \left[ k_1 \arctan(\xi_1) + \frac{k_1 \xi_1}{1 + \xi_1^2} \right] + z_2$$

$$\dot{V}_2 = -W_1 - \frac{k_3 c_2 z_1^2}{k_2^2 - z_1^2} + \frac{k_3 z_1 z_2}{k_2^2 - z_1^2}$$
Step II: Boundedness of state $\xi_2$

Assume for a moment that $\xi_2$ is directly controlled and one can set $z_2 = 0$. Then

$$\dot{V}_2 = -W_1 - \frac{k_3 c_2 z_1^2}{k_2^2 - z_1^2} \Rightarrow V_2(t) < V_2(0)$$

It follows that for all time $t > 0$

$$|z_1| < k_2$$

and consequently that

$$|\xi_2| = |z_1 + \xi_{2,ref}| < k_2 + \frac{\pi}{2} c_1$$

Choosing $k_2$ and $c_1$ such that

$$k_2 + \frac{\pi}{2} c_1 \leq \Xi_2$$

guarantees the first state bound holds for all time.
Step III: Continue back-stepping process

Define the error signal variable

\[ z_3 = \xi_4 - \xi_{4_{\text{ref}}} \]

A candidate cLf is

\[ V_3 = V_2 + \frac{1}{2}k_5 \log \left( \frac{k^2_4}{k^2_4 - z^2_2} \right) \]

The time derivative of \( V_3 \) is

\[ \dot{V}_3 = -W_1 - \frac{k_3c_2z^2_1}{k^2_2 - z^2_1} + \frac{k_3z_1z_2}{k^2_2 - z^2_1} + \frac{k_5z_2}{k^2_4 - z^2_2} \dot{z}_2 \]

Choose stabilising function

\[ \xi_{4_{\text{ref}}} = -c_3z_2 + \dot{\xi}_{3_{\text{ref}}}, \quad c_3 > 0, \quad \Rightarrow \quad \dot{z}_2 = -c_3z_2 + z_3 \]
Step III: Stability analysis

The cross-term \( \frac{k_3 z_1 z_2}{k_2^2 - z_1^2} \) cannot be directly canceled.

The cLf time derivative is given by

\[
\dot{V}_3 = -W_1 - \frac{k_3 c_2 z_1^2}{k_2^2 - z_1^2} + \frac{k_3 z_1 z_2}{k_2^2 - z_1^2} - \frac{k_5 c_3 z_2^2}{k_4^2 - z_2^2} + \frac{k_5 z_2 z_3}{k_4^2 - z_2^2}
\]

need to show negative semi-definite

Case I: If \((k_2 - |z_1|)\) is large relative to \((k_4 - |z_2|)\) then it is possible to choose gains to ensure that the three terms are negative using a ‘completion of squares’ argument.

Case II: If \((k_2 - |z_1|)\) is small relative to \((k_4 - |z_2|)\) then \(|z_1| > > 0\). Since \(|z_4| < k_4\), it is possible to choose the gains such that

\[-k_3 c_2 z_1^2 + k_3 z_1 z_2 < 0\]
Step IV: Crank the handle

\[
\begin{bmatrix}
\dot{\xi}_1 \\
\dot{z}_1 \\
\dot{z}_2 \\
\dot{z}_3 \\
\end{bmatrix} =
\begin{bmatrix}
-A_{11} & 1 & 0 & 0 \\
-A_{21} & -c_2 & 1 & 0 \\
0 & 0 & -c_3 & 1 \\
0 & 0 & 0 & -c_4 \\
\end{bmatrix}
\begin{bmatrix}
\xi_1 \\
z_1 \\
z_2 \\
z_3 \\
\end{bmatrix}
\]

where

\[
A_{11} = c_1 \frac{\arctan(\xi_1)}{\xi_1}
\]

\[
A_{21} = \left(\frac{k_2^2 - z_1^2}{k_3}\right)
\left[k_1 \frac{\arctan(\xi_1)}{\xi_1} + \frac{k_1}{1 + \xi_1^2}\right]
\]

For suitable choice of \(c_1 \cdots c_4\) it is straightforward to show stability of the error system.

However, to guarantee the constraints are satisfied at all times then it is necessary to show a non-negative decrease property of constructed cLf \(V_4\).
Step V: State bounds and control tuning

Let \( Z = (k_1, \ldots, k_{2n}, c_1, \ldots, c_n) \) denote the set of gains used.

At each back-stepping step there were two bounds derived from the stability analysis

\[
\begin{align*}
c_i &\geq \frac{k_{2i}^2}{[k_i \alpha_{i-1}]^2} + \frac{1}{2} \quad \Rightarrow \quad \Phi_i(Z) \geq 0 \\
\frac{k_{2j+1} c_{j+1}}{k_{2j}^2} &\geq \frac{k_{j+1}}{k_j^2 - [k_j \alpha_{j-1}]^2} \quad \Rightarrow \quad \Psi_j(Z) \geq 0
\end{align*}
\]

There are also the original bounds on the states that are interpreted as constraints on \( Z \)

\[
|\xi_k(t)| = |z_{k-1}(t) + \xi_{kref}(t)| = |X_k(k_1, \ldots, k_{2n}, c_1, \ldots, c_n)| < \Xi_k, \quad k = 2, \ldots, n
\]

Optimisation problem:

Minimise \( F(Z) = \sum (\Xi_i - |X_i(Z)|) / (\Xi_i) \) over \( Z \) subject to \( \{\Phi_i(Z) \geq 0\} \) and \( \{\Psi_j(Z) \geq 0\} \) and \( \{|X_k(Z)| < \Xi_k\} \).
Simulation for initial position error of 10m

Multi-state constrained backstepping.
Simulation for initial position error of 100m
Conclusions

➤ Use the flexibility of the backstepping procedure to introduce barrier function characteristics in the constructed control Lyapunov function. This introduces parameterized soft bounds on the state error variables in the transformed system.

➤ Derive stability conditions on the gain choices at each iteration of back-stepping procedure to guarantee stability.

➤ Derive a set of parameter bounds for the control tuning based on the state bounds.

➤ Pose a constrained optimisation problem in the control parameters whose solution leads to good closed-loop performance while preserving the state constraints.