Application of Trefftz Method to Heat Conduction Problem in Functionally Graded Materials§

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Abstract: In this paper we present a Trefftz method for solving steady and transient heat conduction problems in FGMs, including nonlinear FGMs whose properties are dependent on temperature. For the case of steady heat transfer, T-complete solutions of governing equation for nonlinear exponential FGMs are derived by Kirchhoff transformation and coordinate transformation, and then, they can be used to model the temperature fields. For transient case, the analog equation method is used to convert the original governing equation to an equivalent Poisson’s equation. Then, the homogeneous solution is obtained by linear combination of a set of T-complete solutions while the radial basis functions (RBF) are employed to approximate the inhomogeneous terms. Finally, by enforcing satisfaction of the governing equation and boundary conditions at collocation points of the original problem, in which the time domain is discretized by time-stepping method, a Trefftz-RBF scheme is established. The performance of the proposed methods are assessed through three numerical examples. The results are presented for illustrating the accuracy and efficacy of the proposed numerical models.

Keywords: Functionally graded material (FGM), steady heat conduction, transient heat conduction, T-complete solution, Trefftz method, RBF.

1. INTRODUCTION

Functionally graded materials (FGM) [1] are a new generation of composite material whose material properties continuously from one surface to another unlike a composite which has stepped (or discontinuous) material properties. The gradation of properties in a FGM reduces the thermal stresses, residual stresses, and stress concentrations found in traditional composites. For instance, a material that transitions smoothly from a pure metal to a pure ceramic would integrate the advantages of the two components (metal and ceramic here) [2]; the refractory coatings made of FGM in turbine engine can protect engine form high-temperature [3-5]. As a consequence, FGMs have been widely used in aerospace technology and space structures such as the Space Solar Power System which is to supply a large quantity of energy from space to the ground, and which is expected to become operational in 10-20 years [6]. In addition, a FGM’s gradation in material properties allows the designer to tailor material response to meet design criteria. For example, the Space Shuttle utilizes FGM tiles as thermal protection from heat generated during re-entry into the Earth’s atmosphere, as a FGM can alleviate stress concentrations caused by traditional materials by gradually changing material properties through-the-thickness of the material but still provide the thermal protection found in conventional thermal shielding. Since FGMs are always working under harsh conditions in aerospace, including high temperature and large temperature gradients, it is necessary to know the thermal properties of the materials and thermal performance of the structures. However, thermal analysis of FGMs is more complex than that of homogeneous materials due to their continuously varying material properties. Moreover, heat conduction in nonlinear FGMs whose material properties vary not only with Cartesian coordinates but also with temperature together with the heat conduction being transient makes the analysis far more difficult as both the temperature-dependent and time-dependent terms are involved in heat conduction equations.

During the past decades various numerical models have been developed for analyzing the mechanical behaviour of FGMs. For example, finite element method (FEM)[7], the boundary element method (BEM) [8, 9] or dual reciprocity BEM [10, 11] have been widely used to analyze the static or dynamic thermal response of FGMs. Additionally, meshless or meshfree approaches, such as the meshless local Petrov-Galerkin (MLPG) method [12], the meshless local boundary integral equation (LBIIE) method [13] and the method of fundamental solution (MFS) [14-18] have also been developed for solving heat conduction problems in both isotropic or anisotropic, single or multi-materials, linear or nonlinear FGMs. Among the above methods, FEM critically depends on the quality of meshes; however, generating a good quality of mesh for complicated geometry can be time-consuming. BEM involves only discretization of the boundaries which is an important advantage over FEM. However, the classical use of BEM for transient fields [19], based on discretization in time, usually results in domain

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integrals which may increase computing time and even cause some numerical problems and make BEM relatively inefficient compared to FEM. Alternatively, an attractive option is the meshless discretization approach which has received considerable attention by mathematicians and engineers in recent years. Meshless methods only use a number of nodes scattered within the problem domain and on the boundary. Among the existing meshless methods, the techniques most commonly used are the method of fundamental solutions (MFS) [20-23]. The classical MFS is based on the approximation of the solution of a Boundary Value Problem (BVP) by a linear combination of fundamental solutions to the corresponding differential operator. The boundary conditions are then fitted by solving the linear system formed using a number of a collocation points. However, MFS has its limitation in that it can only solve homogeneous problems efficiently. The combination of MFS and RBF enables one to extend MFS to non-homogeneous and parabolic problems and various problems of type-dependent problems [24, 25]. When using MFS, it requires to place source points outside the domain in order to avoid singularity caused by the fundamental solutions, but the location of virtual boundary is still an open question.

Unlike MFS which needs source points to be placed outside the domain in order to avoid singularity, the Trefftz method is formulated using the family of T-complete functions which are non-singularity inside and on the boundary of the given region. The Trefftz method was initiated in 1926 [26]. Since then, it has been studied by many researchers (Cheung et al. [27], Zielinski [28], Qin [29-31], Kita [32]). Moreover, T-complete function can reflect various properties of FGMs. In this paper, we develop a Trefftz method for analysing both steady and transient heat conduction problems in FGMs, including nonlinear FGMs.

First, the derivation of the T-complete function for steady heat conduction equation is carried out then the Trefftz method for steady heat conduction problem in FGMs is introduced. Next, Trefftz-RBF model is developed by means of analog equation method (AEM) to analyse transient heat conduction in FGMs. Typically, the paper is organized as follows. The Trefftz method for steady heat conduction and Trefftz-RBF scheme for transient heat conduction in FGMs are introduced in Sections 2 and 3 respectively, followed by numerical validations in terms of some steady and transient heat conduction problem in FGMs in Section 4, and finally, some concluding remarks are presented based on the reported results in Section 5.

2. TREFFTZ METHOD FOR STEADY HEAT CONDUCTION IN FGMs

2.1. Basic Formulas of Steady Heat Conduction

Consider a two-dimensional (2D) heat conduction problem defined in an anisotropic inhomogeneous medium without heat sources:

\[ \sum_{j=1}^{2} \frac{\partial}{\partial x_j} (K_{xy}(\mathbf{X}, \mu) \frac{\partial u(\mathbf{X})}{\partial x_j}) = 0 \quad \forall \mathbf{X} \in \Omega \]  

(1)

where the nonlinear functionally graded material considered here whose thermal conductivity varies exponentially with position vector and also be a function of temperature. That is

\[ K_y(\mathbf{X}, \mu) = \alpha(\mu) \bar{K}_y \exp(2\beta \cdot \mathbf{X}) \]  

(2)

where \( \alpha(\mu) > 0 \) is a temperature-dependent function. Eq.(1) is reduced to linear FGM heat conduction equation when \( \alpha(\mu)=1 \) in Eq.(2). Vector \( \beta = (\beta_1, \beta_2) \) is a dimensionless graded parameter. Matrix \( \bar{K} = \begin{bmatrix} \bar{K}_{yy} \\ \bar{K}_{xy} \\ \bar{K}_{yx} \\ \bar{K}_{xx} \end{bmatrix} \) is a symmetric, positive-definite constant matrix \( \bar{K}_{12} = \bar{K}_{21}, \det \bar{K} = \bar{K}_{11} \bar{K}_{22} - \bar{K}_{12}^2 > 0 \).

The boundary conditions are as follows:

1. Dirichlet boundary condition
   \[ u = \bar{u} \quad \text{on} \quad \Gamma_u \]  

(3)

2. Neumann boundary condition
   \[ q = \bar{q} \quad \text{on} \quad \Gamma_q \]  

(4)

where \( K_y \) denotes the thermal conductivity which is the function of spatial variable \( X \) and unknown temperature field \( u \), \( q = -\sum_{j=1}^{2} K_{xy} \frac{\partial u}{\partial X_j} n_j \) represents the boundary heat flux. \( n \) is the direction cosine of the unit outward normal vector \( n \) to the boundary \( \Gamma = \Gamma_u \cup \Gamma_q \). \( \bar{u} \) and \( \bar{q} \) are specified functions on the related boundaries, respectively.

2.2. T-Complete Solutions

The nonlinear and anisotropic properties of Eq. (1) make it difficult to generate the related Trefftz functions. To bypass this problem, the Kirchhoff transformation and mathematical variable transformation are used in the derivation [33].

By employing the Kirchhoff transformation

\[ \Psi(u) = \psi(\alpha(X)) = \int \omega(\mu) du \]  

(5)

Eqs. (1) can be reduced to the following form

\[ \sum_{j=1}^{2} K_{xy} \frac{\partial^2 \Psi(X)}{\partial X_j \partial X_j} + 2\beta \cdot (K \nabla \Psi(X)) \exp(2\beta \cdot X) = 0 \]  

(6)

Similarly, Eqs.(3)-(4) are transformed into the corresponding boundary conditions in terms of \( \Psi \) under Kirchhoff transformation

\[ \Psi(\bar{u}) = \psi(\bar{X}) \quad \text{on} \quad \Gamma_u \]  

(7)

\[ p = -\sum_{j=1}^{2} K_{xy} \frac{\partial \Psi}{\partial X_j} n_j = -\sum_{j=1}^{2} \bar{K}_{xy} \frac{\partial u}{\partial X_j} n_j = q = \bar{q} \quad \text{on} \quad \Gamma_q \]  

(8)

To simplify the expression of Eqs. (6)-(8), set

\[ \Psi(X) = \Phi \exp(\beta \cdot X), \]  

there Eqs. (6)-(8) can be rewritten as follows:

\[ \sum_{j=1}^{2} \bar{K}_{xy} \frac{\partial \Phi(X)}{\partial X_j} - \lambda^2 \Phi(X) \exp(\beta \cdot X) = 0 \]  

(9)

\[ \Phi = \psi(\bar{u}) \exp(\beta \cdot X) \quad \text{on} \quad \Gamma_v \]  

(10)
\[ q(x) = -\sum_{i,j=1}^{2} K_{ij} \left( \frac{\partial \Phi(X)}{\partial x_j} - \beta \Phi(X) \right) n_i(X) \exp(\beta \cdot X) = \bar{q} \quad \text{on} \quad \Gamma \quad (11) \]

in which
\[ \lambda = \sqrt{\beta K \beta} \quad (12) \]

Making the change of variables \[ [16], \]
\[ Y = QX \] (13)
where
\[ QKQ^T = I \] (14)
so that
\[ Q = \begin{bmatrix} \frac{1}{\sqrt{K_{11}}} & 0 \\ -\frac{1}{\sqrt{K_{12}}} & \sqrt{\frac{K_{11}}{K_{12}}} \Delta \end{bmatrix} \] (15)
where \( \Delta \) = det \( (K) = K_{11} K_{22} - K_{12}^2 > 0 \)

Then Eq. (9) is transformed into the modified Helmholtz equation
\[ \sum_{i=1}^{2} \frac{\partial^2 \Phi(Y)}{\partial Y_i^2} + \lambda^2 \Phi(Y) = 0 \] (16)
whose T-complete solutions are already known [34], hence we have T-complete solutions of Eq. (9) as
\[ N = \{ I_n(\lambda r) \cos n\theta \}_{n=0}^{\infty} \cup \{ I_n(\lambda r) \sin n\theta \}_{n=1}^{\infty} \] (17)
where polar coordinates \( (r, \theta) \) with \( r = 0 \) at the centroid of \( \Omega \) is used, and \( r = \sqrt{Y_1^2 + Y_2^2} \), \( \theta = \arctan \left( \frac{Y_2}{Y_1} \right) \), \( I_n \) denotes the \( n \)-order modified Bessel function of first kind.

2.3. Trefftz Method

The homogeneous solution to (9)-(11) is approximated as
\[ \Phi(X) = \sum_{n=0}^{\infty} \alpha_n N_n(X) \] (18)
where \( \alpha_n \) are the coefficients to be determined and \( \epsilon \) is the number of components. The terms \( N_n(X) = N(r) = N([X - X]) \) are the T-complete solutions of Eq.(9) and given in Eq.(17). \( \{X_i\}_{n=0}^{\infty} \) are collocation points placed on the physical boundary of the solution domain. As an illustration, the internal function \( N_i \) in Eq. (18) can be given in the form
\[ N_i = I_0(\lambda r), N_2 = I_1(\lambda r) \cos \theta, N_3 = I_1(\lambda r) \sin \theta, \ldots, \] (19)

Eq. (18) can, thus, be written as
\[ \Phi(X) = \sum_{n=0}^{\infty} \sum_{n=1}^{x} c_n I_n(\lambda r) \cos n\theta + \sum_{n=1}^{x} d_n I_n(\lambda r) \sin n\theta \] (20)
where \( \epsilon = 2k + 1 \). Noted that \( \Phi(X) \) in Eq. (20) automatically satisfies the given differential equation (9), all we need to do is to enforce \( \Phi(X) \) to satisfy the modified boundary conditions (10)-(11). To do this, collocation points \( \{X_{i}\}_{n=0}^{\infty} \) are placed on the physical boundary to fit the boundary conditions (10)-(11). It leads to a system of linear algebraic equations in matrix form:
\[ [A]_{m \times m} [c]_{m \times 1} = [b]_{m \times 1} \] (21)
with
\[ [c] = \{ c_0, c_1, \ldots, c_m \}, \quad [b] = \{ b_1, b_2, \ldots, b_m \} \] (22)

If the number of unknown components equals to the number of collocation points on the physical boundary \( e = m \), this leads to properly determined equations. Alternatively, in the case of the number of unknown components being smaller than the number of collocation points \( e < m \), this results in over-determined equations. The least square method can be used to solve the over-determined equations. Once \( \{x\} \) is obtained, \( \Phi(X) \) can be computed at any location in the domain using Eq. (18).

It should be pointed out that the potential field obtained by the above Trefftz method is the solution of Eqs.(9)-(11). Therefore, it needs to use two inverse transformations in obtaining the temperature field \( T \):
1) \( \Psi(X) = \Phi \exp(-\beta \cdot X) \), and 2) \( u(X) = \psi^{-1}(\Psi(X)) \).

3. TRANSIENT HEAT CONDUCTION IN FGMs

3.1. Basic Formulas of Transient Heat Conduction

Consider a two-dimensional (2D) transient heat conduction problem:
\[ \nabla \cdot (k(X) \nabla u(X, t)) = k(X) \nabla^2 u(X, t) + \nabla k(X) \cdot \nabla u(X, t) \]
\[ = \rho(X)c(X) \frac{\partial u(X, t)}{\partial t} \quad \forall X \in \Omega \] (23)

where \( t \) denotes the time variable \( (t > 0) \), \( \rho \) is the mass density, \( c \) the specific heat. Besides the boundary conditions (3)-(4), an initial condition must be given for the time dependent problem. In this paper, a zero initial temperature distribution is considered, i.e.
\[ u(X, 0) = u_0(X) = 0 \] (24)

For convenience, boundary conditions (3)-(4) are expressed in a general form as
\[ B_1 u(X, t) + B_2 q(X, t) = B_3(X, t) \] (25)
where \( B_1, B_2, \) and \( B_3 \) are known coefficients and can be written respectively as
\[ \begin{bmatrix} B_1 & B_2 & B_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = \frac{\alpha}{\Gamma_\alpha} \quad \text{on} \quad \Gamma_\alpha \]
\[ \begin{bmatrix} B_1 & B_2 & B_3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} = \frac{\alpha}{\Gamma_\beta} \quad \text{on} \quad \Gamma_\beta \] (26)

3.2. Analog Equation Method (AEM)

The analog equation method (AEM) can be used to convert Eq. (23) into a Poisson-type equation, we have
\[ V^2u(\mathbf{X},t) = \frac{\rho(\mathbf{X})c(\mathbf{X})}{k(\mathbf{X})} \frac{\partial u(\mathbf{X},t)}{\partial t} - \frac{1}{k(\mathbf{X})} \nabla k(\mathbf{X}) \cdot \nabla u(\mathbf{X},t) \]  

(27)

Eq. (23) can be restated as

\[ V^2u(\mathbf{X},t) = b(\mathbf{X},t) \]  

(28)

where

\[ b(\mathbf{X},t) = \frac{\rho(\mathbf{X})c(\mathbf{X})}{k(\mathbf{X})} \frac{\partial u(\mathbf{X},t)}{\partial t} - \frac{1}{k(\mathbf{X})} \nabla k(\mathbf{X}) \cdot \nabla u(\mathbf{X},t) \]  

(29)

The solution of Eq. (28) can be expressed as a summation of a particular solution \( u_p \) and a homogeneous solution \( u_h \), that is:

\[ u(\mathbf{X}) = u(\mathbf{X})_h + u(\mathbf{X})_p \]  

(30)

where \( u_p \) satisfies the inhomogenous equation

\[ V^2u_p(\mathbf{X}) = b(\mathbf{X}) \]  

(31)

but does not necessarily satisfy the boundary conditions (3)-(4), and \( u_h \) satisfies

\[ V^2u_h(\mathbf{X}) = 0 \]  

(32)

and modified boundary conditions

\[ \begin{align*}
&u_h(\mathbf{X},t) = \overline{u}(\mathbf{X},t) - u_p(\mathbf{X},t) & \text{on } \Gamma_u \\
&q_h(\mathbf{X},t) = \overline{q}(\mathbf{X},t) - q_p(\mathbf{X},t) & \text{on } \Gamma_q
\end{align*} \]  

(33, 34)

### 3.3. T-Complete Solutions for Homogeneous Solution

In the implementation of Trefftz method, the homogeneous solution is approximated in a standard collocation fashion

\[ u_h(\mathbf{X},t) = \sum_{i=1}^{m} \alpha_i(t) N_i(\mathbf{X}, \mathbf{X}_i) \]  

(35)

where \( m \) is the number of collocation points placed on the physical boundary. \( \alpha_i \) are the coefficients to be determined. The terms \( N_i \) are the T-complete solutions of Eq. (32). As an illustration, the internal function \( N_j \) in Eq. (35) is given in the form [34]

\[ N_i = 1, N_j = r \cos \theta, N_k = r \sin \theta, \ldots, \]  

(36)

So, Eq. (35) can be written as

\[ u_h(\mathbf{X},t) = \sum_{n=0}^{m} r^n \cos n\theta + \sum_{n=1}^{m} r^n \sin n\theta \]  

(37)

where \( m = 2k + 1 \).

### 3.4. Radial Basis Functions (RBF) for Particular Solution

The particular solution \( u_p \) can be obtained by means of RBF. To do this, the right-hand side term of Eq. (28) is approximated by using RBF as [35]

\[ b(\mathbf{X},t) = \sum_{j=1}^{n} \beta_j(t) \varphi_j(\mathbf{X}, \mathbf{X}_j) \]  

(38)

where \( n \) is the number of interpolation points in the domain under consideration. Here, \( \varphi_j(\mathbf{X}, \mathbf{X}_j) = \varphi_j(r) = \varphi_j(\|\mathbf{X} - \mathbf{X}_j\|) \) denotes radial basis functions with the reference point \( \mathbf{X}_j \) and \( \beta_j \) are interpolating coefficients to be determined.

Simultaneously, the particular solution \( u_p \) is similarly expressed as

\[ u_p(\mathbf{X},t) = \sum_{j=1}^{n} \beta_j(t) \varphi_j(\mathbf{X}, \mathbf{X}_j) \]  

(39)

where \( \varphi_j \) represent corresponding approximated particular solutions which satisfy the following differential equation

\[ V^2 \varphi_j = \varphi_j \]  

(40)

noting the relation between Eq. (31), (38) and Eq. (39).

The choice of the RBF’s \( \varphi \) is very important because it affects the effectiveness and accuracy of the interpolation. There are several types of RBF, like ad hoc basis function \((1 + r)\), the polyharmonic splines, power spline (PS), thin plate spline (TPS) and multiquadrics (MQ). In this study, we choose \( \varphi \) as a PS:

\[ \varphi_j = r^2 \ln r \]  

(41)

Form Eq. (40), \( \varphi \) can be determined as

\[ \psi_j = \frac{\mathcal{L}^{2k+1}}{(2n+1)^2} \]  

(42)

### 3.5. Time Stepping Method and Construction of Solving Equations

Based on the equations derived above, the solutions \( u(\mathbf{X}) \) to differential equation (23) can be written as

\[ u(\mathbf{X},t) = u(\mathbf{X},t)_h + u(\mathbf{X},t)_p = \sum_{i=1}^{m} \alpha_i(t) N_i(\mathbf{X}, \mathbf{X}_i) + \sum_{j=1}^{n} \beta_j(t) \varphi_j(\mathbf{X}, \mathbf{X}_j) \]  

(43)

and heat flux is

\[ q(\mathbf{X},t) = -k(\mathbf{X}) \left( \sum_{i=1}^{m} \alpha_i(t) \frac{\partial N_i}{\partial n} + \sum_{j=1}^{n} \beta_j(t) \frac{\partial \varphi_j}{\partial n} \right) \]  

(44)

It is convenient for computer programming to utilize the vector form. Therefore, Eq. (43) can be written as

\[ u(\mathbf{X},t) = \{U(\mathbf{X})\} \{A(t)\} \]  

(45)

where

\[ \{U(\mathbf{X})\} = \begin{bmatrix} N_1(\mathbf{X}), & N_2(\mathbf{X}), & \ldots, & N_m(\mathbf{X}), & \varphi_1(\mathbf{X}), & \varphi_2(\mathbf{X}), & \ldots, & \varphi_n(\mathbf{X}) \end{bmatrix}_{m+n} \]  

(46)

\[ \{A(t)\} = \begin{bmatrix} \alpha_1(t), & \alpha_2(t), & \ldots, & \alpha_m(t), & \beta_1(t), & \beta_2(t), & \ldots, & \beta_n(t) \end{bmatrix}_{m+n} \]  

(47)
Accordingly, the heat flux in the normal direction of the boundary can be expressed as follows:
\[
q(X,t) = \{Q(X)\} \{A(t)\}
\]  
where
\[
\{Q(X)\} = -k(X) \left\{ \begin{array}{c}
\frac{\partial N_1(X)}{\partial n_1} \\
\frac{\partial N_2(X)}{\partial n_2} \\
\vdots \\
\frac{\partial N_m(X)}{\partial n_m}
\end{array} \right\}
\]  
(49)

Hence, we reconstruct Eq. (23) and obtain
\[
\rho(x)\dot{c}(x) \cdot \{U(X_i)\} \{\dot{A}(t)\}
\]
\[
- \left\{k(X) \cdot \nabla^2 \{U(X_i)\} + \nabla k(X) \cdot \nabla \{U(X_i)\}\right\} \{\dot{A}(t)\} = 0
\]  
\(j = 1, 2, \ldots, n\)  
(50)

with the boundary condition (25) which can also be reformulated as
\[
\left\{B_i \{U(X_i)\} + B_i \{Q(X_i)\}\right\} \{\dot{A}(t)\}
\]
\[
= B_i \{A(X_i)\} \quad (i = 1, 2, \ldots, m)
\]  
(51)

Equation (43) can be set to satisfy the requirements of the governing equation (23) at n points \(x_j\) in the domain and the boundary conditions at m boundary points \(x_i\), so as to solve the unknowns can be determined. Comparing Eq. (50) with Eq. (51), a typical general form of linear equation system can be generated for the sake of computer programming convenience as shown below:
\[
[M] \{\dot{A}(t)\} + [K] \{A(t)\} = \{F\}
\]  
(52)

where
\[
[M] = \rho(x)\dot{c}(x) \left\{ \begin{array}{c}
\{U(X_1)\} \\
\{U(X_2)\} \\
\vdots \\
\{U(X_n)\}
\end{array} \right\}
\]  
\(\begin{array}{c}
0 \\
0 \\
\vdots \\
0
\end{array} \right\}_{(n+1) \times (n+1)}
\]

\[
[K] = \left\{ \begin{array}{c}
-\left\{k(X) \cdot \nabla^2 \{U(X_1)\} + \nabla k(X) \cdot \nabla \{U(X_1)\}\right\} \\
-\left\{k(X) \cdot \nabla^2 \{U(X_2)\} + \nabla k(X) \cdot \nabla \{U(X_2)\}\right\} \\
\vdots \\
-\left\{k(X) \cdot \nabla^2 \{U(X_n)\} + \nabla k(X) \cdot \nabla \{U(X_n)\}\right\}
\end{array} \right\}
\]  
\(\begin{array}{c}
\beta_i \{U(X_1)\} + \beta_i \{Q(X_1)\} \\
\beta_i \{U(X_2)\} + \beta_i \{Q(X_2)\} \\
\vdots \\
\beta_i \{U(X_n)\} + \beta_i \{Q(X_n)\}
\end{array} \right\}_{(n+1) \times (n+1)}
\]

From Eq. (52), it is apparent that both coefficient vectors \([M]\) and \([K]\) are independent of time. The interpolating coefficients can be determined by Eq. (52), which is solved by using time stepping method [36]. Applying time stepping method to Eq. (52), yields
\[
\{A\}^{n+1} = \{A\}^n + \tau \left[ (1 - \theta)\{\dot{A}\}^n + \theta \{\dot{A}\}^{n+1} \right]
\]  
(57)

and
\[
\left[ \frac{[M]}{\tau} + \theta[K] \right] \{A\}^{n+1} = \left[ \frac{[M]}{\tau} - (1 - \theta)[K] \right] \{A\}^n + (1 - \theta)\{F\}^n + \theta\{F\}^{n+1}
\]  
(58)

where the superscripts \(n\) and \(n+1\) refer to subsequent time instances and \(\tau = t^{n+1} - t^n\) is the time step size. \(\theta (0 \leq \theta \leq 1)\) is a real parameter that determines whether the method is explicit (\(\theta = 0\)), implicit (\(\theta = 1\)) or a linear combination of both types [23]. The special choice of \(\theta = 1/2\) is known in the literature as the Crank-Nicolson scheme. It is easily verified that the conditions which prevent oscillation in the explicit case are exactly the same as the commonly cited sufficient conditions which ensure that it is stable. Furthermore, even though a Crank-Nicolson approach is unconditionally stable, it permits the development of spurious oscillation unless the time step size is no more than twice that required for an explicit method to be stable. Although an implicit scheme is only first-order accurate in time, it has been proved that the parital differential equation can be solved accurately using the implicit scheme [37]. Hence, we use \(\theta = 1\) in our analysis, Eq. (58) reduces to
\[
\left[ [M] + [K] \Delta t \right] \{A\}^{n+1} = [M] \{A\}^n
\]  
(59)

Note that the right-hand side of Eq. (59) is well defined at the previous time step \(t = t^n\). The initial coefficients \(\{A\}^0\) can be calculated from initial condition. Start the procedure with \(\{A\}^0\), the unknown coefficients \(\{A\}\) at step \(t = t^{n+1}\) can be gained by solving Eq. (59). Once interpolating
coefficients are determined, the temperature \( u(\mathbf{X}, t) \) can be known by Eq. (43) at each time step.

4. NUMERICAL EXAMPLES

In order to demonstrate the efficiency and accuracy of the proposed meshless method for heat conduction in FGMs, three benchmark numerical examples of heat conduction problems are considered for which corresponding exact solutions are known and can be used for verification. These examples cover the cases of steady heat conduction in FGM plate whose thermal conductivities vary spatially (example 1), nonlinear FGM plate whose properties are dependent of temperature (example 2) and transient heat conduction in FGM plate (example 3).

In addition, to provide a quantitative understanding of the results, the average relative error \( Arrerr(f) \) and normalized error defined respectively by

\[
Arrerr(f) = \frac{\sqrt{\sum_{i=1}^{N} \left( f^{(num)}_i - f^{(ex)}_i \right)^2}}{\sqrt{\sum_{i=1}^{N} \left( f^{(ex)}_i \right)^2}}
\]

\[
Nerr(w) = \frac{\max_{0 \leq (x,y) \leq 1} \left| f^{(num)} - f^{(ex)} \right|}{\max_{0 \leq (x,y) \leq 1} \left| f^{(ex)} \right|}
\]

where \( N \) is the number of test points and \( f^{(num)} \) and \( f^{(ex)} \) are the numerical and exact result of the field variable.

**Example 1** Steady heat conduction in anisotropic FGM plate

Assume a square plate \( \Omega = \{ X = (X_1, X_2) \mid 0 < X_1, X_2 < 1 \} \) graded along direction \( X_2 \). The thermal conductivity is \( K_y(X) = K_y \exp(2\beta_1 X_2) \), the corresponding value in Eq. (2) is \( a(T) = 1, \kappa = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \beta_1 = 0, \beta_2 = 0.2 \).

The analytical solution is

\[
u(X) = T - \frac{1}{1 - \exp(-2(\beta_1 X_2 + \beta_2 X_2))} \frac{1 - \exp(-2(\beta_1 + \beta_2))}{1 - \exp(-2(\beta_1))}
\]

subject to the following boundary conditions

\( q(X) = 0, X_1 = 0 \), \( q(X) = 0, X_1 = 1 \), \( T(X) = 0, X_2 = 0 \), \( T(X) = 100, X_2 = 1 \).

Since the properties of the FGM are independent of temperature, there is no need to use the Kirchhoff transformation in this example.

In order to investigate the effect of the component number \( e \) in Eq. (18) the number of components is chosen as 10, 20, 30 and 40, respectively, and 40 collocation points in the calculation. Table 1 presents respectively the effect of the terms \( e \) of T complete solutions on the average relative error \( Arrerr \). It can be seen from Table 1 that the results gradually converge to the exact values as the number of components \( (e) \) increases. This can be explained by that the least square method can achieve better numerical accuracy in solving the over-determined equations when the number of unknowns is close to the number of equations. But, from Table 1 we also observe that a larger \( e \) leads to a larger condition number of matrix \( A \), which is not beneficial to some complex problem. So, the optimal value of \( e \) should be found by numerical experimentation. The value of \( e \) is taken to be the same as the number of collocation points on the boundary in the following numerical simulation.

**Table 1. Numerical Results of Example 1 with Different Terms \( e \) of T-Complete Solutions**

<table>
<thead>
<tr>
<th>( e )</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Arrerr )</td>
<td>4.1044e-7</td>
<td>4.0998e-7</td>
<td>4.0893e-7</td>
<td>4.0889e-7</td>
</tr>
<tr>
<td>( Cond(A) )</td>
<td>129.72e8</td>
<td>21283e19</td>
<td>37026e31</td>
<td>10001e51</td>
</tr>
</tbody>
</table>

**Fig. (1).** Normalized error distribution for temperature for Example 1.

**Example 2** Steady heat conduction in nonlinear FGM plate

Consider the steady heat transfer in a nonlinear FGM whose coefficients of heat conduction are defined by Eq. (2) with \( a(u) = 1 + \gamma u \). This problem usually occurs in high-temperature environments. By using the Kirchhoff transformation, we can obtain

\[
\Psi(u) = u + \frac{1}{2} \gamma u^2 \quad u = u^{(1)} \quad \Psi = \frac{-1 + \sqrt{1 + 2\gamma\Psi}}{\gamma}
\]

Let us consider an orthotropic material [16] in the square \( \Omega = (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{2}, \frac{1}{2}) \) in which \( \kappa = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \) and \( \beta_1 = 0, \beta_2 = 1 \). When \( \gamma = 0.5 \), its analytical solution is...
where \( c = X_1 / \sqrt{2} - 1 \), \( p = \sqrt{c^2 + X_2^2} \).

20 collocation points are used in the calculation. Fig. (2) illustrates the temperature distribution in the FGM plate and Fig. (3) shows the corresponding isothermal distribution. It can be seen that the numerical solution matches very well with the analytical solution.

Fig. (2). Numerical temperature distribution for Example 1.

Fig. (3). Numerical isothermals for Example 2.

**Example 3** Transient heat conduction in FGM plate

A 0.04 × 0.04 square plate graded along \( X_1 \) direction is considered. The thermal conductivity and specific heat are defined as \( k = k_0 e^{2\beta x} \) and \( c = c_0 e^{2\beta x} \), respectively. Where \( k_0 = 17 W / m \cdot ^oC \), \( c_0 = 1.0 \times 10^6 J / kg \cdot ^oC \). The boundary condition is shown in Fig. (4) Heaviside function as \( u = T \cdot H(t) \), where \( T = 10^3 C \). The analytical solution is [9]

\[
u = T \left( 1 - e^{2\beta x} \right) \sum_{n=1}^{\infty} \frac{2x e^{-\beta n x}}{\beta^n L} \sin \frac{n\pi X_1}{L} e^{-\beta n^2 \gamma}
\]

Fig. (4). Square functionally graded plate and boundary condition.

Table 2 lists the temperature history at the point (0.02, 0.02) m. It can be seen that the numerical results are in excellent agreement with the analytical results. Also, as expected, with an increase in the value of \( \beta \) a higher temperature at the position of interest is obtained over a longer duration after the increased thermal conductivity.

5. CONCLUSION

A Trefftz collocation method for solving steady and transient heat conduction problems in FGMs are developed. The T-complete solutions for steady heat conduction problem in nonlinear exponential FGMs are derived by way of the Kirchhoff transformation and coordinate transformation. The Trefftz-RBF scheme is established in conjunction with analog equation method (AEM) and time stepping method. Numerical results demonstrate that the proposed method performs very well in terms of numerical accuracy and can converge to the analytical solution and are competitive in solving heat conduction problems in FGMs. Furthermore, the methods described in this paper can easily be extended to three-dimensional problems. This work is underway.

**NOTATIONS**

- \( c \) = Specific heat (J/kg/°C)
- \( k \) = Thermal conductivity (W/m/°C)
- \( q \) = Normal heat flux (W/m²)
- \( t \) = Time (s)
- \( u \) = Temperature (°C)
- \( u_0 \) = Initial temperature (°C)
- \( n \) = Number of interpolation points in the domain
- \( m \) = Number of collocation points on the physical boundary
- \( e \) = Number of components for T-complete solution
### Table 2. Temperature History at the Point (0.02,0.02)\(m\)

<table>
<thead>
<tr>
<th>t(s)</th>
<th>(\beta = 0)</th>
<th>(\beta = 10)</th>
<th>(\beta = 25)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Numerical</td>
<td>Analytical</td>
<td>Numerical</td>
</tr>
<tr>
<td>12</td>
<td>0.3142</td>
<td>0.3191</td>
<td>0.3826</td>
</tr>
<tr>
<td>24</td>
<td>0.4493</td>
<td>0.4486</td>
<td>0.5396</td>
</tr>
<tr>
<td>32</td>
<td>0.4861</td>
<td>0.4854</td>
<td>0.5818</td>
</tr>
<tr>
<td>48</td>
<td>0.4961</td>
<td>0.4959</td>
<td>0.5932</td>
</tr>
<tr>
<td>60</td>
<td>0.4989</td>
<td>0.4988</td>
<td>0.5962</td>
</tr>
</tbody>
</table>

### GREEK SYMBOLS

- \(\lambda\) = Frequency of the modified Helmholtz equation
- \(\alpha\) = Interpolating coefficient defined in Eq. (35)
- \(\beta\) = Interpolating coefficient defined in Eq. (38)
- \(\tau\) = Time step size
- \(\rho\) = Density (kg/m\(^3\))
- \(\theta\) = Temporal weighting in time-stepping method

### REFERENCE


