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Boundary integral equation supported differential quadrature method to solve problems over general irregular geometries

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Abstract Based on the interpolation technique with the aid of boundary integral equations, a new differential quadrature method has been developed (boundary integral equation supported differential quadrature method, BIE-DQM) to solve boundary value problems over generally irregular geometries. The quadrature rule of the BIE-DQM is that the first and the second derivatives of a function with respect to independent variables are approximated by a weighted linear combination of the function values at all discrete nodal points and the corresponding normal derivatives at all boundary points. Several numerical examples are considered to verify the feasibility and effectiveness of the proposed algorithm.

Keywords Differential quadrature · Boundary integral equation · Irregular geometry · Interpolation · Fundamental solution

1 Introduction

Since the pioneer work of Bellman and his co-workers [1] in the early 1970s, the differential quadrature method (DQM) has been recognized as a numerically accurate and computationally efficient numerical technique, and has been successfully used to deal with a large number of problems of physical and engineering science [2–5]. The basic idea of DQM is that the values of the derivatives at each sampling grid point are expressed as weighted linear combinations of the function values at all sampling grid points within the domain under consideration. However, as pointed out by Bert and Malik [2], the DQM is limited to applications to those domains having boundaries that are aligned with the coordinate axes. Thus, in addition to rectangular domains, the DQM has been applied to the line domains of the axisymmetric flexure of circular solid [6] and the annular domain [7] and to the parallelogram domains of skew plates [8]. It is also obvious that the DQM can be used for the analysis of irregular domains which assemblies of such regular domains using domain decomposition [9] or quadrature element [10] approaches. The quadrature rule can be formulated for curvilinear quadrilateral domains using an orthogonal curvilinear coordinate system [11] or using the natural-to-Cartesian geometric mapping technique [12] to transform irregular physical domains into square computational domains. However, irregularly shaped domains which are not parallel to the coordinate axes, and which cannot be segmented into regular shaped sub-domains, would be generally inaccessible to solution by these applications of the quadrature method. Bert and Malik [2] pointed out that extension of the DQM to general quadrilateral domains in conjunction with domain decomposition and the quadrature element concepts should go a long way in the development of the DQM for its employment in a larger class of problems which presently are considered to be the territory of the finite element method or the boundary element method [13].

Very recently, Wu and Shu [7] proposed a radial-basis-function-based DQM where three kinds of radial basis functions were used as the test functions to construct the quadrature rule. Although the interpolation with radial basis functions has potential for dealing with higher dimensional problems with irregular boundaries, the numerical examples were limited to the simple geometry of annular domains.

To circumvent the geometric difficulty inherent in the traditional DQM, a boundary integral equation supported differential quadrature method (BIE-DQM) was developed in the present work, based on the interpolation technique with the aid of boundary integral equations, to solve boundary value problems with irregular geometries. The technique of interpolation with the BIE

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was first suggested by Ochiai and Sekiya [14] for the generation of free-form surfaces in CAD for dies in industry. Shortly afterwards, this technique was employed successfully to deal with the domain integrals encountered in the BIE method when solving a number of problems such as static heat conduction [15], static thermal stress [16] and elasto-plastic analysis with initial strain formulation [17]. From the point of view of avoiding domain integrals, this technique can be considered as a supplement to the dual reciprocity [18] or multiple reciprocity [19] boundary element method. The technique has an essential feature that the errors of the approximated surfaces are almost zero at the place of collocation points, which is called the multidimensional interpolation [20]. The application of Ochiai's work was, however, limited to the functional interpolation only. In the present work, the technique was extended to be able to approximate function derivatives with function values and their normal derivatives at boundary, which results in a new quadrature rule for solving boundary value problems with general irregular domains. In particular, in the BIE-DQM, the first and the second derivatives of a function with respect to independent variables can be approximated by a linear combination of function values at all discrete nodal points and the normal derivatives at all boundary points.

The layout of the paper is as follows. In Sect. 2, we introduce in detail the BIE-DQM. In Sect. 3, simple numerical examples are presented to show the effectiveness of the interpolation technique for derivatives with boundary integral equations and to verify the feasibility and flexibility of the new algorithm with the BIE-DQM by solving a Poisson equation, a convection-diffusion equation with varying parameters and a non-linear equation over irregular geometry. In Sect. 4, discussions are presented to clarify what are the distinction and the advantages of the proposed method over the previous techniques such as BIE, DQM and Ochiai's method.

2 Differential quadrature with BIE

In this section the BIE-DQM is introduced in detail in order to establish notation and to provide a common source for reference in later sections. We first describe the basic BIEs, based on which the interpolation of a smooth function over domains with arbitrary geometry can be realized. Then we describe the quadrature rules for derivatives with BIE.

2.1 Boundary integral equations

Consider a smooth function $u(x)$ defined in the domain Ω with a piecewise smooth boundary Γ. Suppose the following relation exists:

$$u^{[2]}(x) = u^{[1]}_{,kk}(x) = u_{,jjkk}(x) \tag{1}$$

where $u^{[2]}(x)$ represents an unknown strength of a Dirac-type function [20]. In the one-dimensional case, $u(x)$ can be considered as the deflection of a simply supported beam and $u^{[1]}_{,kk}(x)$ corresponds to the moment of the beam, which is the response to the point load $u^{[2]}(x)$. In the two-dimensional case, $u(x)$ can be viewed as the deformation of an imaginary thin plate and $u^{[1]}_{,kk}(x)$ corresponds to the moment of the plate, all resulting from the Dirac-type function $u^{[2]}(x)$, i.e., the point load [16].

In order to realize the interpolation with the BIE, the fundamental solution $g(x, y)$ of Laplace equation is employed, which satisfies

$$g_{,kk}(x, y) + \delta(x, y) = 0 \tag{2}$$

where $\delta(x, y)$ is the Dirac-δ function and x and y are the field and the source points, respectively. The fundamental solution $g(x, y)$ has relations with its higher-order fundamental solution

$$g^0 = g^1_{,kk} \tag{3}$$

In the two-dimensional case, the fundamental solutions are as follows:

$$g(x, y) = \frac{1}{2\pi}\ln\left(\frac{1}{r}\right) \tag{4}$$

$$g^1(x, y) = \frac{r^2}{2\pi}\left(\ln\frac{1}{r} + 1\right) \tag{5}$$

where r is the distance between x and y. Supposing the number of the point load $u^{[2]}(x)$ is M in the domain Ω, we can write

$$\int_{\Omega} u^{[2]}(x) f(x) d\Omega(x) = \sum_{l=1}^{M} u^{[2]}\left(x^l\right) f\left(x^l\right) \tag{6}$$

where $f(x)$ represents any continuous function in the neighborhood of x^l in Ω. Now consider the following integral:

$$\int_{\Omega} u(x)\left[g_{,kk}(x, y) + \delta(x, y)\right] \mathrm{d}\Omega(x) = 0 \tag{7}$$

Using the Green's second identity [13] and from (6) and (7), we obtain the following boundary integral equation:

$$\begin{aligned} \gamma(y)u(y) &+ \int_{\langle\Gamma\rangle} u(x)h(x, y)d\Gamma(x) \\ &- \int_{\Gamma} q(x)g(x, y)d\Gamma(x) \\ = &- \int_{\Gamma} u^{[1]}(x)h^{[1]}(x, y)d\Gamma(x) \\ &+ \int_{\Gamma} q^{[1]}(x)g^{[1]}(x, y)d\Gamma(x) \\ &- \sum_{l=1}^{M} u^{[2]}\left(x^l\right)g^{[1]}\left(x^l, y\right) \end{aligned} \tag{8}$$

where

$$\gamma(y) = \begin{cases} 1 & y \in \Omega \\ 0.5 & y \in \Gamma \quad \text{if } \Gamma \text{ is smooth} \\ 0 & y \in \overline{\Omega \cup \Gamma} \end{cases} \tag{9}$$

and

$$h = \frac{\partial g}{\partial n}, \quad h^{[1]} = \frac{\partial g^{[1]}}{\partial n}, \quad q = \frac{\partial u}{\partial n}, \quad q^{[1]} = \frac{\partial u^{[1]}}{\partial n} \tag{10}$$

where n is the outward normal to the boundary. The first boundary integral in the left-hand side of Eq. (8) is strong singular, which is denoted by corner bracket $\langle\Gamma\rangle$, and should be evaluated in the sense of the Cauchy principal value [13] when $y \in \Gamma$. Taking derivatives of both sides of (8) with respect to y when $y \notin \Gamma$ and then taking the limit process [21], we have

$$\begin{aligned} \gamma(y)u_{,k}(y) = & - \int_{[\Gamma]} u(x)h_k(x,y)d\Gamma(x) \\ & + \int_{\langle\Gamma\rangle} q(x)g_k(x,y)d\Gamma(x) \\ & - \int_{\Gamma} u^{[1]}(x)h_k^{[1]}(x,y)d\Gamma(x) \\ & + \int_{\Gamma} q^{[1]}(x)g_k^{[1]}(x,y)d\Gamma(x) \\ & - \sum_{l=1}^{M} u^{[2]}(x^l)g_k^{[1]}(x^l,y) \end{aligned} \tag{11}$$

where the square bracket $[\Gamma]$ indicates that the first boundary integral in the right-hand side of (11) is hypersingular and evaluated in the sense of the Hadamard finite part [21] when $y \in \Gamma$. The relations of the derived fundamental solutions in Eq. (11) are as follows:

$$g_k = -\frac{\partial g}{\partial x_k}, h_k = -\frac{\partial h}{\partial x_k}, g_k^1 = -\frac{\partial g^1}{\partial x_k}, h_k^1 = -\frac{\partial h^1}{\partial x_k} \tag{12}$$

The explicit expressions of the derived fundamental solutions in (12) are presented in Appendix I. Eq. (11) can be written in the flux form by multiplying the outward normal $n_k(y)$ at both sides:

$$\begin{aligned} & \gamma(y)q(y) + n_k(y) \int_{[\Gamma]} u\ (x)\ h_k(x,y)d\Gamma(x) \\ & \quad - n_k(y) \int_{\langle\Gamma\rangle} q(x)g_k(x,y)d\Gamma(x) \\ & = -n_k(y) \int_{\Gamma} u^{[1]}(x)h_k^{[1]}(x,y)d\Gamma(x) \\ & \quad + n_k(y) \int_{\Gamma} q^{[1]}(x)g_k^{[1]}(x,y)d\Gamma(x) \\ & \quad - n_k(y) \sum_{l=1}^{M} u^{[2]}(x^l)g_k^{[1]}(x^l,y) \quad (y \in \Gamma) \end{aligned} \tag{13}$$

We again take derivatives of both sides of (13) with respect to y by setting $y \in \Omega$ and then take the limit process to obtain

$$\begin{aligned} \gamma(y)q_{,k}(y) = & -n_j(y) \int_{\{\Gamma\}} u(x)h_{jk}(x,y)d\Gamma(x) \\ & - n_j(y) \int_{[\Gamma]} q(x)g_{jk}(x,y)d\Gamma(x) \\ & - n_j(y) \int_{\langle\Gamma\rangle} u^{[1]}(x)h_{jk}^{[1]}(x,y)d\Gamma(x) \\ & + n_j(y) \int_{\Gamma} q^{[1]}(x)g_{jk}^{[1]}(x,y)d\Gamma(x) \\ & - n_j(y) \sum_{l=1}^{M} u^{[2]}(x^l)g_{jk}^{[1]}(x^l,y) \quad (y \in \Gamma) \end{aligned} \tag{14}$$

where the bracket $\{\Gamma\}$ indicates that the first boundary integral in the right-hand side of (14) is supersingular and should be evaluated in the sense of principal value when $y \in \Gamma$. The relations of derived fundamental solutions in (14) are as follows:

$$\begin{aligned} & g_{jk} = -\frac{\partial g_k}{\partial x_j}, h_{jk} = -\frac{\partial h_k}{\partial x_j}, \\ & g_{jk}^{[1]} = -\frac{\partial g_k^{[1]}}{\partial x_j}, h_{jk}^{[1]} = -\frac{\partial h_k^{[1]}}{\partial x_j} \end{aligned} \tag{15}$$

The explicit expressions of the derived fundamental solutions in (15) are also given in Appendix I. The orders of singularity of various kernels in the two-dimensional BIE (8), (11), (13) and (14) are listed in the Table A of Appendix II.

2.2 Function interpolation

Combining (8) and (13) we can obtain an interpolation formula with BIEs in discrete forms. To simplify the expression, supposing that the boundary has been divided into N constant boundary elements, $\Delta\Gamma$, we have from (8)

$$\begin{aligned} \mathrm{H}_\Gamma \mathrm{u}_\Gamma - \mathrm{G}_\Gamma \mathrm{q}_\Gamma = & -\mathrm{H}_\Gamma^{[1]} \mathrm{u}_\Gamma^{[1]} + \mathrm{G}_\Gamma^{[1]} \mathrm{q}_\Gamma^{[1]} \\ & - \mathrm{G}_\Gamma^{\mathrm{M}} \mathrm{u}^{[2]} \quad (y \in \Gamma) \end{aligned} \tag{16}$$

$$\begin{aligned} \mathrm{u}_\Omega + \mathrm{H}_\Omega \mathrm{u}_\Gamma - \mathrm{G}_\Omega \mathrm{q}_\Gamma = & -\mathrm{H}_\Omega^{[1]} \mathrm{u}_\Gamma^{[1]} + \mathrm{G}_\Omega^{[1]} \mathrm{q}_\Gamma^{[1]} \\ & - \mathrm{G}_\Omega^{\mathrm{M}} \mathrm{u}^{[2]} \quad (y \in \Omega) \end{aligned} \tag{17}$$

The subscripts Γ and Ω represent, respectively, the values on boundary (N nodes) or in domain (M nodes) for function or function derivative vectors, u and q, in (16) and (17). For matrices, the subscripts Γ and Ω represent, respectively, the locations of the source points, y, on boundary or in domain. Similarly, from (13) we can write

$$\mathrm{H_N u_\Gamma - G_N q_\Gamma = -H_N^{[1]} u_\Gamma^{[1]} + G_N^{[1]} q_\Gamma^{[1]} - G_N^{\mathit{M}} u^{[2]}} \quad (y \in \Gamma) \tag{18}$$

The entries of the matrices in (16)–(18) are presented in Appendix III. Rewriting (16)–(18) in compact matrix form, we have

$$\begin{bmatrix} \mathrm{H_N} & -\mathrm{G_N} & 0 \\ \mathrm{H_\Gamma} & -\mathrm{G_\Gamma} & 0 \\ \mathrm{H_\Omega} & -\mathrm{G_\Omega} & \mathrm{I} \end{bmatrix} \begin{Bmatrix} \mathrm{u_\Gamma} \\ \mathrm{q_\Gamma} \\ \mathrm{u_\Omega} \end{Bmatrix} = \begin{bmatrix} -\mathrm{H_N^{[1]}} & \mathrm{G_N^{[1]}} & -\mathrm{G_N^M} \\ -\mathrm{H_\Gamma^{[1]}} & \mathrm{G_\Gamma^{[1]}} & -\mathrm{G_\Gamma^M} \\ -\mathrm{H_\Omega^{[1]}} & \mathrm{G_\Omega^{[1]}} & -\mathrm{G_\Omega^M} \end{bmatrix} \begin{Bmatrix} \mathrm{u_\Gamma^{[1]}} \\ \mathrm{q_\Gamma^{[1]}} \\ \mathrm{u^{[2]}} \end{Bmatrix} \tag{19}$$

where I is a $M \times M$ unit matrix. (19) can be concisely written as

$$\mathrm{M_0 b^{[0]} = M_1 b^{[1]}} \tag{20}$$

where

$$\mathrm{M_0} = \begin{bmatrix} \mathrm{H_N} & -\mathrm{G_N} & 0 \\ \mathrm{H_\Gamma} & -\mathrm{G_\Gamma} & 0 \\ \mathrm{H_\Omega} & -\mathrm{G_\Omega} & \mathrm{I} \end{bmatrix}, \mathrm{b^{[0]}} = \begin{Bmatrix} \mathrm{u_\Gamma} \\ \mathrm{q_\Gamma} \\ \mathrm{u_\Omega} \end{Bmatrix}, \mathrm{M_1} = \begin{bmatrix} -\mathrm{H_N^{[1]}} & \mathrm{G_N^{[1]}} & -\mathrm{G_N^M} \\ -\mathrm{H_\Gamma^{[1]}} & \mathrm{G_\Gamma^{[1]}} & -\mathrm{G_\Gamma^M} \\ -\mathrm{H_\Omega^{[1]}} & \mathrm{G_\Omega^{[1]}} & -\mathrm{G_\Omega^M} \end{bmatrix}, \mathrm{b^{[1]}} = \begin{Bmatrix} \mathrm{u_\Gamma^{[1]}} \\ \mathrm{q_\Gamma^{[1]}} \\ \mathrm{u^{[2]}} \end{Bmatrix} \tag{21}$$

If the vector $\mathrm{b^{[0]}}$ is prescribed, we can determine the values of vector $\mathrm{b^{[1]}}$ by

$$\mathrm{b^{[1]} = M_1^{-1} M_0 b^{[0]}} \tag{22}$$

Then the function values at any point, $x \in \Omega \cup \Gamma$, can be computed by using Eq. (8) in a discrete form. The procedure mentioned above is one of the forms of interpolation with the BIE. It should be pointed out that interpolation with the BIE may have many forms. The interpolation can be realized by different combinations of integral Eqs. [14–17]. In the present work, the purpose in using the hypersingular Eq. (13) is to improve the accuracy of the interpolation for function derivatives on the boundary.

2.3 Quadrature rules for derivatives

Assuming that the boundary has been divided into N constant boundary elements, $\Delta\Gamma$, we can write (11), (13) and (14) in discrete forms, respectively, as follows:

$$\mathrm{u}_{\Gamma,k} = -\mathrm{H}_{\Gamma k}\mathrm{u}_\Gamma + \mathrm{G}_{\Gamma k}\mathrm{q}_\Gamma - \mathrm{H}_{\Gamma k}^{[1]}\mathrm{u}_\Gamma^{[1]} + \mathrm{G}_{\Gamma k}^{[1]}\mathrm{q}_\Gamma^{[1]} - \mathrm{G}_{\Gamma k}^{\mathrm{M}}\mathrm{u}^{[2]} \quad (y \in \Gamma) \tag{23}$$

$$\mathrm{u}_{\Omega,k} = -\mathrm{H}_{\Omega k}\mathrm{u}_\Gamma + \mathrm{G}_{\Omega k}\mathrm{q}_\Gamma - \mathrm{H}_{\Omega k}^{[1]}\mathrm{u}_\Gamma^{[1]} + \mathrm{G}_{\Omega k}^{[1]}\mathrm{q}_\Gamma^{[1]} - \mathrm{G}_{\Omega k}^{\mathrm{M}}\mathrm{u}^{[2]} \quad (y \in \Omega) \tag{24}$$

$$\mathrm{q}_{\Gamma,k} = -\mathrm{H}_{\mathrm{N}k}\mathrm{u}_\Gamma + \mathrm{G}_{\mathrm{N}k}\mathrm{q}_\Gamma - \mathrm{H}_{\mathrm{N}k}^{[1]}\mathrm{u}_\Gamma^{[1]} + \mathrm{G}_{\mathrm{N}k}^{[1]}\mathrm{q}_\Gamma^{[1]} - \mathrm{G}_{\mathrm{N}k}^{\mathrm{M}}\mathrm{u}^{[2]} \quad (y \in \Gamma) \tag{25}$$

The entries of the matrices in (23)–(25) are listed in Appendix III. We can rewrite (23)–(25), respectively, in compact matrix forms as

$$\begin{Bmatrix} \mathrm{u}_{\Gamma,k} \\ \mathrm{u}_{\Omega,k} \end{Bmatrix} = \begin{bmatrix} -\mathrm{H}_{\Gamma k} & \mathrm{G}_{\Gamma k} \\ -\mathrm{H}_{\Omega k} & \mathrm{G}_{\Omega k} \end{bmatrix} \begin{Bmatrix} \mathrm{u_\Gamma} \\ \mathrm{q_\Gamma} \end{Bmatrix} + \begin{bmatrix} -\mathrm{H}_{\Gamma k}^{[1]} & \mathrm{G}_{\Gamma k}^{[1]} & -\mathrm{G}_{\Gamma k}^{\mathrm{M}} \\ -\mathrm{H}_{\Omega k}^{[1]} & \mathrm{G}_{\Omega k}^{[1]} & -\mathrm{G}_{\Omega k}^{[1]} \end{bmatrix} \begin{Bmatrix} \mathrm{u_\Gamma^{[1]}} \\ \mathrm{q_\Gamma^{[1]}} \\ \mathrm{u^{[2]}} \end{Bmatrix} \tag{26}$$

$$\{\mathrm{q}_{\Gamma,k}\} = [-\mathrm{H}_{\mathrm{N}k} \quad \mathrm{G}_{\mathrm{N}k}] \begin{Bmatrix} \mathrm{u_\Gamma} \\ \mathrm{q_\Gamma} \end{Bmatrix} + \left[-\mathrm{H}_{\mathrm{N}k}^{[1]} \quad \mathrm{G}_{\mathrm{N}k}^{[1]} \quad -\mathrm{G}_{\mathrm{N}k}^{[1]}\right] \begin{Bmatrix} \mathrm{u_\Gamma^{[1]}} \\ \mathrm{q_\Gamma^{[1]}} \\ \mathrm{u^{[2]}} \end{Bmatrix} \tag{27}$$

Inserting (19) or (20) into (26) and (27), respectively, to eliminate vector $\mathrm{b^{[1]}}$, we obtain

$$\begin{Bmatrix} \mathrm{u}_{\Gamma,k} \\ \mathrm{u}_{\Omega,k} \end{Bmatrix} = \begin{bmatrix} -\mathrm{H}_{\Gamma k} & \mathrm{G}_{\Gamma k} \\ -\mathrm{H}_{\Omega k} & \mathrm{G}_{\Omega k} \end{bmatrix} \begin{Bmatrix} \mathrm{u_\Gamma} \\ \mathrm{q_\Gamma} \end{Bmatrix} + \begin{bmatrix} -\mathrm{H}_{\Gamma k}^{[1]} & \mathrm{G}_{\Gamma k}^{[1]} & -\mathrm{G}_{\Gamma k}^{\mathrm{M}} \\ -\mathrm{H}_{\Omega k}^{[1]} & \mathrm{G}_{\Omega k}^{[1]} & -\mathrm{G}_{\Omega k}^{[1]} \end{bmatrix} \mathrm{M_1^{-1} M_0} \times \begin{Bmatrix} \mathrm{u_\Gamma} \\ \mathrm{q_\Gamma} \\ \mathrm{u_\Omega} \end{Bmatrix} \tag{28}$$

$$\{\mathrm{q}_{\Gamma,k}\} = [-\mathrm{H}_{\mathrm{N}k} \quad \mathrm{G}_{\mathrm{N}k}] \begin{Bmatrix} \mathrm{u_\Gamma} \\ \mathrm{q_\Gamma} \end{Bmatrix} + \left[-\mathrm{H}_{\mathrm{N}k}^{[1]} \quad \mathrm{G}_{\mathrm{N}k}^{[1]} \quad -\mathrm{G}_{\mathrm{N}k}^{[1]}\right] \mathrm{M_1^{-1} M_0} \times \begin{Bmatrix} \mathrm{u_\Gamma} \\ \mathrm{q_\Gamma} \\ \mathrm{u_\Omega} \end{Bmatrix} \tag{29}$$

Equations (28) and (29) can be rewritten in a concise form, respectively, as

$$\mathrm{u}_{,k} = \mathrm{U}_k \mathrm{b}^{[0]} \tag{30}$$

$$\mathrm{q}_{,k} = \mathrm{Q}_k \mathrm{b}^{[0]} \tag{31}$$

Notice that (30) is just the quadrature rule with the BIE for the first derivatives over irregular geometries, which can be written in a similar form with the conventional DQM as

$$u_{,k}(y^n) = \sum_{m=1}^{N+M} A_k^{nm} u^m + \sum_{m=1}^{N} B_k^{nm} q^m \quad (n = 1, 2, \ldots, N+M) \tag{32}$$

where $A_k^{nm} = A_k(x^m, y^n)$ and $B_k^{nm} = B_k(x^m, y^n)$ are the corresponding entries in the matrix U_k , u^m represents the function values at each of all the nodal points, q^m the values of outward normal at all the boundary nodes. In Eq. (32), both A_k^{nm} and B_k^{nm} serve as the weighting coefficients. Now taking derivatives at both sides of (30), we can write

$$\mathrm{u}_{,ik} = \mathrm{U}_k \mathrm{b}_{,i}^{[0]} \tag{33}$$

where

$$\mathrm{b}_{,i}^{[0]} = \begin{Bmatrix} \mathrm{u}_{\Gamma,i} \\ \mathrm{q}_{\Gamma,i} \\ \mathrm{u}_{\Omega,i} \end{Bmatrix} \tag{34}$$

Substituting (30) and (31) into (33) and after some rearrangement, we have the quadrature rule with BIE for the second-derivatives over irregular geometries as follows:

$$\mathrm{u}_{,ik} = \mathrm{U}_{ik} \mathrm{b}^{[0]} \tag{35}$$

We also rewrite (35) in a similar form with the conventional DQM as

$$u_{,ik}(y^n) = \sum_{m=1}^{N+M} A_{ik}^{nm} u^m + \sum_{m=1}^{N} B_{ik}^{nm} q^m \quad (n = 1, 2, \ldots, N+M) \tag{36}$$

where $A_{ik}^{nm} = A_{ik}(x^m, y^n)$ and $B_{ik}^{nm} = B_{ik}(x^m, y^n)$ are the corresponding entries in the matrix U_{ik} . Here A_{ik}^{nm} and B_{ik}^{nm} serve also as the weighting coefficients in (36). It can be seen from both Eqs. (32) and (36) that the function derivatives with respect to independent variables are approximated by a weighted linear combination of function values at all discrete nodal points as well as their normal derivatives at all boundary points. This feature entails all the merits of ordinary DQM for solution of partial differential equations. Moreover, it can obviously be seen that there are two distinct advantages resulting from the formation of (32) and (36). The first is that the BIE-supported quadrature rule can deal with boundary value problems over domains with generally irregular geometries. The second is that the boundary condition of the problem can be treated naturally without any difficulty. The difference of the BIE-DQM from the conventional DQM lies in the way to determine the weighting coefficients.

3 Numerical examples

In this section, an interpolation problem as well as the solutions of a Poisson equation, a convection-diffusion equation and a non-linear equation are considered using the proposed BIE-DQM. In all the numerical examples, five-node boundary elements are applied to improve the accuracy of derivative approximations on boundary. The unconformity boundary elements are used only at corner points to avoid indefiniteness of the outward normal at these places. For the evaluation of hypersingular and supersingular boundary integrals with kernels h_k and h_{jk}, the following relations are used, respectively:

$$\int_\Gamma h_k(x, y) d\Gamma(x) = 0 \tag{37}$$

$$\int_\Gamma h_{jk}(x, y) d\Gamma(x) = 0 \tag{38}$$

For the evaluation of hypersingular boundary integrals with other kernels in (14), the singular integrals are approximated by the mean values of the two corresponding nearly singular boundary integrals with distance transformation techniques [22–23]. In the numerical examples, four-point Gauss quadrature is used in general for evaluation of ordinary integrals and eight-point Gauss quadrature is used for singular integrals, but at most 16-point Gauss quadrature is used for nearly singular integrals according to the distances between x and y.

3.1 Derivative approximation

The first and second derivatives are approximated, respectively, by the quadrature rules (32) and (36). The function values at all nodal points and the normal derivative values at all boundary points are taken as known over a square domain of $x_1 \in [0, 1]$ and $x_2 \in [0, 1]$. The function approximated is as follows:

$$u(x_1, x_2) = R^2 \exp(x_1 - x_2) \tag{39}$$

where

$$R = \sqrt{x_1^2 + (x_2 - 2)^2} \tag{40}$$

The boundary is divided into eight elements with a total of 36 boundary nodes. Thirty-six equally-spaced internal nodes are employed in the domain. The relative errors in absolute values along all the boundary nodes are shown in Fig. 1. The relative error-surfaces of $u_{,2}$ and $u_{,12}$ are shown in Figs. 2 and 3, respectively. It can be seen from Figs. 1–3 that the accuracy for the first derivatives is very good both in domain and on boundary. The accuracy for the second-derivatives is acceptable

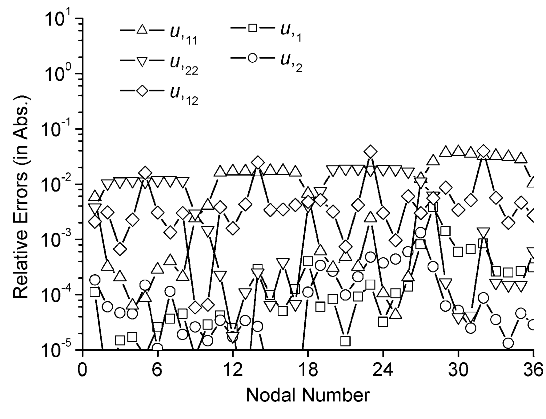

Fig. 1 Relative errors along the boundary of the square

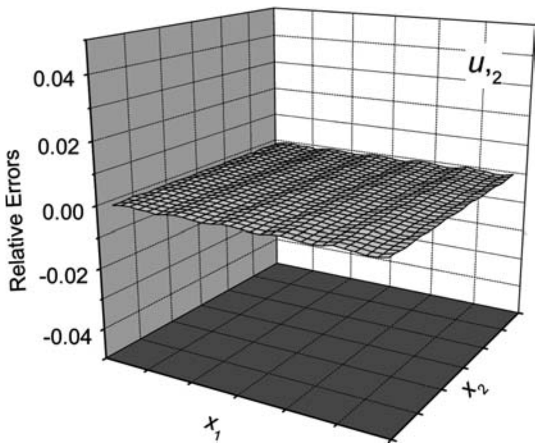

Fig. 2 Relative error surface of $u_{,2}$ in the domain of the square

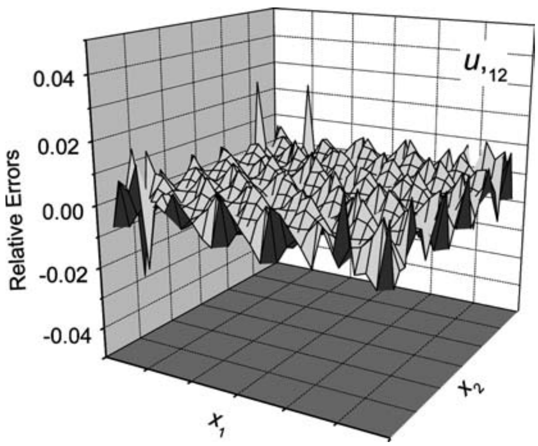

Fig. 3 Relative error surface of $u_{,12}$ in the domain of the square

although it is not as good as that for the first derivatives, as expected. The maximum relative errors of the second-derivatives occur on the boundaries at certain directions (Fig. 1).

3.2 Poisson equation

We first consider the following Poisson equation:

$$u_{,kk} = 28x_1 \quad (x_1, x_2 \in \Omega) \tag{41}$$

subject to the boundary conditions

$$u(x_1,x_2) = \bar{u}(x_1,x_2) \quad (x_1,x_2 \in \Gamma_u),$$
$$q(x_1,x_2) = \bar{q}(x_1,x_2) \quad (x_1,x_2 \in \Gamma_q) \tag{42}$$

where $\bar{u}(x_1,x_2)$ and $\bar{q}(x_1,x_2)$ are the prescribed values of functions and fluxes, respectively. The analytical solution is as follows:

$$u = x_1^3 - 3x_1^2x_2 + 2x_1x_2^2 + x_2^3 \tag{43}$$

The above Dirichlet and Neumann boundary conditions $\bar{u}(x_1,x_2)$ and $\bar{q}(x_1,x_2)$, respectively, on Γ_u and Γ_q can be easily evaluated using the analytical solution (43).

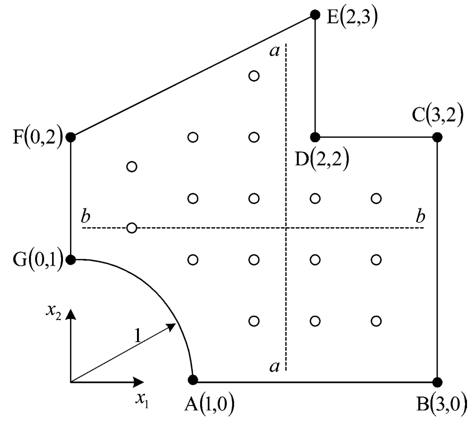

Fig. 4 Configuration of 2D irregular geometry

The domain of an irregular geometry is shown in Fig. 4. The Dirichlet conditions are prescribed along the boundary C-D-E (Fig. 4). The Neumann conditions are prescribed along the other boundaries. The boundary is modeled by 12 boundary elements and a total of 55 boundary nodes. There are 16 internal nodes used as shown in Fig. 4. By applying the quadrature rule (36) into the Poisson Eq. (41), we can easily obtain the algebraic equations

$$\sum_{m=1}^{N+M} A_{kk}^{nm} u^m + \sum_{m=1}^{N} B_{kk}^{nm} q^m$$
$$= 28x_1^n \ (n = 1,2,\ldots,N+M) \tag{44}$$

Suppose that there are N_1 and N_2 $(N_1 + N_2 = N)$ boundary nodes, respectively, on Dirichlet and Neumann prescribed boundaries. Substituting boundary conditions (42)–(44), we obtain the system equations as follows:

$$\sum_{m=1}^{N_2+M} A_{kk}^{nm} u^m + \sum_{m=1}^{N_1} B_{kk}^{nm} q^m = 28x_1^n - \sum_{m=1}^{N_1} A_{kk}^{nm} \bar{u}^m$$
$$- \sum_{m=1}^{N_2} B_{kk}^{nm} \bar{q}^m \ (n = 1,2,\ldots,N+M) \tag{45}$$

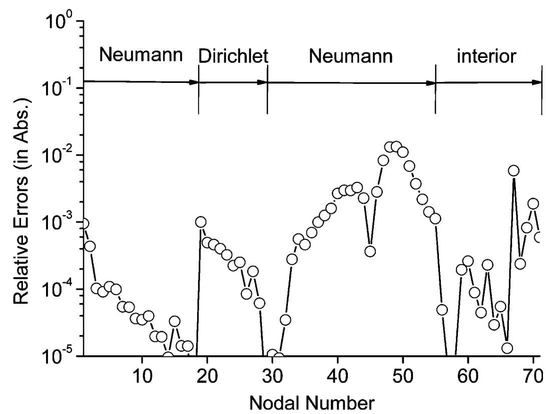

Fig. 5 Relative errors along the boundary and interior nodes for Poisson equation

which can be written concisely in matrix form as

$$\mathrm{Ax} = \mathrm{b} \tag{46}$$

Compared with the conventional DQM [1–12], it is obvious that the treatment of boundary conditions with the BIE-DQM is very simple and convenient, similar to that in the boundary element method [13]. The relative errors along the boundary and at the interior nodes are shown in Fig. 5 in which the maximum relative error is 1.3% over the Nuemann boundary. It can be seen from Fig. 8 that the proposed approach can provide a very good accuracy both in the domain and on the boundary. The computed values of domain variables are compared with the exact ones in the domain along the horizontal line $b-b$ (see Fig. 6) and along the vertical line $a-a$ (Fig. 7), respectively. It is found from Figs. 6 and 7 that the computed results of both the function and the derivative are in good agreement with the analytical results in the domain.

3.3 Convection-diffusion equation with varying parameters

Next we consider the following convection-diffusion equation:

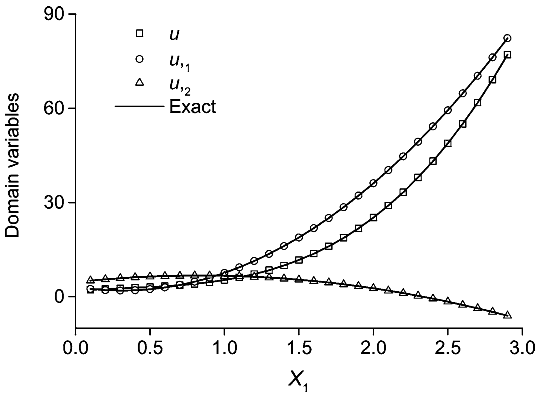

Fig. 6 Domain variables along the $b-b$ line for Poisson equation

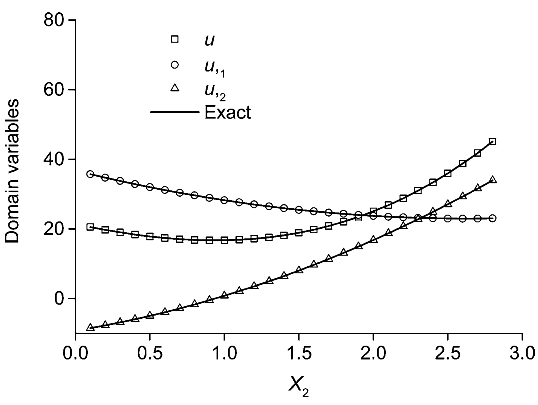

Fig. 7 The domain variables along the $a-a$ line for Poisson equation

$$u_{,kk} + x_2 u_{,1} + x_1 u_{,2} = (1-x_2)\exp(-x_1) + (1-x_1)\exp \times (-x_2)(x_1, x_2 \in \Omega) \tag{47}$$

subject to the boundary conditions still represented by (42). The solution domain in this example is the same as that used in Sect. 3.2, as shown in Fig. 4. The Dirichlet conditions, however, are prescribed along the arc G-A (Fig. 4) for the convection-diffusion problem. The Neumann conditions are prescribed along the other boundaries. The analytical solution for the convection-diffusion equation is as follows:

$$u = \exp(-x_1) + \exp(-x_2) \tag{48}$$

By applying the quadrature rules (32) and (36) into the convection diffusion Eq. (47) at any nodal point y^n in conjunction with the boundary conditions (42), we obtain the system equations as follows:

$$\begin{aligned}
&\sum_{m=1}^{N_2+M} \left(A_{kk}^{nm} + x_1^n A_1^{nm} + x_1^n A_2^{nm}\right) u^m \\
&\quad + \sum_{m=1}^{N_1} \left(B_{kk}^{nm} + x_2^n B_1^{nm} + x_1^n B_2^{nm}\right) q^m \\
&\quad = \left(1 - x_2^n\right)\exp\left(-x_1^n\right) + \left(1 - x_1^n\right)\exp\left(-x_2^n\right) \\
&\quad - \sum_{m=1}^{N_1} \left(A_{kk}^{nm} + x_1^n A_1^{nm} + x_1^n A_2^{nm}\right) \bar{u}^m \\
&\quad - \sum_{m=1}^{N_2} \left(B_{kk}^{nm} + x_2^n B_1^{nm} + x_1^n B_2^{nm}\right) \bar{q}^m \\
&\quad \times (n = 1, 2, \ldots, N+M)
\end{aligned} \tag{49}$$

The relative errors along the boundary and interior nodes are shown in Fig. 8 in which the maximum relative error is 0.08% over the Nuemann boundary. The computed values of domain variables are compared with the exact ones in the domain, respectively, along the horizontal line $b-b$ (see Fig. 9) and along the vertical line $a-a$ (see Fig. 10). It can also be seen from this example that, with the BIE-DQM, the transform of

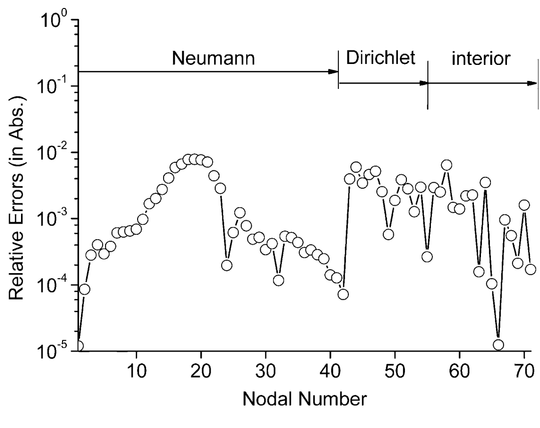

Fig. 8 Relative errors along the boundary and interior nodes for convection-diffusion equation

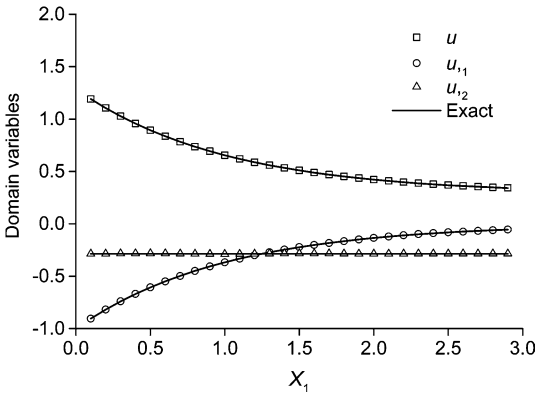

Fig. 9 Domain variables along the $b-b$ line for convection-diffusion equation

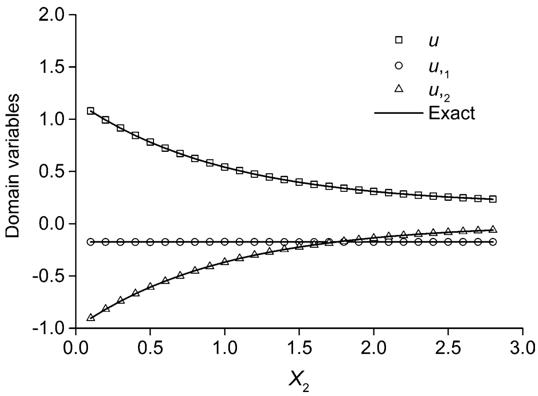

Fig. 10 Domain variables along the $a-a$ line for convection-diffusion equation

differential equation to algebraic equations is very simple, and the treatment of boundary conditions is convenient and natural. It can be seen also from Figs. 8–10 that the accuracy of the numerical results is good both in the domain and on the boundary for both the function and derivative values, although the accuracy of derivatives is generally not as good as that of functions, which is also true for other numerical methods.

3.4 A non-linear differential equation

The last example considered is a non-linear differential equation as follows

$$L(u) = 4uu_{,kk} - 10u_{,1}u_{,2} + 5 = 0 \tag{50}$$

The boundary conditions is still defined by (42). The solution domain here is the same as that used in Sect. 3.2, as shown in Fig. 4. The Dirichlet conditions, however, are prescribed along the arc C-D-E-F-G (Fig. 4) for the non-linear differential equation. The exact solution to the above problem is given as

$$u = \cos\,(x_1 + 0.5x_2) \tag{51}$$

It is well-known that the difficulty in solving this non-linear problem by the conventional BIE is to find a corresponding fundamental solution. In contrast, with the proposed BIE-DQM, this non-linear problem can be solved by applying the quadrature rules (32) and (36) without any difficulty. In this case, an iterative process is required in handling the nonlinear term in Eq. (50). For this purpose, we adopt Newton's approach wherein, beginning with an assumed field $u^{(0)}$ consistent with the boundary conditions (42), the successively refined solutions can be obtained through the following iterative scheme

$$u^{(j+1)} = u^{(j)} + \theta^{(j)} \tag{52}$$

where $\theta = \theta(x_1, x_2)$ is the refinement of u and j is the iteration count. The refinement θ is determined by the solution of the following equation written in the operator form as

$$\theta L'(u) + L(u) = 0 \tag{53}$$

where L' represents the Frechet derivative defined as [2]

$$\theta L'(u) = \frac{\partial}{\partial \varepsilon} L(u + \varepsilon\theta)|_{\varepsilon=0} \tag{54}$$

Substituting the Frechet derivative (54) and (50) into (53), we obtain

$$\begin{aligned} &4u\theta_{,kk} - 10u_{,2}\theta_{,1} - 10u_{,1}\theta_{,2} + 4u_{,kk}\theta \\ &= -\left(4uu_{,kk} - 10u_{,1}u_{,2} + 5\right) \end{aligned} \tag{55}$$

The above equation is a linear one in terms of the refinement θ. By applying the quadrature rules (32) and (36) into (55) at any nodal point y^n yields the system equations as follows:

$$\begin{aligned} &\sum_{m=1}^{N_2+M} \left(4u^n A_{kk}^{nm} - 10u_{,2}^n A_1^{nm} - 10u_{,1}^n A_2^{nm}\right)\theta^m \\ &\quad + \sum_{m=1}^{N_1} \left(4u^n B_{kk}^{nm} - 10u_{,2}^n B_1^{nm} - 10u_{,1}^n B_2^{nm}\right)\frac{\partial\theta^m}{\partial n} \\ &\quad + 4u_{,kk}^n\theta^n = -\left(4u^n u_{,kk}^n - 10u_{,1}^n u_{,2}^n + 5\right) \\ &\qquad (n = 1, 2, \ldots, N + M) \end{aligned} \tag{56}$$

where the following boundary conditions for the θ-variables have been included in (56).

$$\begin{aligned} &\theta(x_1, x_2) = 0 \qquad (x_1, x_2 \in \Gamma_u), \\ &\frac{\partial\theta}{\partial n}(x_1, x_2) = 0 \quad \left(x_1, x_2 \in \Gamma_q\right) \end{aligned} \tag{57}$$

In the calculation, the following convergence criterion was employed.

$$\sum_{m=1}^{N_2+M} (\theta^m)^2 + \sum_{m=1}^{N_1} \left(\frac{\partial\theta^m}{\partial n}\right)^2 \le 10^{-6} \tag{58}$$

With an initial guess, say $u^{(0)} = \lambda\cos(x_1 + 0.5x_2)$ where $0.8 \le \lambda \le 1.2$, to which the convergence of Newton's method is very sensitive, a converged solution could always be obtained in a maximum of six iterations.

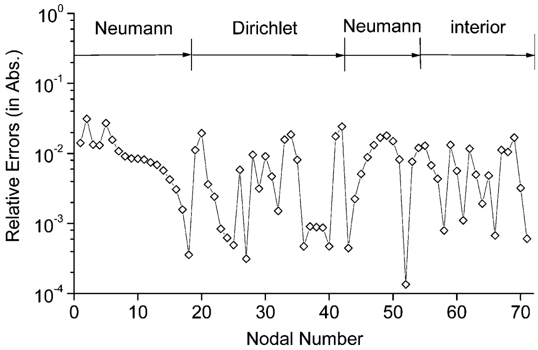

Fig. 11 Relative errors along the boundary and interior nodes for non-linear equation

The relative errors along the boundary and interior nodes are shown in Fig. 11 in which the maximum relative error is 3.1% over the Nuemann boundary. The computed values of domain variables are compared with the exact ones in the domain along the horizontal line $b-b$ in Fig. 12 and along the vertical line $a-a$ in Fig. 13, respectively. It can be seen again from Figs. 11–13 that the accuracy of the numerical results is good in the domain and on the boundary for both the function and derivative values. This example shows the feasibility and convenience to deal with non-linear problems using the proposed BIE-DQM method.

4 Discussions

In the previous section, the feasibility, versatility and numerical accuracy of the proposed BIE-DQM are verified by three numerical examples using relatively small number of interior nodal points. In this section, further discussions are made to clarify some important questions about the BIE-DQM for the correlation with, the difference from as well as the similarity to its two origins, the DQM and the BIE method to identify what are the distinction and the advantages of the proposed method over previous approached such as Ochiai's method.

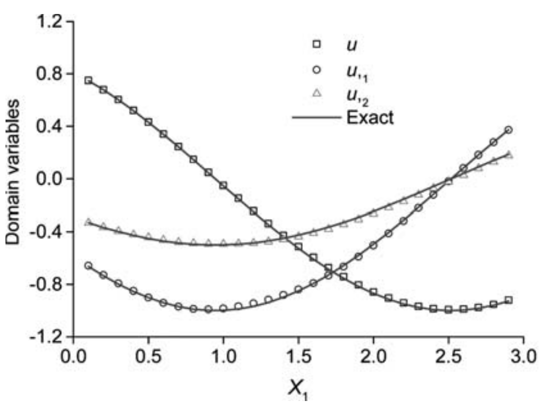

Fig. 12 Domain variables along the $b-b$ line for non-linear equation

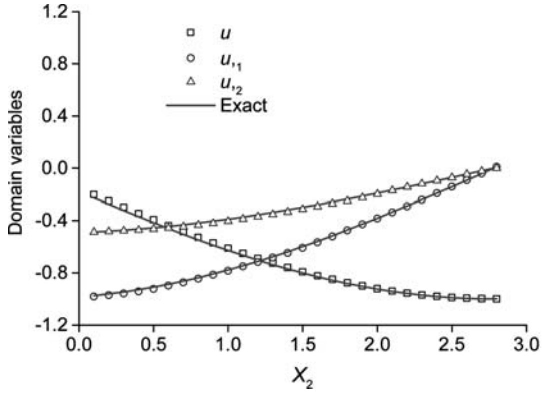

Fig. 13 Domain variables along the $a-a$ line for non-linear equation

It is well known that, in the conventional DQM, the values of derivative at each sampling grid point is approximated as weighted linear conbination of all the function values at nodal points in the whole domain including boundary [1, 2]. The DQM uses a set of test functions to determine the weighting coefficients where the polynomials are the most frequently used test functions. When applied to problems with globally smooth solutions, DQM can yield very accurate numerical results using a considerably small number of grid points. It has been shown [2, 6, 24–25] that the DQM is essentially equivalent to the general collocation method. In other words, numerical interpolations are the mathematical basis of the DQM. The quadrature rule comes from the interpolation, which is in fact to one-dimensional only. When solving a two- or three-dimensional problem, however, the interpolations, or more particularly, the quadrature rules, have to be performed independently at each dimensional direction for the controlling differential equation, which results in the limitation of requiring a regular geometry of the problem in the DQM. Others are also affected by the grid distributions and treatments of boundary conditions in some cases (Bert and Malik [2]).

It should be pointed out that the mathematical basis of the BIE-DQM is also the numerical interpolation of the function itself and its derivatives, which can be done with the aid of BIE method presented in [14] and named as multidimensional interpolation [20] by Ochiai. The purpose of Ochiai's work was to deal with the domain integrals encountered in the BIE method when solving a number of problems [15–17], which can be considered as a supplement to the dual reciprocity [18] or multiple reciprocity [19] boundary element method. In Ochiai's work, the application was, however, limited to the functional interpolation only. In the present work, the limitation of Ochiai's technique has been removed and it can thus successively to approximate function derivatives over general irregular domains. Concretely, in the BIE-DQM, the first and the second derivatives of a function with respect to independent variables are approximated by a weighted linear conbination of function values at all discrete nodal points as well as the normal derivatives at all boundary nodal points.

Similar to the conventional DQM, there are two main steps to solve a problem using the BIE-DQM. The first is to compute the weighting coefficients and the second is to apply the quadrature rules combined with boundary conditions. The difference between the BIE-DQM and the conventional DQM lies in how to determine the weighting coefficients. Instead of using any test functions, the weighting coefficients in the Eqs. (32) and (36) are derived from the BIE-aided numerical interpolations as described the Sect. 2. Here the BIE takes the role of test functions in a general sense. Therefore, the functional derivatives with respect to independent variables are approximated by a weighted linear conbination of not only the functions but also the boundary normal derivatives in the BIE-DQM. Owing to this reason, the BIE-DQM gets two distinct merits: it can be performed very easily and naturally both in dealing with the problems with irregular geometries and in treating boundary conditions. It is noticed that the form of the quadrature rules (32) and (36) is almost similar to that of the conventional DQM. Therefore, when solving a problem, incorporated with the boundary conditions, one need simply to apply the quadrature rules (32) and (36) to the differential equation to transform it into the linear algebraic equations as shown in the three numerical examples in the previous section. This is just the same practice performed in the conventional DQM. From this point, the BIE-DQM, as a newly developed numerical method, would be suitable theoretically to solve quite a number of second-order differential equations including the initial problems.

Here the difference between the BIE-DQM and the conventional BIE method should also be mentioned. In the BIE method [13], the corresponding fundamental solution is required before the differential equations can be solved. Moreover, using the BIE method yields a domain-type integral in most of the cases, although there are a number of efficient procedures to treat it [18–19, 26–27]. In contrast, one does not worry about the fundamental solution in the use of the BIE-DQM. In determining the weighting coefficients using the BIE method, one needs only the knowledge of fundamental solution of Laplace's equation and its higher-order form (see (4) and (5)), which are easy to obtain. In fact, from the main features of the solution procedure, the BIE-DQM belongs to the category of DQM, which is intrinsically a cell-free boundary-type numerical method. It has been shown that no domain-type integral takes place in deriving the quadrature rules (32) and (36), although the boundary elements are still required.

In summary, as the BIE-DQM can be looked as a combination of the DQM and the BIE method, it has common favorable features of the two origins while bypassing some shortcomings. It must be pointed out that the procedure in deriving the quadrature rules (32) and (36) is slightly complicated because the hypersingular and supersingular boundary integrals have to be treated numerically. This is a limitation in the application of the BIE-DQM. However, these boundary integrals can be computed indirectly with the techniques of nearly singular boundary integrals [22–23] without any difficulty.

5 Conclusions

A new differential quadrature method, the BIE-DQM, is developed in this paper, based on the interpolation technique with the BIE for the solution of problems over domains with generally irregular geometries. With the BIE-DQM, the first and the second derivatives of a function with respect to independent variables are approximated by a weighted sum of the function values at all discrete nodal points and the normal derivatives at all boundary points. As a combination of the conventional DQM and the BIE method, the BIE-DQM has common favorable features of the two origin methods while avoiding some shortcomings of them. The feasibility and versatility of the new algorithm with the BIE-DQM are assessed through three examples of a Poisson equation, a convection-diffusion equation with varying parameters and a non-linear differential equation over irregular geometry.

Acknowledgements The authors wish to thank the Australian Research Council for supporting this work.

Appendix I

The explicit formulations of derived fundamental solutions in (8), (11), (13) and (14) are as follows:

$$g_k(x,y) = \frac{1}{2\pi r} r_{,k} \tag{A1}$$

$$g_{jk}(x,y) = \frac{1}{2\pi r^2}\left(2r_{,j}r_{,k} - \delta_{jk}\right) \tag{A2}$$

$$h(x,y) = -\frac{1}{2\pi r}\frac{\partial r}{\partial n} \tag{A3}$$

$$h_k(x,y) = \frac{1}{2\pi r^2}\left(n_k - 2\frac{\partial r}{\partial n} r_{,k}\right) \tag{A4}$$

$$h_{jk}(x,y) = \frac{1}{\pi r^3}\left[\left(\delta_{jk} - 4r_{,j}r_{,k}\right)\frac{\partial r}{\partial n} + r_{,j}n_k + r_{,k}n_j\right] \tag{A5}$$

$$g_k^{[1]}(x,y) = -\frac{r}{4\pi}\left(\ln\frac{1}{r} + \frac{1}{2}\right) r_{,k} \tag{A6}$$

$$g_{jk}^{[1]}(x,y) = \frac{1}{4\pi}\left[\left(\ln\frac{1}{r} + \frac{1}{2}\right)\delta_{jk} - r_j r_k\right] \tag{A7}$$

$$h^{[1]}(x,y)=\frac{r}{4\pi}\left(\ln\frac{1}{r}+\frac{1}{2}\right)\frac{\partial r}{\partial n} \tag{A8}$$

$$h_k^{[1]}(x,y)=\frac{1}{4\pi}\left[\frac{\partial r}{\partial n}r_{,k}-\left(\ln\frac{1}{r}+\frac{1}{2}\right)n_k\right] \tag{A9}$$

$$h_{jk}^{[1]}(x,y)=\frac{1}{4\pi r}\left[\left(2r_{,j}r_{,k}-\delta_{jk}\right)\frac{\partial r}{\partial n}-r_{,j}n_k-r_{,k}n_j\right] \tag{A10}$$

Appendix II

The singularity orders of various kernels in the two-dimensional BIE are listed in Table A. The difference among them lies in the denominators in the kernels as a function of r and the distance between x and y.

Appendix III

Suppose that the boundary is smooth if the source point y is placed on the boundary in the following equations. The entries of the matrices in Eq.(16) can be expressed as follows:

$$H_{\Gamma nm}=\gamma(y^n)+\int_{\langle\Delta\Gamma_m\rangle}h(x,y^n)d\Gamma(x)\quad(y^n\in\Delta\Gamma_m) \tag{A11}$$

$$H_{\Gamma nm}=\int_{\Delta\Gamma_\mathrm{m}}h(x,y^n)d\Gamma(x)\quad(y^n\in\Gamma/\Delta\Gamma_m) \tag{A12}$$

$$G_{\Gamma nm}=\int_{\Delta\Gamma_m}g(x,y^n)d\Gamma(x)\quad(y^n\in\Gamma) \tag{A13}$$

$$H_{\Gamma nm}^{[1]}=\int_{\Delta\Gamma_m}h^{[1]}(x,y^n)d\Gamma(x)\quad(y^n\in\Gamma) \tag{A14}$$

$$G_{\Gamma nm}^{[1]}=\int_{\Delta\Gamma_m}g^{[1]}(x,y^n)d\Gamma(x)\quad(y^n\in\Gamma) \tag{A15}$$

$$\mathrm{G}_{\Gamma nl}^{\mathrm{M}}=g^{[1]}\left(x^l,y^n\right)\quad\left(x^l\in\Gamma,y^n\in\Omega\right) \tag{A16}$$

The entries of the matrices in Eq.(17) are expressed as follows:

$$H_{\Omega nm}=\int_{\Delta\Gamma_m}h(x,y^n)d\Gamma(x)\quad(y^n\in\Omega) \tag{A17}$$

$$G_{\Omega nm}=\int_{\Delta\Gamma_m}g(x,y^n)d\Gamma(x)\quad(y^n\in\Omega) \tag{A18}$$

$$H_{\Omega nm}^{[1]}=\int_{\Delta\Gamma_m}h^{[1]}(x,y^n)d\Gamma(x)\quad(y^n\in\Omega) \tag{A19}$$

$$G_{\Omega nm}^{[1]}=\int_{\Delta\Gamma_m}g^{[1]}(x,y^n)d\Gamma(x)\quad(y^n\in\Omega) \tag{A20}$$

$$\mathrm{G}_{\Omega nl}^{\mathrm{M}}=g^{[1]}\left(x^l,y^n\right)\quad\left(x^l\in\Omega,y^n\in\Omega\right) \tag{A21}$$

The entries of the matrices in Eq.(18) are expressed as follows:

$$H_{\mathrm{N}nm}=n_k(y^n)\int_{[\Delta\Gamma_m]}h_k(x,y^n)d\Gamma(x)\times(y^n\in\Gamma/\Delta\Gamma_m) \tag{A22}$$

$$H_{\mathrm{N}nm}=n_k(y^n)\int_{\Delta\Gamma_m}h_k(x,y^n)d\Gamma(x)\quad(y^n\in\Delta\Gamma_m) \tag{A23}$$

$$G_{\mathrm{N}nm}=-\gamma(y^n)+n_k(y^n)\int_{\langle\Delta\Gamma_m\rangle}g_k(x,y^n)d\Gamma(x)\times(y^n\in\Delta\Gamma_m) \tag{A24}$$

$$G_{\mathrm{N}nm}=n_k(y^n)\int_{\Delta\Gamma_m}g_k(x,y^n)d\Gamma\times(x)(y^n\in\Gamma/\Delta\Gamma_m) \tag{A25}$$

$$H_{\mathrm{N}nm}^{[1]}=n_k(y^n)\int_{\Delta\Gamma_m}h_k^{[1]}(x,y^n)d\Gamma(x)\quad(y^n\in\Gamma) \tag{A26}$$

$$G_{\mathrm{N}nm}^{[1]}=n_k(y^n)\int_{\Delta\Gamma_m}g_k^{[1]}(x,y^n)d\Gamma(x)\quad(y^n\in\Gamma) \tag{A27}$$

$$\mathrm{G}_{\mathrm{N}nl}^{\mathrm{M}}=n_k(y^n)g_k^{[1]}\left(x^l,y^n\right)\quad\left(x^l\in\Gamma,y^n\in\Omega\right) \tag{A28}$$

The entries of the matrices in Eq.(23) are expressed as follows:

$$H_{\Gamma knm}=2\int_{\Delta\Gamma_m}h_k(x,y^n)d\Gamma(x)\quad(y^n\in\Gamma/\Delta\Gamma_m) \tag{A29}$$

$$H_{\Gamma knm}=2\int_{[\Delta\Gamma_m]}h_k(x,y^n)d\Gamma(x)\quad(y^n\in\Delta\Gamma_m) \tag{A30}$$

$$G_{\Gamma knm}=2\int_{\Delta\Gamma_m}g_k(x,y^n)d\Gamma(x)\quad(y^n\in\Gamma/\Delta\Gamma_m) \tag{A31}$$

Table A Singularity orders of the kernels in the two-dimensional BIE

Singularity order		Eq. (8)	Eq. (11)	Eq. (13)	Eq. (14)
Weakly singular	$\ln\left(\frac{1}{r}\right)$	g	$h_k^{[1]}$	$h_k^{[1]}$	$g_{jk}^{[1]}$
Strong singular	$\frac{1}{r}$	h	g_k	g_k	$h_{jk}^{[1]}$
Hypersingular	$\frac{1}{r^2}$		h_k	h_k	g_{jk}
Supersingular	$\frac{1}{r^3}$				h_{jk}

$$G_{\Gamma knm} = 2 \int_{\langle \Delta\Gamma_m \rangle} g_k(x, y^n) d\Gamma(x) \quad (y^n \in \Delta\Gamma_m) \tag{A32}$$

$$H^{[1]}_{\Gamma knm} = 2 \int_{\Delta\Gamma_m} h^{[1]}_k(x, y^n) d\Gamma(x) \quad (y^n \in \Gamma) \tag{A33}$$

$$G^{[1]}_{\Gamma knm} = 2 \int_{\Delta\Gamma_m} g^{[1]}_k(x, y^n) d\Gamma(x) \quad (y^n \in \Gamma) \tag{A34}$$

$$\mathrm{G}^{\mathrm{M}}_{\Gamma knl} = 2 g^{[1]}_k\left(x^l, y^n\right)\left(x^l \in \Gamma, y^n \in \Omega\right) \tag{A35}$$

The entries of the matrices in Eq.(24) are expressed as follows:

$$H_{\Omega knm} = \int_{\Delta\Gamma_m} h_k(x, y^n) d\Gamma(x) \quad (y^n \in \Omega) \tag{A36}$$

$$G_{\Omega knm} = \int_{\Delta\Gamma_m} g_k(x, y^n) d\Gamma(x) \quad (y^n \in \Omega) \tag{A37}$$

$$H^{[1]}_{\Omega knm} = \int_{\Delta\Gamma_m} h^{[1]}_k(x, y^n) d\Gamma(x) \quad (y^n \in \Omega) \tag{A38}$$

$$G^{[1]}_{\Omega knm} = \int_{\Delta\Gamma_m} g^{[1]}_k(x, y^n) d\Gamma(x) \quad (y^n \in \Omega) \tag{A39}$$

$$\mathrm{G}^{\mathrm{M}}_{\Omega knl} = g^{[1]}_k\left(x^l, y^n\right) \quad \left(x^l \in \Omega, y^n \in \Omega\right) \tag{A40}$$

The entries of the matrices in Eq.(25) are expressed as follows:

$$H_{\mathrm{N}knm} = 2 n_j(y^n) \int_{\Delta\Gamma_m} h_{jk}(x, y^n) d\Gamma(x) \quad (y^n \in \Gamma/\Delta\Gamma_m) \tag{A41}$$

$$H_{\mathrm{N}knm} = 2 n_j(y^n) \int_{\{\Delta\Gamma_m\}} h_{jk}(x, y^n) d\Gamma(x) \quad (y^n \in \Delta\Gamma_m) \tag{A42}$$

$$G_{\mathrm{N}knm} = 2 n_j(y^n) \int_{\Delta\Gamma_m} g_{jk}(x, y^n) d\Gamma(x) \quad (y^n \in \Gamma/\Delta\Gamma_m) \tag{A43}$$

$$G_{\mathrm{N}knm} = 2 n_j(y^n) \int_{[\Delta\Gamma_m]} g_{jk}(x, y^n) d\Gamma(x) \quad (y^n \in \Delta\Gamma_m) \tag{A44}$$

$$H^{[1]}_{\mathrm{N}knm} = 2 n_j(y^n) \int_{\Delta\Gamma_m} h^{[1]}_{jk}(x, y^n) d\Gamma(x) \quad (y^n \in \Gamma/\Delta\Gamma_m) \tag{A45}$$

$$H^{[1]}_{\mathrm{N}knm} = 2 n_j(y^n) \int_{\langle \Delta\Gamma_m \rangle} h^{[1]}_{jk}(x, y^n) d\Gamma(x) \quad (y^n \in \Delta\Gamma_m) \tag{A46}$$

$$G^{[1]}_{\mathrm{N}knm} = 2 n_j(y^n) \int_{\Delta\Gamma_m} g^{[1]}_{jk}(x, y^n) d\Gamma(x) \quad (y^n \in \Gamma) \tag{A47}$$

$$\mathrm{G}^{\mathrm{M}}_{\mathrm{N}knl} = 2 n_j(y^n) g^{[1]}_k\left(x^l, y^n\right) \quad \left(x^l \in \Gamma, y^n \in \Omega\right) \tag{A48}$$

References

1. Bellman RE, Kashef BG, Casti J (1972) Differential quadrature: a technique for rapid solution of nonlinear partial differential equations; J Comput Phys 10:40–52
2. Bert CW, Malik M (1996) Differential quadrature method in computational mechanics: A review. Appl Mech Rev 49:1–27
3. Civan F (1994) Rapid and accurate solution of reactor models by the quadrature method. Computers in Chem Eng 18:1005–1009
4. Tomasiello S (1998) Differential quadrature method: application to initial-boundary-value problems J Sound and Vibration 218:573–585
5. Malik M, Civan F (1995) Comparatively study of differential quadrature and cubature methods vis-à-vis some conventional techniques in context of convection-diffusion-reaction problems Chem Eng Sci 50:531–547
6. Jang SK, Bert CW, Striz AG (1989) Application of differential quadrature to static analysis of structural components. Int J Numer Meth Eng 28:561–577
7. Wu L, Shu C (2002) Development of RBF-DQ method for approximation and its application to simulate natural convection in concentric annuli. Comput Mech 29:477–485
8. Wang X, Bert CW, Striz AG (1994) Buckling and vibration analysis of skew plates by differential quadrature method. AIAA Journal 32:886–889
9. Zong Z, Lam Y (2002) A localized differential quadrature (LDQ) method and its application to the 2D wave equation. Comput Mech 29:382–391
10. Chen WL, Striz AG, Bert CW (2000) High-accuracy plane stress and plate elements in the quadrature element method. Int J Solids and Struc 37:627–647
11. Choi ST, Chou YT (2002) Vibration analysis of non-circular curved panels by the differential quadrature method. J Sound and Vibration 259:525–539
12. Karami G, Malekzadeh P (2003) An efficient differential quadrature methodology for free vibration analysis of arbitrary straight-sided quadrilateral thin plates. J Sound and Vibration 263:415–442
13. Brebbia CA, Telles JCF, Wrobel LC (1984) Boundary element techniques-theory and applications in engineering. Springer, Heidelberg
14. Ochiai Y, Sekiya T (1995) Generation of free-form surface in CAD for dies. Adv Eng 22:113–118
15. Ochiai Y, Sekiya T (1995) Steady heat conduction analysis by improved multiple-reciprocity boundary element method. Eng Anal Bound Elem 22:113–118
16. Ochiai Y, Sekiya T (1995) Steady thermal stress analysis by improved multiple-reciprocity boundary element method. J Thermal Stress 18:603–620
17. Ochiai Y, Kobayashi T (2001) Initial strain formulation without internal cells for elasto-plastic analysis by triple-reciprocity BEM. Int J Numer Metho Eng 50:1877–1892
18. Partridge PW, Brebbia CA, Wrobel LW (1992) The Dual Reciprocity Boundary Element Method. Southampton: Computational Mechanics Publication

19. Nowak AJ, Neves AC (1994) The Multiple Reciprocity Boundary Element Method. Southampton: Computational Mechanics Publication
20. Ochiai Y (2003) Multidimensional numerical integration for meshless BEM. Eng Anal Bound Elem 27:241–249
21. Guiggiani M, Krishnasamy G, Rudolghi TJ, Rizzo FJ (1992) A general algorithm for the numerical solution of hypersingular boundary integral equations. ASME J Appl Mech 59:604–614
22. Ma H, Kamiya N (2002) A general algorithm for the numerical evaluation of nearly singular boundary integrals of various orders for two- and three-dimensional elasticity. Comput Mech 29:277–288
23. Ma H, Kamiya N (2002) Distance transformation for the numerical evaluation of near singular boundary integrals with various kernels in boundary element method. Eng Anal Bound Elem 26:329–339
24. Quan JR, Chang CT (1989) New insight in solving distributed system equations by the quadrature method-I. Analysis. Computers in Chem Eng 13:779–788
25. Quan JR, Chang CT (1989) New insight in solving distributed system equations by the quadrature method-II. Numerical experiments. Computers in Chem Eng 13:1017–1024
26. Gao XW (2002) A boundary element method without internal cells for two-dimensional and three-dimensional elastoplastic problems. ASME J Appl Mech. 69:154–160
27. Gao XW (2002) The radial integration method for evaluation of domain integrals with boundary-only discretization. Eng Anal Bound Elem 26:905–916