Penny-shaped Crack in a Solid Piezoelectric Cylinder
With Two Typical Boundary Conditions

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Abstract: The response of elastic stress and electric displacement in a long piezoelectric cylinder with a concentric penny-shaped crack is investigated in this paper. The long piezoelectric cylinder is subjected to two types of boundary conditions: ① the piezoelectric cylinder is inserted in a smooth rigid bore of radius b, ② the surface of the piezoelectric cylinder is stress and electric charge free. Based on the potential function approach and Hankel transform, a system of dual integral equation is obtained, and then reduced to a Fredholm integral equation of the second kind. Numerical results of various field intensity factors for cylinder are obtained to show the effect of the ratio a/b on fracture behaviours of the cracked piezoelectric cylinder (a is the radius of the crack and b is the radius of the PZT-63 cylinder).

Key words: Piezoelectric cylinder; fracture; penny-shaped crack


1 Introduction

It is recognized that piezoelectric materials have been extensively used as transducers and sensors in a composite configuration in adaptive structures due to their intrinsic direct and converse piezoelectric effect[11]. In recent years, significant efforts had been made to the study of fracture behaviors of piezoelectric materials in the presence of cracks[2-4]. Among various crack problems, a penny-shaped crack in a cylinder is most popular and has many reports in the literature[5-10]. Sneddon[5-7] analyzed the distribution of stress in a long cylinder of elastic material when it is deformed by the application of pressure to the inner surfaces of a penny-shaped crack situated with its centre on the axis of the cylinder and its plane perpendicular to that axis under two types of boundary condition. Parhi and Atsumi[8] considered the same problem of[5-7] in a transversely isotropic infinite cylinder. Dhaliwal[9] studied the state of stress in a long elastic cylinder with a concentric penny-shaped crack bonded to an infinite elastic medium. Case and Reifsnider[10] presented an analysis of a penny-shaped crack in the center of multiple concentric cylinders.

The study of infinite piezoelectric cylinders containing a penny-shaped crack under combined electrical and mechanical loads has received some attention in recent years. Using the Fourier and Hankel transforms, Narita et al[11] obtained the stress intensity factor, the total energy release rate, and the mechanical strain energy release rate for a penny-shaped crack in a piezoceramic cylinder under mode I loading. Yang and Lee[12] based on the potential function approach and Hankel transform and Lin[13] based on Fourier and Hankel trans-
forms investigated a piezoelectric cylinder with a penny-shaped crack embedded in an infinite matrix.

In this paper, the electroelastic problem of a penny-shaped crack in a long piezoelectric cylinder under combined electrical and mechanical loads is considered using the potential function approach and Hankel transform. The emphasis is placed on investigating the fracture behaviors of the penny-shaped crack in a piezoelectric cylinder affected by the following two different boundary conditions, which are similar to that of a piezoelectric cylinder inserted in a smooth rigid bore of radius $b$, or the surface of the piezoelectric cylinder is stress and electric charge free. The mixed boundary conditions lead to a system of dual integral equations, and then these integral equations are reduced to a Fredholm equation of the second kind, which are solved by the use of Gaussian quadrature formulae. The mechanical and electrical fields and all sorts of field intensity factors of mode 1 are obtained, and numerical calculations on a cracked PZT-6B piezoelectric ceramics are carried out. Consequently, the effect of the ratio $a/b$ (the radius of the crack to radius of the cylinder) on crack propagation are shown in several figures, and the effect of boundary condition in radial direction is also discussed.

2 Problem Statement and Basic Equation

Consider a piezoelectric cylinder of radius $b$ containing a centered penny-shaped crack of radius $a$ under axisymmetric electromechanical loads (Fig. 1). For convenience, a cylindrical coordinate system $(r, \theta, z)$ originated at the center of the crack is used, with the $z$-axis along the axis of symmetry of the cylinder. The cylinder is assumed to be a transversely isotropic piezoelectric material with the poling direction parallel to the $z$-axis. It is subjected to the far-field of a normal stress, $\sigma_r = \sigma(r)$ and a normal electric displacement, $D_r = D(r)$.

The constitutive equations for piezoelectric materials which are transversely isotropic and poled along the $z$-axis can be written as:

\begin{align}
\sigma_r &= c_{12} \frac{\partial u_r}{\partial r} + c_{11} \frac{u_r}{r} + c_{13} \frac{\partial u_z}{\partial z} + e_{31} \frac{\partial \varphi}{\partial z} \\
\sigma_z &= c_{13} \frac{\partial u_r}{\partial r} + c_{13} \frac{u_r}{r} + c_{33} \frac{\partial u_z}{\partial z} + e_{33} \frac{\partial \varphi}{\partial z} \\
\sigma_{rz} &= c_{13} \left( \frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z} \right) + e_{15} \frac{\partial \varphi}{\partial r} \\
D_r &= e_{15} \left( \frac{\partial u_r}{\partial r} + \frac{u_r}{r} \right) - d_{11} \frac{\partial \varphi}{\partial r} \\
D_z &= e_{33} \left( \frac{\partial u_r}{\partial r} + \frac{u_r}{r} \right) + d_{33} \frac{\partial \varphi}{\partial z} - d_{33} \frac{\partial \varphi}{\partial z}
\end{align}

where $\sigma_{ij}, D_i$ are the components of stress tensor and electric displacement vector respectively, and $u_r, u_z$ denote the displacements in the $r$- and $z$-directions respectively.

In the derivation of the analytic solution, the following potential functions are introduced:

\begin{equation}
\begin{align*}
    u_r &= \sum_{i=1}^{1} k_{11} \frac{\partial \Phi_i}{\partial r}, \quad u_z = \sum_{i=1}^{1} k_{11} \frac{\partial \Phi_i}{\partial z}, \quad \varphi = -\sum_{i=1}^{2} k_{2i} \frac{\partial \Phi_i}{\partial z}
\end{align*}
\end{equation}

where $u_i (k = r, z)$ are displacements, $\varphi$ is electric potential, $\Phi_i(r, z) (i = 1, 2, 3)$ are the potential functions, $k_{1i}$, and $k_{2i} (i = 1, 2, 3)$ are unknown constants in the piezoelectric medium.

Combining the constitutive equations (1a)-(1e), the field equations and gradient equations and use the
potential functions (2), we have the following equations

\[ c_{11} \left( \frac{\partial \Phi_i}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi_i}{\partial r} \right) + \left( c_{44} + k_1 (c_{13} + c_{44}) + k_2 (e_{31} + e_{15}) \right) \frac{\partial \Phi_i}{\partial z^2} = 0 \]  
(3a)

\[ \left[ c_{44} k_1 + c_{13} + c_{44} + e_{15} k_2 \right] \frac{\partial \Phi_i}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi_i}{\partial r} + \left[ c_{33} k_1 + e_{33} k_2 \right] \frac{\partial \Phi_i}{\partial z^2} = 0 \]  
(3b)

\[ \left[ e_{15} k_1 + e_{31} + e_{15} - d_{11} k_2 \right] \frac{\partial \Phi_i}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi_i}{\partial r} + \left[ e_{33} k_1 - d_{33} k_2 \right] \frac{\partial \Phi_i}{\partial z^2} = 0 \]  
(3c)

Following Yang and Lee\[^{[12]}\), we take the solution of the equations (3a)-(3c) in the form

\[ \Phi_i (r, z) = \int_0^\infty \frac{1}{\xi} \left[ A_i (\xi) I_0 \left( \frac{\xi r}{s_i} \right) \cos (\xi z) + B_i (\xi) \exp (- \xi s) J_0 (\xi r) \right] d \xi \]  
(4)

where \( A_i (\xi), B_i (\xi) \) (\( i = 1, 2, 3 \)) are the unknown functions to be determined. \( J_n \) is the Bessel functions of the first kind of order \( n \), and \( I_n \) is the modified Bessel function of the first kind and the second kind of \( n \) order.

Then we have the expressions of components of displacement, stress and electric displacement in the following form.

\[ u_r (r, z) = - \sum_{i=1}^3 k_i \int_0^\infty A_i (\xi) I_0 \left( \frac{\xi r}{s_i} \right) \sin (\xi z) d \xi - \sum_{i=1}^3 k_i A_i \int_0^\infty B_i (\xi) J_0 (\xi r) e^{-\xi z} d \xi + \frac{a (r) z}{r} \]  
(5a)

\[ u_r (r, z) = \sum_{i=1}^3 \frac{1}{s_i} \int_0^\infty A_i (\xi) I_1 \left( \frac{\xi r}{s_i} \right) \cos (\xi z) d \xi - \sum_{i=1}^3 \frac{1}{s_i} \int_0^\infty B_i (\xi) J_1 (\xi r) e^{-\xi z} d \xi \]  
(5b)

\[ \varphi (r, z) = \sum_{i=1}^3 k_2 \int_0^\infty A_i (\xi) I_1 \left( \frac{\xi r}{s_i} \right) \sin (\xi z) d \xi + \sum_{i=1}^3 k_2 s_i \int_0^\infty B_i (\xi) J_1 (\xi r) e^{-\xi z} d \xi - \frac{b (r) z}{r} \]  
(5c)

\[ \sigma_r = - \sum_{i=1}^3 \int_0^\infty A_i (\xi) I_0 \left( \frac{\xi r}{s_i} \right) \cos (\xi z) d \xi + \sum_{i=1}^3 \int_0^\infty B_i (\xi) J_0 (\xi r) e^{-\xi z} d \xi + \frac{c (r)}{r} \]  
(5d)

\[ \sigma_r = - \sum_{i=1}^3 \int_0^\infty A_i (\xi) I_1 \left( \frac{\xi r}{s_i} \right) \sin (\xi z) d \xi + \sum_{i=1}^3 \int_0^\infty B_i (\xi) J_1 (\xi r) e^{-\xi z} d \xi + \frac{c_1 - c_2}{2} \sum_{i=1}^3 \int_0^\infty A_i (\xi) I_2 \left( \frac{\xi r}{s_i} \right) \cos (\xi z) d \xi + \frac{c_1 - c_2}{2} \sum_{i=1}^3 \int_0^\infty B_i (\xi) J_2 (\xi r) e^{-\xi z} d \xi \]  
(5e)

\[ \sigma_r = - \sum_{i=1}^3 \int_0^\infty A_i (\xi) I_1 \left( \frac{\xi r}{s_i} \right) \sin (\xi z) d \xi + \sum_{i=1}^3 \int_0^\infty B_i (\xi) J_1 (\xi r) e^{-\xi z} d \xi + \frac{c_1 + c_2}{2} \sum_{i=1}^3 \int_0^\infty A_i (\xi) I_2 \left( \frac{\xi r}{s_i} \right) \cos (\xi z) d \xi + \frac{c_1 + c_2}{2} \sum_{i=1}^3 \int_0^\infty B_i (\xi) J_2 (\xi r) e^{-\xi z} d \xi \]  
(5f)

\[ D_r = - \sum_{i=1}^3 \int_0^\infty A_i (\xi) I_1 \left( \frac{\xi r}{s_i} \right) \cos (\xi z) d \xi + \sum_{i=1}^3 \int_0^\infty B_i (\xi) J_1 (\xi r) e^{-\xi z} d \xi + \frac{d (r)}{r} \]  
(5g)

\[ D_r = - \sum_{i=1}^3 \int_0^\infty A_i (\xi) I_1 \left( \frac{\xi r}{s_i} \right) \sin (\xi z) d \xi + \sum_{i=1}^3 \int_0^\infty B_i (\xi) J_1 (\xi r) e^{-\xi z} d \xi + \frac{d (r)}{r} \]  
(5h)

in which

\[ F_{1i} = (c_{33} k_1 - e_{33} k_2), \quad F_{2i} = (e_{33} k_1 + d_{33} k_2), \quad F_{3i} = (c_{44} (1 + k_1) - e_{15} k_2), \quad F_{4i} = (e_{15} (1 + k_1) + d_{11} k_2), \quad F_{5i} = (c_{13} k_1 - e_{33} k_2), \quad \frac{c_{11} + c_{12}}{2} \]  
(6)

\[ \frac{a (r)}{r} = \frac{d_{33} \sigma (r) + e_{33} D (r)}{c_{33} d_{33} + e_{33}^2}, \quad \frac{b (r)}{r} = \frac{c_{33} D (r) + e_{33} \sigma (r)}{c_{33} d_{33} + e_{33}^2}, \quad \frac{c (r)}{r} = \frac{a (r)}{r}, \quad \frac{d (r)}{r} = \frac{D (r)}{r} \]  
(7)

3 Derivation of Integral Equations

We shall consider separately two sets of boundary conditions.

Case 1. In the first case we assume that the piezoelectric cylindrical surface is free from shear and is supported in such a way that the radial component of the displacement vector vanishes on the surface. Such a situ-
ation would arise physically if the piezoelectric cylinder was resting in a cylindrical hollow (of exactly the same radius) in a rigid body and was then deformed by the application of a known stress and an electric displacement to the end of the piezoelectric cylinder. The problem of determining the distribution of stress and electric displacement in the vicinity of the crack is equivalent to that of finding the distribution of stress and electric displacement in the semi-infinite cylinder \( z \geq 0 \), \( 0 \leq r < a \), when its plane boundary \( z = 0 \) is subjected to the condition

\[
\sigma_z (r, 0) = 0 \quad (0 \leq r < a) \quad (8a)
\]

\[
u_z (r, 0) = 0 \quad (a < r < b) \quad (8b)
\]

\[
\varphi (r, 0) = 0 \quad (a < r < b) \quad (8c)
\]

\[
\sigma_{\varphi} (r, 0) = 0 \quad (0 \leq r < b) \quad (8d)
\]

\[
D_{\varphi} (r, 0) = D_{\varphi} (r, 0) \quad (0 \leq r < a) \quad (8e)
\]

\[
E_{\varphi} (r, 0) = E_{\varphi} (r, 0) \quad (0 \leq r < a) \quad (8f)
\]

and its curved boundary \( r = b \) is subjected to the conditions

\[
u_z (b, z) = 0, \quad \sigma_{\varphi} (b, z) = 0, \quad D_{\varphi} (b, z) = 0, \quad z \geq 0
\]

(9)

From the boundary conditions (8c)-(8f) and (9) and making use of the Fourier inversion theorem and the Hankel inversion theorem we can obtain a system of dual integral equation

\[
- \int_0^\infty \xi \left[ F_{11} I_0 \left( \frac{\xi r}{s_1} \right) A_1 (\xi) + F_{12} I_0 \left( \frac{\xi r}{s_2} \right) A_1 (\xi) + F_{13} I_0 \left( \frac{\xi r}{s_3} \right) A_1 (\xi) \right] d\xi + \int_0^\infty \xi [M_1 F_{11} + M_2 F_{12} + M_3 F_{13}] B_1 (\xi) I_0 (\xi r) d\xi = - \varphi (r) \quad (0 \leq r < a)
\]

\[
\int_0^\infty \left[ M_1 I_{11} s_1 + M_2 I_{12} s_2 + M_3 I_{13} s_3 \right] B_1 (\xi) J_0 (\xi r) d\xi = 0 \quad (a < r < b)
\]

(10)

This equation can be solved by using the function \( \psi (a) \) defined by

\[
B_1 (\xi) = \int_0^a \psi (a) \sin (\xi a) da
\]

(11)

where \( \psi (0) = 0 \).

We can obtain a Fredholm integral equation of the second kind in the following form

\[
\varphi (a) + \int_0^a \psi (\beta) L (a, \beta) d\beta = \frac{2}{\pi m_0} \int_0^a \frac{\varphi (r)}{\sqrt{a^2 - r^2}} dr
\]

(12)

In which

\[
L (a, \beta) = \frac{4}{\pi m_0} \sum_{i=1}^3 \frac{F_{ij}}{s_j} \int_0^\infty \frac{1}{\Delta (\xi)} \sinh \left( \frac{\xi a}{s_j} \right) \sum_{i=1}^3 \frac{1}{s_i} N_{ij} (\xi) \sinh \left( \frac{\xi b}{s_i} \right) K_1 \left( \frac{\xi b}{s_i} \right) d\xi
\]

(13)

**Case 2.** In the second case we assume that the piezoelectric cylindrical surface is stress free. The conditions (8a)-(8f) remain the same, and the boundary conditions (9) are replaced by the following conditions:

\[
\sigma_{\varphi} (b, z) = 0, \quad \sigma_{\varphi} (b, z) = 0, \quad D_{\varphi} (b, z) = 0 \quad (z \geq 0)
\]

Conducting the procedure similar to that in case 1, we can obtain a Fredholm integral equation of the second kind which is exactly the same as that given in Eq. (12), except the kernel \( L (a, \beta) \) takes the following form.

\[
L (a, \beta) = \frac{4}{\pi m_0} \sum_{i=1}^3 \frac{F_{ij}}{s_j} \int_0^\infty \frac{1}{\Delta (\xi)} \sinh \left( \frac{\xi a}{s_j} \right) \sum_{i=1}^3 \frac{1}{s_i} \sinh \left( \frac{\xi b}{s_i} \right) \left[ N_{ij} (\xi) K_1 \left( \frac{\xi b}{s_i} \right) - \frac{\xi P_{ij} (\xi)}{s_i} K_0 \left( \frac{\xi b}{s_i} \right) + \frac{\xi W_{ij} (\xi)}{s_i} K_2 \left( \frac{\xi b}{s_i} \right) \right] d\xi
\]

(14)

Each kind of the field intensity factors is obtained in the form
\[ K' = K_1 = \lim_{r \to a} \frac{2\pi (r-a) \sigma_r (r,0)}{a} = \sqrt{\frac{\pi}{a}} m_0 \psi (a) \]

\[ K^D = \lim_{r \to a} \frac{2\pi (r-a) D_x (r,0)}{a} = \sqrt{\frac{\pi}{a}} m_1 \psi (a) \]

\[ K^' = \lim_{r \to a} \frac{2\pi (r-a) \varepsilon_x (r,0)}{a} = \sqrt{\frac{\pi}{a}} m_2 \psi (a) \]

\[ K^E = \lim_{r \to a} \frac{2\pi (r-a) E_x (r,0)}{a} = \sqrt{\frac{\pi}{a}} m_3 \psi (a) \]

(15)

in which

\[ m_0 = -(M_1 F_{11} + M_2 F_{12} + M_3 F_{13}) \]

\[ m_1 = -(F_{21} M_1 + F_{22} M_2 + F_{23} M_3) \]

\[ m_2 = -(k_{11} s_{12}^2 M_1 + k_{12} s_{12}^2 M_2 + k_{13} s_{12}^2 M_3) \]

\[ m_3 = -(k_{21} s_{12}^2 M_1 + k_{22} s_{12}^2 M_2 + k_{23} s_{12}^2 M_3) \]

(16)

and \( K', K^D, K^E \) are the stress intensity factor, electric displacement intensity factor, strain intensity factor and electric field intensity factor, respectively.

4 Numerical Results and Discussion

Material properties of PZT-6B ceramic are as follows, elastic constants \((10^6 \text{ N/m}^2)\) \(c_{11} = 16.8, c_{12} = 6.0, c_{33} = 16.3, c_{44} = 2.71; \) piezoelectric constants \((\text{C/m}^2)\) \(e_{13} = 4.6, e_{31} = -0.9, e_{33} = 7.1; \) dielectric permittivities \((10^{-10} \text{ F/m})\) \(d_{11} = 36, \alpha_{13} = 34.\)

From Eqs. (5a)-(5h), it is clear that once the functions \(A_i (\xi), B_i (\xi)\) are known, the stress and electric displacement inside the piezoelectric cylinder can be obtained. The determination of the stress’s intensity factor requires the solution of the function \(\psi (\xi)\). The Fredholm integral equation of the second kind (16) can be solved numerically using Gaussian quadrature formula. Then we can obtain all sorts of intensity factors using equation (20).

It can be found easily that the stress intensity factor is only dependent on the mechanical loading if the piezoelectric cylinder is under the far-field stress and electric displacement. The change of the normalized stress intensity factor, electric displacement intensity factor and strain intensity factor with the ratio of crack radius to PZT-6B cylinder radius is shown in Fig. 2, where the normalized variables are introduced \(T_1 = \frac{k^r}{2c_0 (a/u)^{1/2}}, T_2 = \frac{k^D}{2c_0 (a/p)^{1/2}}, T_3 = \frac{k^e}{2c_0 (a/p)^{1/2}}.\)

It can be found easily that the stress intensity factor is only dependent on the mechanical loading if the piezoelectric cylinder is under the far-field stress and electric displacement in these two loading cases. This behavior is same to those of Narita\(^{[11]}\) and Yang and Lee\(^{[12]}\). The change of the normalized stress intensity factor, electric displacement intensity factor and strain intensity factor with the ratio of crack radius to PZT-6B cylinder radius is shown in Fig. 2. It can be seen that all these intensity factors have similar distribution along the dimensionless crack radius. When the value of \(a/b\) becomes from 0 to 0.65, the normalized intensity factors keep constant, but when that becomes larger than 0.65, these intensity factors increase with the increase in \(a/b\). However, in case 2 the intensity factors increases more rapidly than that in case 1, which may be caused by the different loading condition in surface of the piezoelectric cylinder in radial direction.

5 Conclusion

In this paper the solutions of the field equations and the field intensity factors for a penny-shaped crack in
a piezoelectric cylinder under two different loading cases are obtained by the potential theory and Hankel transform using the electric continuous boundary condition on the crack surfaces. The stress intensity factor is only dependent on the mechanical loading if the piezoelectric cylinder is under the far-field stress and electric displacement. The tendency of the field intensity factors with the dimensionless crack radius \( a/b \) is also plotted and discussed, then it comes to an conclusion that different loading condition in radial direction may cause different increase rate in these intensity factors with the value of \( a/b \).

References:

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Analysis of Electromagnetic Wave Guide Based on Symplectic System

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Abstract: Based on the Hamilton formulation for electro-magnetic wave guides, the finite element semi-analytical transverse discretization under the symplectic formulation is used to solve the problems of wave guide with complicated cross sections. This method can be applied to anisotropic materials, easier for treating interface conditions between different materials. The symplectic structure should be kept unchanged for various interface conditions. Examples of rectangular wave guide, rectangular waveguide with T-septum, layered waveguide, anisotropic waveguide are presented. The numerical results approach the theoretical results.

Key words: waveguide; Hamilton system; symplectic; semianalytical method

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2 种典型边界条件下压电圆柱体中的变形裂纹

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摘 要：研究了含有同心变形裂纹的压电圆柱体的弹性应力和电位移响应。长圆柱体承受2种典型的边界条件，将其分别为压电圆柱插入到刚性简内及压电圆柱的表面无应力电荷。基于势函数法和 Hankel 变换，得到对偶积分方程，然后缩减为第二类 Fredholm 积分方程，最终得到了 PZT-6B 圆柱的场强度因子的数值结果。研究结果表明，裂纹与圆柱的半径比对断裂行为有显著的影响。

关键词：压电圆柱体；断裂；变形裂纹

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