Trefftz plane elements of elastoplasticity with \( p \)-extension capabilities

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The present investigation is concerned with the development of Trefftz element formulation of two-dimensional elastoplastic solid with \( p \)-method capabilities. A modified variational functional is introduced and used to derive hybrid Trefftz finite element formulation for elastoplasticity of bulky solids. The formulation is applicable to either strain hardening or elastic perfectly plastic materials. A solution algorithm based on initial stress formulation is implemented into the new element model. The performance of the proposed element model is assessed by two examples and comparison is made with results obtained by other approaches.

**Key words:** Trefftz method, finite element method, elastoplasticity, \( p \)-extension

1. Introduction

The hybrid Trefftz (HT) finite element (FE) model, originating nearly thirty years ago [1, 2], has been considerably improved and has now become a highly efficient computational tool for the solution of complex boundary value problems. In history, the first attempt to generate a general-purpose HT FE formulation occurred in the study by Jirousek and Leon [1] of the effect of mesh distortion on thin plate elements. It was immediately noticed that Trefftz-complete functions represented an optimal expansion basis for hybrid-type elements where inter-element continuity need not be satisfied *a priori*. Since then, the Trefftz element concept has become increasingly popular, attracting a growing number of researchers into this field [3, 4]. Trefftz elements have been successfully applied to problems of elasticity [5], Kirchhoff plates [6, 7], moderately thick Reissner-Mindlin plates [8], thick plates [9], general 3-D solid mechanics [9, 10], axisymmetric solid mechanics [11], potential problems [12, 13], shells [14], elastodynamic problems [15, 16], transient heat conduction analysis [17], and geometrically nonlinear plate bending [18]. Although remarkable progress has been made in developing HT FE formulation for

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analysing linear elastic problems, comparatively little progress has been made in applications of HT FE formulation to materially nonlinear problems. The earliest Trefftz formulation for elastoplasticity appears to be due to Zielinski [19]. He applied a globally based Trefftz-type method to plasticity, in which the fundamental solutions with singularities outside the given area were used as shape functions and the iterative algorithm was based on the initial stress approach. Freitas and Wang [20] presented a stress model of Trefftz elements (or T-elements for short) for analysing quasi-static, gradient-dependent elastoplasticity problems. The FE approximation consists of direct estimation of the stress and plastic multiplier fields in the domain of the element as well as of the displacements and plastic multiplier gradients on its boundary. In their analysis, the model is assumed to be of geometrically linear response. The elastoplastic constitutive relations are uncoupled into elastic and plastic deformation modes.

The present paper reports development of a new HT FE model with p-extension which is suitable for practical engineering analysis and is easy to implement into standard FE computer programming code. A modified variational functional is introduced and used to derive the corresponding HT FE formulation of elastoplastic materials. The formulation is applicable to strain hardening and elastic perfectly plastic materials. The Mises yield criteria of these materials and the initial stress scheme are employed to calculate the so-called plastic stresses and strains. Two numerical examples are considered to demonstrate the efficiency of the proposed element formulation in nonlinear analysis of elastoplastic problems.

2. Basic governing equations of elastoplasticity

Consider an elastoplastic solid, occupying a two-dimensional arbitrary shaped domain $\Omega$ bounded by its boundary $\Gamma$. Throughout this paper, repeated indices $i$, $j$ and $k$ imply the summation convention of Einstein. In the “small” deformation range, compatibility and equilibrium are expressed in the incremental form

$$\begin{align*}
[L]^T\{\dot{\sigma}\} + \{\dot{b}\} &= 0 \quad \text{in} \ \Omega, \\
\{\dot{\varepsilon}\} &= [L]\{\dot{u}\} \quad \text{in} \ \Omega, \\
\{\dot{\sigma}\} &= [D_{ep}][\dot{\varepsilon}] \quad \text{in} \ \Omega, \\
\{\dot{u}\} &= \{\dot{\hat{u}}\} \quad \text{on} \ \Gamma_u, \\
\{\dot{t}\} &= [\dot{\sigma}][n] = \{\dot{\hat{t}}\} \quad \text{on} \ \Gamma_\sigma,
\end{align*}$$

where

$$\begin{align*}
\{\dot{u}\} &= \{\dot{u}_1, \dot{u}_2\}_T, \quad \{\dot{\varepsilon}\} = \{\dot{\varepsilon}_{11}, \dot{\varepsilon}_{22}, 2\dot{\varepsilon}_{12}\}_T, \quad \{\dot{\sigma}\} = \{\dot{\sigma}_{11}, \dot{\sigma}_{22}, \dot{\sigma}_{12}\}_T,
\end{align*}$$
\[ \{ \dot{u} \} = \{ \dot{u}_1, \dot{u}_2 \}^T, \quad \{ \dot{\bar{t}} \} = \{ \dot{\bar{t}}_1, \dot{\bar{t}}_2 \}^T, \quad \{ \dot{b} \} = \{ \dot{b}_1, \dot{b}_2 \}^T, \quad \{ n \} = \{ n_1, n_2 \}^T, \quad (7) \]

\[ [L] = \begin{bmatrix} \frac{\partial}{\partial x_1} & 0 \\ 0 & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} \end{bmatrix}, \quad [\dot{\sigma}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}, \quad (8) \]

\[ [D_{ep}] = [D_e] - [D_e] \left\{ \frac{\partial F}{\partial \{\sigma\}} \right\}^T \left[ [D_e] \left\{ A + \left\{ \frac{\partial F}{\partial \{\sigma\}} \right\} \right\} \right]^{-1} \left[ [D_e] \left\{ \frac{\partial F}{\partial \{\sigma\}} \right\} \right], \quad (9) \]

with

\[ [D_e] = \frac{E}{1-\mu^2} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix}, \quad (10) \]

for plane stress, and

\[ [D_e] = \frac{E(1-\mu)}{(1+\mu)(1-2\mu)} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1-\mu & 0 \\ 0 & 0 & \frac{1-2\mu}{2(1-\mu)} \end{bmatrix}, \quad (11) \]

for plain strain, in which \( E \) and \( \mu \) are Young's modulus and Poisson's ratio, a dot over a variable indicates an increment, \( \dot{\sigma}_{ij} \) and \( \dot{\epsilon}_{ij} \) are stresses and strains, \( \dot{b}_i \) body forces, \( \dot{u}_i \) displacements, \( \dot{\bar{t}}_j \) prescribed boundary displacements and tractions, \( \Gamma_u \) and \( \Gamma_\sigma \) the constrained and free parts of the boundary \( \Gamma \), \( n_i \) the direction cosines of the outward normal to \( \Gamma \), uniquely defined everywhere since \( \Gamma \) will be assumed to be smooth, for simplicity. The yield function \( F \) and parameter \( [A] \) are defined in Appendix A. The strain \( \{ \dot{\epsilon} \} \) may be decomposed into its elastic and plastic parts:

\[ \{ \dot{\epsilon} \} = \{ \dot{\epsilon}^e \} + \{ \dot{\epsilon}^p \}. \quad (12) \]

With expression (12), the stress-strain equation can be written as:

\[ \{ \dot{\sigma} \} = [D_e] (\{ \dot{\epsilon} \} - \{ \dot{\epsilon}^p \}) = \{ \dot{\sigma}^e \} + \{ \dot{\sigma}^p \}, \quad (13) \]

where \( \{ \dot{\sigma}^e \} \) and \( \{ \dot{\sigma}^p \} \) are defined by

\[ \{ \dot{\sigma}^e \} = [D_e] \{ \dot{\epsilon} \}, \quad \{ \dot{\sigma}^p \} = -[D_e] \{ \dot{\epsilon}^p \}. \quad (14) \]
Substituting Eq. (13) into (1) and using Eq. (2), the governing differential equation (1) can be rewritten as [21]

\[
[L]^T[D_e][L]\{\dot{u}\} + \{\dot{b}^p\} = 0 \text{ in } \Omega
\]

(15)

and the natural boundary condition (5) is replaced by

\[
\{\dot{t}_0\} = [\dot{\sigma}^e]\{n\} = ([\dot{\sigma}] - [\dot{\sigma}^p])\{n\} = \{\dot{t}\} - [\dot{\sigma}^p]\{n\} = \{\dot{t}_0\},
\]

(16)

where \(\{\dot{b}^p\} = \{\dot{b}\} + [L]^T[\dot{\sigma}^p]\). The plastic part of stresses \(\{\dot{\sigma}^p\}\) is described in Appendix A.

Moreover, in the Trefftz FE approach, Eqs. (1)–(16) should be completed by adding the following inter-element continuity requirements:

\[
\{\dot{u}\}_e = \{\dot{u}\}_f \quad \text{(on } \Gamma_e \cap \Gamma_f, \text{ conformity)}
\]

(17)

\[
\{\dot{t}\}_e + \{\dot{t}\}_f = 0 \quad \text{(on } \Gamma_e \cap \Gamma_f, \text{ reciprocity)}
\]

(18)

where ‘e’ and ‘f’ stand for any two neighbouring elements. Equations (1)–(18) are taken as a basis to establish the modified variational principle for Trefftz FE analysis in elastoplasticity.

3. Modified variational principles

The Trefftz FE equation for an elastic solid can be established by the variational approach [4]. Since the stationary conditions of the traditional potential and complementary variational functional cannot guarantee the satisfaction of the inter-element continuity condition which is required in the Trefftz FE analysis, some new variational functionals need to be developed. Piltner [22] presented two different variational formulations to treat special elements with holes or cracks. The formulation consists of a conventional potential energy and a least square functional. The least square functional was not added as a penalty function to the potential functional, but is minimized separately for the special elements considered. Jirousek [23] developed a variational functional in which either the displacement conformity or the reciprocity of the conjugate tractions is enforced at the element interfaces. Jirousek and Zielinski [24] obtained two complementary HT formulations based on weighted residual method. The dual formulations enforced the reciprocity of boundary tractions more strongly than the conformity of the displacement fields. Qin [4] presented a modified variational principle based HT displacement frame.
Here we extend the modified variational functional to the case of elastoplastic problem which is based on the total potential energy as:

\[ \Psi_m = \sum_e \Psi_{me} = \sum_e \left\{ \Psi_e + \int_{\Gamma_{ue}} \left( \{ \hat{\mathbf{u}} \}^T - \{ \mathbf{i} \}^T \right) \{ \mathbf{i} \} ds - \int_{\Gamma_{1e}} \{ \mathbf{i} \}^T \{ \hat{\mathbf{u}} \} ds \right\}, \quad (19) \]

where

\[ \Psi_e = \iint_{\Omega_e} (\Psi(\{ \dot{\varepsilon} \}) - \{ \mathbf{b}^{ep} \}^T \{ \hat{\mathbf{u}} \}) d\Omega - \int_{\Gamma_{te}} \{ \hat{\mathbf{i}} \}^T \{ \hat{\mathbf{u}} \} ds, \quad (20) \]

with

\[ \Psi(\{ \dot{\varepsilon} \}) = \frac{1}{2} \{ \dot{\varepsilon} \}^T [\mathbf{D}_e] \{ \dot{\varepsilon} \}, \quad (21) \]

in which Eq. (15) is assumed to be satisfied, a priori. The terminology “modified principle” refers, here, to the use of a conventional functional \( \Psi_e \) and some modified terms for the construction of a special variational principle to account for additional requirements such as the condition defined in Eqs. (17) and (18).

The boundary \( \Gamma_e \) of a particular element consists of the following parts:

\[ \Gamma_e = \Gamma_{ue} \cup \Gamma_{te} \cup \Gamma_{1e}, \quad (22) \]

where

\[ \Gamma_{ue} = \Gamma_u \cap \Gamma_e, \quad \Gamma_{te} = \Gamma_t \cap \Gamma_e, \quad (23) \]

and \( \Gamma_{1e} \) is the inter-element boundary of the element ‘e’. We now show that the stationary condition of the functional (19) leads to Eqs. (4), (5), (17), (18), and \( \{ \dot{\mathbf{u}} \} = \{ \hat{\mathbf{u}} \} \) on \( \Gamma_t \cup \sum_e \Gamma_{1e} \), and present the theorem on the existence of extremum of the functional, which ensures that an approximate solution can converge to the exact one. For the variational functional \( \Psi_m \), we have the following two statements:

(a) Modified complementary principle

\[ \delta \Psi_m = 0 \Rightarrow (4), (5), (17), (18) \] and \( \{ \dot{\mathbf{u}} \} = \{ \hat{\mathbf{u}} \} \) on \( \Gamma_t \cup \sum_e \Gamma_{1e} \),

where \( \delta \) stands for the variation symbol.

(b) Theorem on the existence of extremum

If the expression

\[ \iint_{\Omega} \delta^2 \Psi(\{ \dot{\varepsilon} \}) d\Omega - \int_{\Gamma_{na}} \delta \{ \hat{\mathbf{i}} \}^T \delta \{ \hat{\mathbf{u}} \} ds - \sum_e \int_{\Gamma_{1e}} \delta \{ \hat{\mathbf{i}} \}^T \{ \hat{\mathbf{u}} \} ds \quad (25) \]
is uniformly positive (or negative) in the neighborhood of \( \{u\}_0 \), where the displacement \( \{u\}_0 \) has such a value that \( \Psi_m(\{\hat{u}\}_0) = (\Psi_m)_0 \), and where \( (\Psi_m)_0 \) stands for the stationary value of \( \Psi_m \), we have

\[
\Psi_m \geq (\Psi_m)_0 \quad \text{[or \( \Psi_m \leq (\Psi_m)_0 \)]},
\]

in which the relation that \( \{\hat{u}\}_e = \{\hat{u}\}_I \) is identical on \( \Gamma_e \cap \Gamma_I \) has been used. This is due to the definition in Eq. (37) of Section 4.

**PROOF:** First, we derive the stationary conditions of functional (19). To this end, performing a variation of \( \Psi_m \) and noting that Eq. (1) holds true *a priori* by the previous assumption, we obtain

\[
\delta \Psi_m = \int_{\Gamma_u} [(\hat{\mathbf{t}})^T \delta(\{\hat{\mathbf{u}}\} - \{\hat{\mathbf{u}}\}) + (\{\hat{\mathbf{u}}\})^T \delta(\tilde{\mathbf{u}})] \, ds + \\
\quad + \int_{\Gamma_t} ((\hat{\mathbf{t}})^T - \{\mathbf{t}\}) \delta(\mathbf{u}) \, ds + \sum_e \int_{\Gamma_{te}} [(\{\mathbf{t}\})^T \delta(\{\hat{\mathbf{u}}\}) - (\mathbf{u})^T \delta(\mathbf{t})] \, ds.
\]

(27)

Therefore, the Euler equations for expression (27) are Eqs. (4), (5), (17), (18) and \( \{\hat{\mathbf{u}}\} = \{\hat{\mathbf{u}}\} \) on \( \Gamma_t \cup \sum_e \Gamma_{te} \) as the quantities \( \delta \{\mathbf{t}\} \), \( \delta \{\mathbf{u}\} \) and \( \delta \{\hat{\mathbf{u}}\} \) may be arbitrary.

The principle (24) has thus been proved. This indicates that the stationary condition of the functional satisfies the required boundary and inter-element continuity equations and can thus be used for deriving Trefftz FE formulation.

As for the proof of the theorem on the existence of extremum, we may complete it by way of the so-called “second variational approach” [25]. In doing this, performing variation of \( \delta \Psi_m \) and using the constrained conditions (1), we find

\[
\delta^2 \Psi_m = \int_{\Omega} \delta^2 \Psi(\{\mathbf{r}\}) \, d\Omega - \int_{\Gamma_u} \delta \{\mathbf{t}\}^T \delta \{\mathbf{u}\} \, ds - \sum_e \int_{\Gamma_{te}} \delta \{\mathbf{t}\}^T \delta \{\hat{\mathbf{u}}\} \, ds = \quad \text{expression (25)}.
\]

(28)

Therefore the theorem has been proved from the sufficient condition of the existence of a local extreme of a functional [25]. This completes the proof.

4. Assumed fields

The main idea of the Trefftz FE approach is to establish a FE formulation whereby intra-element continuity is enforced on a non-conforming internal displacement field chosen so as to *a priori* satisfy the governing differential equation of the problem under consideration [4]. In other words, as an obvious alternative to the Rayleigh-Ritz method as a basis for a FE formulation, the model, here,
is based on the method of Trefftz [26]. With this method the solution domain $\Omega$ is subdivided into elements, and over each element ‘e’ the assumed intra-element fields (for a two-dimensional problem) are

$$\{\dot{\mathbf{u}}\} = \{\ddot{\mathbf{u}}\} + \sum_{i=1}^{m} \{\mathbf{N}_i\}^T \{\ddot{\mathbf{c}}_i\} = \{\ddot{\mathbf{u}}\} + [\mathbf{N}]\{\ddot{\mathbf{c}}\}, \quad (29)$$

where $\{\ddot{\mathbf{c}}_i\}$ stands for undetermined coefficient vector, and $\{\ddot{\mathbf{u}}\} = \{\ddot{u}_1, \ddot{u}_2\}^T$ and $\{\mathbf{N}_i\}$ are known functions. $\{\dot{\mathbf{u}}\} = \{\dot{\mathbf{u}}(x)\}$ and $\{\mathbf{N}_i(x)\}$ in Eq. (29) should be chosen such that

$$[\mathbf{L}]^T[\mathbf{D}_e][\mathbf{L}]\{\ddot{\mathbf{u}}\} = 0 \text{ and } [\mathbf{L}]^T[\mathbf{D}_e][\mathbf{L}]\{\ddot{\mathbf{c}}_i\} = 0, \quad (i = 1, 2, \ldots, m) \quad (30)$$

everywhere in $\Omega_e$. A complete system of homogeneous solutions $\{\mathbf{N}_i\}$ can be generated in a systematic way from Muskhelishvili’s complex variable formulation [5]. For convenience, we list the results presented in [5] as follows:

$$2G\{\mathbf{N}_j\} = \left\{ \begin{array}{l} \text{Re } Z_{1k} \\ \text{Im } Z_{1k} \end{array} \right\} \quad \text{with } Z_{1k} = i\kappa z^k + kiz^{k-1}, \quad (31)$$

$$2G\{\mathbf{N}_{j+1}\} = \left\{ \begin{array}{l} \text{Re } Z_{2k} \\ \text{Im } Z_{2k} \end{array} \right\} \quad \text{with } Z_{2k} = \kappa z^k - kiz^{k-1}, \quad (32)$$

$$2G\{\mathbf{N}_{j+2}\} = \left\{ \begin{array}{l} \text{Re } Z_{3k} \\ \text{Im } Z_{3k} \end{array} \right\} \quad \text{with } Z_{3k} = iz^k, \quad (33)$$

$$2G\{\mathbf{N}_{j+3}\} = \left\{ \begin{array}{l} \text{Re } Z_{4k} \\ \text{Im } Z_{4k} \end{array} \right\} \quad \text{with } Z_{4k} = -z^k, \quad (34)$$

where $z = x + iy$ and $i = \sqrt{-1}$.

The particular solution $\{\ddot{\mathbf{u}}\}$ can be obtained by means of its source function. The source function corresponding to Eq. (15) has been given in [27] as

$$u_{ij}^*(r_{pq}) = \frac{1+\nu}{4\pi E}[-(3-\nu)\delta_{ij} \ln r_{pq} + (1+\nu)r_{pq,ij} r_{pq,j}], \quad (35)$$

where $r_{pq} = [(x_q-x_p)^2+(y_q-y_p)^2]^{1/2}$, $u_{ij}^*(r_{pq})$ denotes $i$th component of displacement at the field point $q$ of the solid under consideration when a unit point force
is applied in the \( j \)th direction at the source point \( p \). Using this source function, the particular solution can be expressed by

\[
\{ \dot{\mathbf{u}} \} = \left\{ \dot{\bar{u}} \right\} = \int\int_{\Omega}^{ep} \left\{ \begin{array}{c} u_1^j \\ u_2^j \end{array} \right\} d\Omega.
\]  

(36)

The unknown coefficient \{\dot{c}\} may be calculated from conditions on the external boundary and/or the continuity conditions on the inter-element boundary. Thus various Trefftz element models can be obtained by using different approaches to enforce these conditions. In the majority of them a hybrid technique is used, whereby the elements are linked through an auxiliary conforming displacement frame which has the same form as in conventional FE method. This means that in the Trefftz FE approach, a conforming displacement field should be independently defined on the element boundary to enforce the field continuity between elements and also to link the coefficient \{\dot{c}\}, appearing in Eq. (29), with nodal displacement \{\dot{d}\}. The frame is defined as

\[
\{ \dot{\bar{u}}(x) \} = [\mathbf{\tilde{N}}(x)]\{\dot{\bar{d}}\}, \quad (x \in \Gamma_e),
\]

(37)

where the symbol “~” is used to specify that the field is defined on the element boundary only, \{\dot{\bar{d}}\} = \{\dot{d}(\dot{c})\} stands for the vector of the nodal displacements which are the final unknowns of the finite element formulation, \( \Gamma_e \) represents the boundary of element \( e \), and \([\mathbf{\tilde{N}}]\) is a matrix of the corresponding shape functions which are the same as those in conventional FE formulation.

In the development of the present \( p \)-element, the following assumptions are adopted. First of all, the problem is assumed to be plane strain (or plane stress) of elastoplasticity. Secondly, the element may be of a general quadrilateral shape or a triangle shape with two degrees of freedom (DOF) \( (u_1, u_2) \) at each corner node (see Fig. 1). Thirdly, to achieve higher order variations, an optional number of extra hierarchic modes is introduced along with the hierarchic DOF, \( a_{ci} \) for \( \bar{u}_1 \), \( b_{ci} \) for \( \bar{u}_2 \), which are conveniently associated with mid-side node C (see Fig. 1). Thus, along the side A-C-B of a particular element (see Fig. 1), a simple interpolation of the frame displacement field can be given in the form

\[
\{ \dot{\mathbf{u}}_{AB} \} = \left\{ \begin{array}{c} \dot{u}_{1AB} \\ \dot{u}_{2AB} \end{array} \right\} = \left[ \begin{array}{cc} \tilde{N}_1 & 0 \\ 0 & \tilde{N}_1 \end{array} \right] \{ \dot{\mathbf{d}}_{AB} \} + \sum_{i=1}^{M} \gamma^{i-1} R_i \left\{ \begin{array}{c} a_{ci} \\ b_{ci} \end{array} \right\},
\]

(38)

where \( M \) is the order of the hierarchic DOF, and \{\dot{\mathbf{d}}_{AB}\} = \{\dot{u}_{1A} \, \dot{u}_{2A} \, \dot{u}_{1B} \, \dot{u}_{2B}\}^T.

\[
\tilde{N}_1 = \frac{1 - \xi}{2}, \quad \tilde{N}_2 = \frac{1 + \xi}{2}, \quad R_i = \xi^{i-1}(1 - \xi^2)
\]

(39)
Fig. 1. The Trefftz p-element in plane elastoplasticity.

and where ξ is defined in Fig. 1.

The coefficient γ is equal to +1 or −1 according to the orientation of the side A-C-B (see Fig. 1) in the global coordinate system (X, Y):

\[
\gamma = \begin{cases} 
+1 & \text{if } X_B - X_A \leq Y_B - Y_A \\
-1 & \text{if } X_B - X_A > Y_B - Y_A
\end{cases}
\] (40)

The purpose of using the coefficient γ is to ensure a univocal definition of the frame functions \( \tilde{u} \) in terms of parameters \( a_{ci} \) and \( b_{ci} \), common to two elements sharing the mid-side node C.

The tractions \( \{\dot{t}_0\} = \{\dot{t}_{10}, \dot{t}_{20}\}^T \) appearing in Eq. (16) can be derived from Eqs. (5), (16), (29) and (36), and denote

\[
\{\dot{t}_0\} = [\dot{\sigma}^e] \{n\} = [Q] \{\dot{c}\} + [\tilde{T}].
\] (41)

5. Element stiffness matrix

The element matrix equation can be generated by setting \( \delta \Psi_{me} = 0 \). To simplify the derivation, we first transform all domain integrals in Eq. (19) into boundary ones. In fact, by reason of solution properties of the intra-element trial functions the functional \( \Psi_{me} \) can be simplified to

\[
\Psi_{me} = -\frac{1}{2} \int_{\Omega_e} \{\dot{b}^p\}^T \{\dot{u}\} d\Omega + \frac{1}{2} \int_{\Gamma_e} \{\dot{t}_0\}^T \{\dot{u}\} ds - \int_{\Gamma_e} (\{\dot{t}_0\}^T \{\dot{u}\}) ds - \int_{\Gamma_e} \{\dot{t}_0\}^T (\{\dot{u}\} - \{\tilde{u}\}) ds + \frac{1}{2} \int_{\Gamma_e} \{\dot{t}_p\}^T \{\dot{u}\} ds - \int_{\Gamma_e} \{\dot{t}_p\}^T (\{\dot{u}\} - \{\tilde{u}\}) ds - \int_{\Gamma_e} \{\dot{t}_p\}^T \{\dot{u}\} ds - \int_{\Gamma_e} \{\dot{t}_p\}^T (\{\dot{u}\} - \{\tilde{u}\}) ds - \int_{\Gamma_e} \{\dot{t}_p\}^T \{\dot{u}\} ds.
\] (42)
Substituting the expressions given in Eqs. (29) and (37) into (41) produces

\[
\Psi_{me} = \frac{1}{2} \{\mathbf{e}\}^T [H] \{\dot{\mathbf{e}}\} + \{\mathbf{e}\}^T [S] \{\dot{\mathbf{d}}\} + \{\dot{\mathbf{c}}\}^T \{\mathbf{r}_1\} + \{\dot{\mathbf{d}}\}^T \{\mathbf{r}_2\}
\]

+ terms without \{\mathbf{e}\} or \{\dot{\mathbf{d}}\},

(43)
in which the matrices \([H], [S]\) and the vectors \{\mathbf{r}_1\}, \{\mathbf{r}_2\} are as follows:

\[
[H] = \int_{\Gamma_u} [Q]^T [N] ds,
\]

\[
[S] = - \int_{\Gamma_{ue}} [Q]^T [\tilde{N}] ds - \int_{\Gamma_{te}} [Q]^T [\tilde{N}] ds,
\]

\[
\{\mathbf{r}_1\} = - \frac{1}{2} \int_{\Omega_u} [N]^T \{\dot{\mathbf{b}}^{ep}\} d\Omega + \frac{1}{2} \int_{\Gamma_u} \{[Q]^T \{\dot{\mathbf{u}}\} + [N]^T (\{\mathbf{T}\} + \{\mathbf{i}^p\})\} ds
\]

+ \int_{\Gamma_{ue}} [Q]^T \{\dot{\mathbf{u}}\} ds - \int_{\Gamma_{te}} [N]^T \{\dot{\mathbf{t}}\} ds,

\[
\{\mathbf{r}_2\} = - \int_{\Gamma_{ue}} [\tilde{N}]^T \{\mathbf{T}\} ds - \int_{\Gamma_{te}} [\tilde{N}]^T (\{\mathbf{T}\} + \{\mathbf{i}^p\}) ds,
\]

(44)

where \{\dot{\mathbf{i}}_0\} = \{\dot{\mathbf{i}}_{10}, \dot{\mathbf{i}}_{20}\}^T is the prescribed traction vector.

To enforce inter-element continuity on the common element boundary, the unknown vector \{\mathbf{e}\} should be expressed in terms of nodal DOF \{\dot{\mathbf{d}}\}. An optional relationship between \{\mathbf{e}\} and \{\dot{\mathbf{d}}\} in the sense of variation can be obtained from

\[
\frac{\partial \Psi_{me}}{\partial \{\dot{\mathbf{e}}\}^T} = [H] \{\dot{\mathbf{e}}\} + [S] \{\dot{\mathbf{d}}\} + \{\mathbf{r}_1\} = 0.
\]

This leads to

\[
\{\dot{\mathbf{e}}\} = - [G] \{\dot{\mathbf{d}}\} - \{\mathbf{g}\},
\]

(46)

where \([G] = [H]^{-1} [S]\) and \{\mathbf{g}\} = \[H]^{-1} \{\mathbf{r}_1\}, and then straightforwardly yields the expression of \(\Psi_{me}\) only in terms of \{\dot{\mathbf{d}}\} and other known matrices

\[
\Psi_{me} = - \frac{1}{2} \{\dot{\mathbf{d}}\}^T [G]^T [H] [G] \{\dot{\mathbf{d}}\} + \{\dot{\mathbf{d}}\}^T \{\mathbf{r}_2\} - [G]^T \{\mathbf{r}_1\}
\]  

+ terms without \{\dot{\mathbf{d}}\},

(47)

Therefore, the element stiffness matrix equation can be obtained by taking the vanishing variation of the functional \(\Psi_{me}\) as

\[
\frac{\partial \Psi_{me}}{\partial \{\dot{\mathbf{d}}\}^T} = 0 \Rightarrow \{\mathbf{K}\} \{\dot{\mathbf{d}}\} = \{\dot{\mathbf{P}}\} = \{\dot{\mathbf{P}}_0\} + \{\mathbf{P}(\{\dot{\mathbf{p}}\})\},
\]

(48)
where $[K] = [G]^T[H][G]$ and $\{P\} = -[G]^T\{r_1\} + \{r_2\}$ are, respectively, the element stiffness matrix and the equivalent nodal force vector. The expression (48) is the elemental stiffness matrix equation for Trefftz FE analysis. $[K]$ and $\{P_0\}$ can be calculated in the usual way, while $\{P(\{\sigma^P\})\}$ contains unknown $\sigma^P_{ij}$. An iterative procedure is thus required. The procedure is briefly described in the next section.

Moreover, to ensure a good numerical conditioning during the inversion of matrix $[H]$ the homogeneous solution $\{N_i\}$ in Eq. (29) has to be expressed in terms of suitably scaled local coordinates $(x_1, x_2)$ originated at the element centroid (Fig. 1):

$$x_1 = (X - X_c)/a, \quad x_2 = (Y - Y_c)/a,$$

where $X_c$ and $Y_c$ stand for global coordinates at centroid of the element, and $a$ is the average distance between the centroid and node $i$ of the element:

$$a = \sum_{i=1}^{N} (x_{1i}^2 + x_{2i}^2)^{1/2}/N$$

and where $N$ is the number of nodes for the element under consideration.

Besides, it should be pointed out that the standard hybrid Trefftz formulation implies that all terms representing the rigid body modes (three in plane problem) have been discarded from the intraelement field to prevent the matrix $[H]$ from being singular. However, after the finite element assembly has been solved for nodal displacements, those missing terms can easily be recovered. The intraelement field $\{\hat{u}\}$ in an element may be augmented by the rigid body modes:

$$\{\hat{u}_e\} = \{\hat{u}_e\} + [N_e]\{\hat{c}_e\} + \begin{bmatrix} 1 & 0 & x_2 \\ 0 & 1 & -x_1 \end{bmatrix} \{\hat{c}_0\},$$

where ‘e’ stands for element. Using a least-square procedure to match the nodal displacements $(\hat{u}_{1i}, \hat{u}_{2i})$ at corner nodes

$$\sum_{i=1}^{N} [(\hat{u}_{1i} - \hat{u}_{1i})^2 + (\hat{u}_{2i} - \hat{u}_{2i})^2] = \text{min (with respect to } \{\hat{c}_0\}).$$

Equation (52) provides

$$\{\hat{c}_0\} = [R]^{-1}\{r\}$$

with

$$[R] = \sum_{i=1}^{N} \begin{bmatrix} 1 & 0 & x_{2i} \\ 0 & 1 & -x_{1i} \end{bmatrix}, \quad \{r\} = \sum_{i=1}^{N} \begin{bmatrix} \hat{u}_{1i} - \hat{u}_{1i} \\ \hat{u}_{2i} - \hat{u}_{2i} \end{bmatrix}.\quad \begin{bmatrix} \hat{u}_{1i} - \hat{u}_{1i} \\ \hat{u}_{2i} - \hat{u}_{2i} \end{bmatrix}$$

where $\hat{u}_{1i}$ and $\hat{u}_{2i}$ are the nodal displacements.
6. Determination of elastoplastic status and iterative scheme

6.1 Determination of elemental stress status

To determine the stress status of an element at any loading step, say \( i \)th step, assuming that \( \{ \sigma \}_i \) stand for the stress at \( i \)th loading step, if \( F(\{ \sigma \}_i) < 0 \), the element is said to be in elastic condition. Then, the stress increment at \( i \)th loading step can be calculated by

\[
\{ \sigma \}_i = \{ \sigma \}_{i-1} + \{ \dot{\sigma} \}_i = \{ \sigma \}_{i-1} + [D_e]\{ \dot{\varepsilon} \}_i.
\]  \( (55) \)

However, if \( F(\{ \sigma \}_{i-1}) < 0 \) and \( F(\{ \sigma \}_i) > 0 \), the element is said to experience transition from elastic to plastic conditions during the loading step. Let \( \xi \{ \dot{\sigma} \}_i \) be the proportion of the stress increment at which the plastic behaviour is encountered \([28]\). Then \( r \) can be determined from

\[
F(\{ \sigma \}_{i-1} + \xi \{ \dot{\sigma} \}_i) = 0.
\]  \( (56) \)

Solving Eq. (56) yields

\[
\xi = \frac{-B + \sqrt{B^2 - 4AC}}{2A},
\]  \( (57) \)

where

\[
A = \frac{1}{2}[(\dot{s}_{11}^e)_i^2 + (\dot{s}_{22}^e)_i^2 + 2(\dot{s}_{12}^e)_i^2], \quad B = (\dot{s}_{11}^e)_i(s_{11})_i + (\dot{s}_{22}^e)_i(s_{22})_i + 2(\dot{s}_{12}^e)_i(s_{12})_i,
\]

\[
C = (s_{11})_i^2 + (s_{22})_i^2 + 2(s_{12})_i^2 - \frac{2}{3}(\sigma_s)_i^2.
\]  \( (58) \)

Having determined the parameter \( \xi \), the stress increment \( \{ \dot{\sigma} \}_i \) is calculated from

\[
\{ \dot{\sigma} \}_i = (\xi [D_e] + (1 - \xi)[D_{ep}])\{ \dot{\varepsilon} \}_i.
\]  \( (59) \)
6.2 Iterative scheme

The steps of the iterative procedure in this analysis are described as follows:

At the first loading step, assume \( P(\{\dot{\sigma}^p\}) = 0 \). Then solve Eq. (48) for \( \{\dot{d}\} \) and calculate the parameter \( \xi \) in section 6.1 above, \( \{\dot{\sigma}^e\} \), and \( \{\dot{\sigma}^p\} \), using Eqs. (14) or (A11) as well as \( P(\{\dot{\sigma}^p\}) \).

Suppose that \( \{d\}^{(k)}, \{\sigma\}^{e(k)} \) and \( \{\sigma\}^{p(k)} \) stand for the \( k \)th approximations, which can be obtained from the proceeding cycle of iteration. The \( (k+1) \)th solution may be calculated as follows:

1. Apply a load increment and take \( \{d\}^{(k)}, \{\sigma\}^{e(k)} \) and \( \{\sigma\}^{p(k)} \) as initial values;
2. Enter the iterative cycle for \( i = 1, 2, \ldots \). Calculate the stress increments in all elements using the formulation in Appendix A. Calculate the total stress and compile a list of yielded elements. Calculate the correct stresses in the elastoplastic elements by using Eqs. (14) or (A11);
3. Calculate \( P(\{\dot{\sigma}^p\}) \) in Eq. (48) using the current value of \( \{\dot{d}\}^{[i-1]} \) and \( \{\dot{\sigma}^p\}^{[i-1]} \), where the superscript \( [i-1] \) stands for the increments at the \( (i-1) \)th iterative cycle. Solve Eq. (48) for \( \{\dot{d}\}^{[i]} \);
4. If \( \eta_i = \|([\{\dot{d}\}^{[i]}]^T \{d\}^{[i]} - (\{d\}^{[i-1]})^T \{\dot{d}\}^{[i-1]})/([\{\dot{d}\}^{[i-1]}]^T \{\dot{d}\}^{[i-1]} \| \leq \eta \) (\( \eta \) is a convergence tolerance), proceed to the next loading step and calculate
   \[
   \{d\}^{(k+1)} = \{d\}^{(k+1)} + \{\dot{d}\}^{[i]}, \\
   \{\sigma\}^{(k+1)} = \{\sigma\}^{(k)} + \{\dot{\sigma}\}^{[i]}, \\
   \{\sigma^p\}^{(k+1)} = \{\sigma^p\}^{(k)} + \{\dot{\sigma}^p\}^{[i]},
   \]
   otherwise, go back to step (2).

It is noted that the required stiffness matrices appearing in Eq. (48) do not change through each step of computation. Hence, once the matrix \( [K] \) has been formed, they can be stored in the core and used in each cycle of iteration without any change. Obviously this can save a large amount of computing time.

7. Numerical applications

Since the main purpose of this paper is to outline the basic principles of the proposed method and demonstrate its feasibility, numerical assessment is limited to an infinitely long thick cylinder under internal pressure and a perforated strip under tension. In all the computation, the convergence tolerance \( \eta = 0.001 \) is used.

Example 1: An infinitely long thick cylinder subjected to internal pressure \( p \). In this example, the plane strain expansion of a thick cylinder under internal pressure is studied (see Fig. 2). An elastic-perfectly plastic material is assumed with the von
Fig. 2. Mesh and geometry of internally pressurized thick cylinder.

Fig. 3. Hoop stress distributions for the thick cylinder along radius: (a) analytical solution [29]; (b) present method.

Mises yield criterion (i.e., $H' = 0$). The initial parameters used are taken to be: $a = 100$ mm, $b = 200$ mm, $E = 20.58 \times 10^4$ MPa, $\nu = 0.3$, uniaxial yield stress $\sigma_s = 235.2$ MPa and the internal pressure $p = 135.3$ MPa. Due to the problem being radially symmetric, one quadrant of the thick cylinder is used in the analysis. The numerical results of stress distributions are shown in Figs. 3 and 4, and comparison is made with the analytical solution [29]. It can be seen from Figs. 3 and 4 that the present results are in good agreement with the analytical solution. Figure 5 shows the results of stress versus $M$, here $M$ is the number of hierarchic degrees of freedom. In Figure 5 $\alpha$ and $\beta$ are defined by

$$
\alpha = 100 \times \frac{(\sigma_{rr}(\text{exact}) - \sigma_{rr}(\text{present FE}))}{\sigma_{rr}(\text{exact})},
$$

$$
\beta = 100 \times \frac{(\sigma_{\theta\theta}(\text{exact}) - \sigma_{\theta\theta}(\text{present FE}))}{\sigma_{\theta\theta}(\text{exact})},
$$

(61)
Fig. 4. Radial stress distributions for the thick cylinder along radius: (a) analytical solution [29]; (b) present method.

Fig. 5. Percentage error of stress versus hierarchic DOF \( M \) \( (r = 0.15 \text{ m}) \).
and the stresses are calculated at radius $r = 0.15$ m. It is evident that the hierarchic DOF can improve the convergent performance. In the course of computation, convergence was achieved with less than 8 iterations.

Example 2: A perforated strip in axial tension under plane stress condition. This example was analysed by using finite element method [28] and boundary element method [30]. A quadrant of the strip is modelled by 214 elements (Fig. 6) and von Mises yield criterion is used in the calculation. The material constants used for the analysis are: $E = 7000$ kgf/mm$^2$ (1 kgf/m$^2 = 9.80665$ Pa), $\nu = 0.2$, uniaxial yield stress $\sigma_s = 24.3$ kgf/mm$^2$, the linear hardening parameter $H = 224.0$ kgf/mm$^2$. The computed results of longitudinal strain coefficient $E \varepsilon_x/\sigma_s$ at the root of the plate versus dimensionless load factor $2\sigma_m/\sigma_s$, where $2\sigma_m = \sigma_x + \sigma_y$, are plotted in Fig. 7 and compared to the boundary element results. The maximum discrepancy of the results from the two methods is observed to be 2.16%. This discrepancy is acceptable.

8. Conclusions

A HT FE formulation for elastoplastic analysis of two-dimensional solids has been developed. In the analysis, a modified variational functional is constructed and
used to derive HT FE formulation for the elastoplasticity of bulky solids. Moreover, we use incremental field equations and have made a modification to the nonlinear boundary equation (16). The numerical results show that this modification is practicable, and are in good agreement with analytical solutions. The results also show that the hierarchic DOF can improve the convergent performance of the Trefftz element model.

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REFERENCES

Appendix A:

Expressions of yield function $F$, matrix $[A]$ and $[D_{ep}]$, and stress vector $\{\dot{\sigma}^p\}$

(1) Yield function $F$

A general form of the yield function used in this analysis can be written as

$$F(\{\sigma\}, \{\varepsilon^p\}, \kappa) = 0,$$  \hspace{1cm} (A1)
where $\kappa$ is a strain hardening parameter. Differentiating (A1) provides consistent condition

$$\begin{bmatrix} \frac{\partial F}{\partial \{\sigma\}} \end{bmatrix}^T \{\dot{\sigma}\} + \left[ \begin{bmatrix} \frac{\partial F}{\partial \{\varepsilon_p\}} \end{bmatrix}^T + \frac{\partial F}{\partial \kappa} \left[ \frac{\partial \kappa}{\partial \{\varepsilon_p\}} \right]^T \right] \{\dot{\varepsilon}_p\} = 0. \quad (A2)$$

When Huber-Mises yield surface equation is used, $F$ is given as

$$F = \sqrt{3\bar{\sigma}} - Y(\kappa), \quad (A3)$$

where $Y(\kappa)$ is the yield stress from uniaxial tests, and

$$\bar{\sigma} = \left[ \frac{1}{2} \left( (s_{11} - s_{22})^2 + (s_{22} - s_{33})^2 + (s_{33} - s_{11})^2 \right) + 3s_{12}^2 + 3s_{23}^2 + 3s_{31}^2 \right]^{1/2}, \quad (A4)$$

where $s_{ij} = \sigma_{ij} - \delta_{ij}(\sigma_{11} + \sigma_{22} + \sigma_{33})/3$.

(2) Matrix $[A], [D_e\varepsilon], \text{and stress vector } \{\dot{\varepsilon}^p\}$

If the Prandtl-Reuss flow theory is considered, the strain can be defined by Eq. (12), while the relationship between elastic strain and stress is given as

$$\{\dot{\sigma}\} = [D_e]\{\dot{\varepsilon}_e\}. \quad (A5)$$

Using Eq. (12), we have

$$\{\dot{\sigma}\} = [D_e]\left(\{\dot{\varepsilon}\} - \{\dot{\varepsilon}^p\}\right). \quad (A6)$$

According to the normality principle which requires the normality of the plastic strain increment vector to the yield surface in the hyper space of $n$ stress dimensions ($n = 3$ for two dimension problem), $\{\dot{\varepsilon}^p\}$ can be expressed as

$$\{\dot{\varepsilon}^p\} = \dot{\lambda} \left\{ \frac{\partial F}{\partial \{\sigma\}} \right\}, \quad (A7)$$

where $\dot{\lambda}$ is a proportionality constant, as yet undetermined.

Substituting Eq. (A6) into Eq. (A2) yields

$$\begin{bmatrix} \frac{\partial F}{\partial \{\sigma\}} \end{bmatrix}^T [D_e]\{\dot{\varepsilon}\} - \left[ \begin{bmatrix} \frac{\partial F}{\partial \{\sigma\}} \end{bmatrix}^T [D_e] - \left\{ \frac{\partial F}{\partial \{\varepsilon_p\}} \right\}^T + \frac{\partial F}{\partial \kappa} \left[ \frac{\partial \kappa}{\partial \{\varepsilon_p\}} \right]^T \right] \{\dot{\varepsilon}^p\} = 0. \quad (A8)$$
Using Eq. (A8), the constant \( \dot{\lambda} \) is determined as

\[
\dot{\lambda} = \left[ [A] + \left\{ \frac{\partial F}{\partial \{\sigma\}} \right\}^T [D_e] \left\{ \frac{\partial F}{\partial \{\sigma\}} \right\} \right]^{-1} \left\{ \frac{\partial F}{\partial \{\sigma\}} \right\}^T [D_e] [\dot{\varepsilon}],
\]

(A9)

where

\[
[A] = - \left[ \left\{ \frac{\partial F}{\partial \{\varepsilon^p\}} \right\}^T + \frac{\partial F}{\partial \kappa} \left\{ \frac{\partial \kappa}{\partial \{\varepsilon^p\}} \right\} \right] \left\{ \frac{\partial F}{\partial \{\sigma\}} \right\}.
\]

(A10)

Therefore \( \dot{\sigma}^p \) is obtained as

\[
\dot{\sigma}^p = -[D_e] \left\{ \frac{\partial F}{\partial \{\sigma\}} \right\} \left\{ \frac{\partial F}{\partial \{\sigma\}} \right\}^T [D_e] \left[ [A] + \left\{ \frac{\partial F}{\partial \{\sigma\}} \right\}^T [D_e] \left\{ \frac{\partial F}{\partial \{\sigma\}} \right\} \right]^{-1} \{\dot{\varepsilon}\}.
\]

For an isotropic hardening material, the parameter \( \kappa \) is assumed to be plastic work \( W_p \). For this, consider a plot of the uniaxial test giving \( \sigma \) versus the plastic uniaxial strain \( \varepsilon^p \) as shown in Fig. 8, we have from Eq. (A3)

\[
\left\{ \frac{\partial F}{\partial \{\varepsilon^p\}} \right\} = 0, \quad \left\{ \frac{\partial F}{\partial \{\sigma\}} \right\} = \frac{3}{2\sigma} \{s'\},
\]

(A11)

\[
\frac{\partial F}{\partial \kappa} \left\{ \frac{\partial \kappa}{\partial \{\varepsilon^p\}} \right\}^T = \frac{\partial F}{\partial W_p} \{\sigma\}^T = - \frac{d\sigma(W_p)}{d\varepsilon_p} \{\sigma\}^T = - \frac{H}{\sigma(W_p)} \{\sigma\}^T,
\]

(A12)

in which the relation \( W_p = \int \{\sigma\}^T \{d\varepsilon^p\} \) has been used, \( \{s'\} = \{s_{11}, s_{22}, s_{33}, 2s_{12}, 2s_{23}, 2s_{31}\}^T \), and \( H = d\sigma(W_p)/d\varepsilon_p \) is the slope of the \( \sigma-\varepsilon^p \) curve which is known as the plastic modulus. Noting that \( \{\sigma\}^T \left\{ \frac{\partial F}{\partial \{\sigma\}} \right\} = \sigma \), we have

\[
[A] = H[I].
\]

(A13)

Substituting Eqs. (A11) and (A13) into Eq. (9), yields

\[
[D_{ep}] = [D_e] - \frac{9G^2}{(3G + H)\sigma^2} \{s\} \{s\}^T,
\]

(A14)

where \( \{s\} = \{s_{11}, s_{22}, s_{33}, s_{12}, s_{23}, s_{31}\}^T \).