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Dual variational formulation for Trefftz finite element method of elastic materials

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Abstract

A dual variational principle is presented for Trefftz finite element analysis. The proof of the stationary conditions of the variational functional and the theorem on the existence of extremum are provided in this paper. They are boundary displacement condition, surface traction condition and interelement continuity condition. Based on the assumed intraelement and frame fields, element stiffness matrix equation is obtained which can easily be implemented into computer programs for numerical analysis with Trefftz finite element method. Two numerical examples are considered to illustrate the effectiveness and applicability of the proposed element model.

Keywords: Variational principle; Finite element method; Elasticity; Trefftz method; Functional

1. Introduction

Variational functionals are essential and play a central role in the formulation of the fundamental governing equations in Trefftz finite element method (TFEM). They are the heart of many numerical methods such as boundary element methods, finite volume methods and Trefftz finite element methods (Qin, 2000). Herrera (1985, 2000) presented a variational formulation which is for problems with or without discontinuities using Trefftz method. Piltner (1985) presented two different variational formulations to treat special elements with holes or cracks. The formulation consists of a conventional potential energy and a least square functional. The least square functional was not added as a penalty function to the potential functional, but is minimized separately for the special elements considered. Jirousek (1978, 1993) developed a variational functional in which either the displacement conformity or the reciprocity of the conjugate tractions is enforced at the element interfaces. Jirousek and Zielinski (1993) obtained two complementary hybrid Trefftz formulations based on weighted residual method. The dual formulations enforced more strongly the reciprocity of boundary tractions than the conformity of the displacement fields. Qin (2000) presented a modified variational principle based hybrid-Trefftz displacement frame. During the past years

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these variational functionals together with Trefftz solutions have been widely used to create new elements in solving problems of elasticity (Jirousek and Venkatesh, 1992), Kirchhoff plates (Jirousek and Guex, 1986; Qin, 1994), moderately thick Reissner-Mindlin plates (Jirousek, 1995; Qin, 1995), thick plates (Piltner, 1992), general 3-D solid mechanics (Piltner, 1989), axisymmetric solid mechanics (Wroblewski et al., 1992), potential problems (Zielinski and Zienkiewicz, 1985), shells (Vörös and Jirousek, 1991), elastodynamic problems (Qin, 1996), transient heat conduction analysis (Jirousek, 1996), geometrically nonlinear plates (Qin, 1997) and materially nonlinear elasticity (Freitas and Wang, 1998). The variational functional of Qin (2000) is, however, limited to the case that nodes containing unknown displacements must connect with at least one inter-element boundary. To remove this limitation, we present a pair of dual variational functional which is based on the total potential energy and complementary energy in this paper.

2. Basic equations for TFEM in solid mechanics

2.1. Basic field equations and boundary conditions

Consider a linear isotropic body, the differential governing equation in the Cartesian coordinates x_i $(i = 1, 2, 3)$ are given by

$$\sigma_{ij,j} + \bar{b}_i = 0 \quad \text{in } \Omega \tag{1}$$

where σ_{ij} is the stress tensor, a comma denotes partial differentiation, $\bar{b}_i$ is body force vector, Ω is the solution domain, and the Einstein summation convention over repeated indices is used. For an isotropic elastic solid, the constitutive relation is

$$\sigma_{ij} = \frac{\partial \Psi(\varepsilon_{ij})}{\partial \varepsilon_{ij}} = s_{ijkl}\varepsilon_{kl} \tag{2a}$$

for ε_{ij} as basic variable, and

$$\varepsilon_{ij} = \frac{\partial \Pi(\sigma_{ij})}{\partial \sigma_{ij}} = c_{ijkl}\sigma_{kl} \tag{2b}$$

for σ_{ij} as basic variable, where s_{ijkl} and c_{ijkl} are stiffness and compliance coefficient tensor, respectively, ε_{ij} is the elastic strain tensor, Π and Γ are, respectively, potential energy and complementary energy functions which defined by

$$\Psi(\varepsilon_{ij}) = \frac{1}{2}s_{ijkl}\varepsilon_{ij}\varepsilon_{kl}, \tag{3a}$$

and

$$\Pi(\sigma_{ij}) = \frac{1}{2}c_{ijkl}\sigma_{ij}\sigma_{kl} \tag{3b}$$

The relation between strain tensor and displacement, u_i, is given by

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{4}$$

The boundary conditions of the boundary value problem (1)–(4) are given by

$$u_i = \bar{u}_i \quad \text{on } \Gamma_u \tag{5}$$

$$t_i = \sigma_{ij}n_j = \bar{t}_i \quad \text{on } \Gamma_t \tag{6}$$

where $\bar{u}_i$ and $\bar{t}_i$ are, respectively, prescribed boundary displacement and traction vector, an overhead bar denotes prescribed value, $\Gamma = \Gamma_u + \Gamma_t$ is the boundary of the solution domain Ω.

Moreover, in the Trefftz finite element approach, Eqs. (1)–(6) should be completed by adding following inter-element continuity requirements:

$$u_{ie} = u_{if}, \quad (\text{on } \Gamma_e \cap \Gamma_f,\ \text{conformity}), \tag{7}$$

$$t_{ie} + t_{if} = 0, \quad (\text{on } \Gamma_e \cap \Gamma_f,\ \text{reciprocity}) \tag{8}$$

where 'e' and 'f' stand for any two neighbouring elements. Eqs. (1)–(8) are taken as a basis to establish the modified variational principle for Trefftz finite element analysis in solid mechanics.

2.2. Assumed fields

The main idea of the TFEM is to establish a finite element formulation whereby the intra-element continuity is enforced on a non-conforming internal displacement field chosen so as to a priori satisfy the governing differential equation of the problem under consideration (Qin, 2000). In other words an obvious alternative to Rayleigh–Ritz method as a basis for a finite element formulation, the model, here, is based on the method of Trefftz (1926). With this method the solution domain Ω is subdivided into elements, and over each element "e", the assumed intra-element fields are

$$\mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \breve{\mathbf{u}} + \sum_{i=1}^{m} \mathbf{N}_i \mathbf{c}_i = \breve{\mathbf{u}} + \mathbf{Nc} \tag{9}$$

where $\mathbf{c}_i$ stands for undetermined coefficient, and $\breve{\mathbf{u}}(= \{\breve{u}_1, \breve{u}_2, \breve{u}_3\}^{\mathrm{T}})$ and $\mathbf{N}_i$ are known functions.

If the governing differential equation (1) is rewritten in a general form

$$\Re \mathbf{u}(\mathbf{x}) + \bar{\mathbf{b}}(\mathbf{x}) = 0, \quad (\mathbf{x} \in \Omega_e) \tag{10}$$

where $\Re$ stands for the differential operator matrix for Eq. (1), $\mathbf{x}$ for position vector, $\bar{\mathbf{b}}(= \{\{\bar{b}_1, \bar{b}_2, \bar{b}_3\}^{\mathrm{T}}$ for known right-hand side term, and Ω_e stands for the sub-domain of eth element, then $\breve{\mathbf{u}} = \breve{\mathbf{u}}(\mathbf{x})$ and $\mathbf{N}_i = \mathbf{N}_i(\mathbf{x})$ in Eq. (9) have to be chosen such that

$$\Re \breve{\mathbf{u}} + \bar{\mathbf{b}} = 0 \quad \text{and} \quad \Re \mathbf{N}_i = 0 \quad (i = 1, 2, \ldots, m) \tag{11}$$

everywhere in Ω_e. The unknown coefficient $\mathbf{c}$ may be calculated from conditions on the external boundary and/or the continuity conditions on the inter-element boundary. Thus various Trefftz element models can be obtained by using different approaches to enforce these conditions. In the majority of them a hybrid technique is usually used, whereby the elements are linked through an auxiliary conforming displacement frame which has the same form as in conventional FE method. This means that, in the Trefftz finite element approach, a conforming displacement field should be independently defined on the element boundary to enforce the field continuity between elements and also to link the coefficient $\mathbf{c}$, appearing in Eq. (9), with nodal displacement $\mathbf{d}(= \{d\})$. The frame is defined as

$$\tilde{\mathbf{u}}(\mathbf{x}) = \widetilde{\mathbf{N}}(\mathbf{x})\mathbf{d}, \quad (\mathbf{x} \in \Gamma_e) \tag{12}$$

where the symbol "$\sim$" is used to specify that the field is defined on the element boundary only, $\mathbf{d} = \mathbf{d}(\mathbf{c})$ stands for the vector of the nodal displacements which are the final unknowns of the finite element formulation, Γ_e represents the boundary of element e, and $\widetilde{\mathbf{N}}$ is a matrix of the corresponding shape functions which are the same as those in conventional finite element formulation.

The tractions $\mathbf{T} = \{t_1, t_2, t_3\}^{\mathrm{T}}$ and $\widetilde{\mathbf{T}}$ can be derived from Eqs. (2), (6) and (9), and denote

$$\mathbf{T} = \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix} = \begin{Bmatrix} \sigma_{1j}n_j \\ \sigma_{2j}n_j \\ \sigma_{3j}n_j \end{Bmatrix} = \mathbf{Qc} + \breve{\mathbf{T}}, \quad \widetilde{\mathbf{T}} = \begin{Bmatrix} \tilde{t}_1 \\ \tilde{t}_2 \\ \tilde{t}_3 \end{Bmatrix} = \begin{Bmatrix} \tilde{\sigma}_{1j}n_j \\ \tilde{\sigma}_{2j}n_j \\ \tilde{\sigma}_{3j}n_j \end{Bmatrix} = \widetilde{\mathbf{Q}}\mathbf{d} \tag{13}$$

3. Modified variational principles

The Trefftz finite element equation for an elastic solid can be established by the variational approach (Qin, 2000). Since the stationary conditions of the traditional potential and complementary variational functional cannot guarantee the satisfaction of the inter-element continuity condition which is required in the Trefftz finite element analysis, some new variational functionals are needed to be developed. For this purpose, we present following two modified variational functionals suitable for Trefftz finite element analysis:

$$\Pi_m = \sum_e \Pi_{me} = \sum_e \left\{ \Pi_e + \int_{\Gamma_{te}} (\bar{t}_i - t_i)\tilde{u}_i \, \mathrm{d}s - \int_{\Gamma_{Ie}} t_i \tilde{u}_i \, \mathrm{d}s \right\} \tag{14a}$$

$$\Psi_m = \sum_e \Psi_{me} = \sum_e \left\{ \Psi_e + \int_{\Gamma_{ue}} (\bar{u}_i - \tilde{u}_i) t_i \, \mathrm{d}s - \int_{\Gamma_{Ie}} t_i \tilde{u}_i \, \mathrm{d}s \right\} \tag{14b}$$

where

$$\Pi_e = \int\int_{\Omega_e} \Pi(\sigma_{ij}) \, \mathrm{d}\Omega - \int_{\Gamma_{ue}} t_i \bar{u}_i \, \mathrm{d}s \tag{15a}$$

$$\Psi_e = \int\int_{\Omega_e} (\Psi(\varepsilon_{ij}) - b_i u_i) \, \mathrm{d}\Omega - \int_{\Gamma_{te}} \bar{t}_i \tilde{u}_i \, \mathrm{d}s \tag{15b}$$

in which Eq. (1) are assumed to be satisfied, a priori. The terminology "modified principle" refers, here, to the use of a conventional functional (Π_e or Ψ_e here) and some modified terms for the construction of a special variational principle to account for additional requirements such as the condition defined in Eqs. (7) and (8).

The boundary Γ_e of a particular element consists of the following parts:

$$\Gamma_e = \Gamma_{ue} \cup \Gamma_{te} \cup \Gamma_{Ie} \tag{16}$$

where

$$\Gamma_{ue} = \Gamma_u \cap \Gamma_e, \quad \Gamma_{te} = \Gamma_t \cap \Gamma_e \tag{17}$$

and Γ_{Ie} is the inter-element boundary of the element 'e'. We now show that the stationary condition of the functional (14a) [or (14b)] leads to Eqs. (5)–(8) and ($u_i = \tilde{u}_i$ on Γ_t), and present the theorem on the existence of extremum of the functional, which ensures that an approximate solution can converge to the exact one. Take Π_m as an example, we have following two statements:

(a) *Modified complementary principle*

$$\delta\Pi_m = 0 \Rightarrow (5)\text{–}(8) \quad \text{and} \quad (u_i = \tilde{u}_i \text{on} \Gamma_t) \tag{18}$$

where δ stands for the variation symbol.

(b) *Theorem on the existence of extremum*

If the expression

$$\int\int_{\Omega} \delta^2 \Pi(\varepsilon_{ij})\,\mathrm{d}\Omega - \int_{\Gamma_t} \delta t_i\,\delta\tilde{u}_i\,\mathrm{d}s - \sum_e \int_{\Gamma_{eI}} \delta\tilde{u}_i\,\delta t_i\,\mathrm{d}s \tag{19}$$

is uniformly positive (or negative) at the neighborhood of $\mathbf{u}_0$, where $\mathbf{u}_0$ is such a value that $\Pi_m(\mathbf{u}_0) = (\Pi_m)_0$, and where $(\Pi_m)_0$ stands for the stationary value of Π_m, we have

$$\Pi_m \geqslant (\Pi_m)_0[\text{or } \Pi_m \leqslant (\Pi_m)_0] \tag{20}$$

in which the relation that $\tilde{\mathbf{u}}_e = \tilde{\mathbf{u}}_f$ is identical on $\Gamma_e \cap \Gamma_f$ has been used.

Proof. First, we derive the stationary conditions of functional (14a). To this end, performing a variation of Π_m and noting that Eq. (1) holds true a priori by the previous assumption, one obtains

$$\delta\Pi_m = \int_{\Gamma_u} (u_i - \bar{u}_i)\,\delta t_i\,\mathrm{d}s + \int_{\Gamma_t} [(\bar{t}_i - t_i)\,\delta\tilde{u}_i + (u_i - \tilde{u}_i)\,\delta t_i]\,\mathrm{d}s + \sum_e \int_{\Gamma_{eI}} [(u_i - \tilde{u}_i)\,\delta t_i - t_i\,\delta\tilde{u}_i]\,\mathrm{d}s \tag{21}$$

Therefore, the Euler equations for expression (21) are Eqs. (5)–(8) and $u_i = \tilde{u}_i$ on Γ_t as the quantities δt_i, δu_i and $\delta\tilde{u}_i$ may be arbitrary. The principle (18) has thus been proved. This indicates that the stationary condition of the functional satisfies the required boundary and inter-element continuity equations and can thus be used for deriving Trefftz finite element formulation.

As for the proof of the theorem on the existence of extremum, we may complete it by way of the so-called "second variational approach" (Simpson and Spector, 1987). In doing this, performing variation of $\delta\Pi_m$ and using the constrained conditions (1), we find

$$\delta^2\Pi_m = \int\int_{\Omega} \delta^2 \Pi(\varepsilon_{ij})\,\mathrm{d}\Omega - \int_{\Gamma_t} \delta t_i\,\delta\tilde{u}_i\,\mathrm{d}s - \sum_e \int_{\Gamma_{eI}} \delta\tilde{u}_i\,\delta t_i\,\mathrm{d}s = \text{expression (19)} \tag{22}$$

Therefore the theorem has been proved from the sufficient condition of the existence of a local extreme of a functional (Simpson and Spector, 1987). This completes the proof. The function Π_m can be stated and proved similarly. We omit those details for the sake of conciseness. □

4. Element stiffness matrix

The element matrix equation can be generated by setting $\delta\Pi_{me} = 0$ or $\delta\Psi_{me} = 0$. To simplify the derivation, we first transform all domain integrals in Eq. (14a) into boundary ones. In fact by reason of solution properties of the intraelement trial functions the functional Π_{me} can be simplified to

$$\Pi_{me} = \frac{1}{2}\int_{\Omega_e} \bar{b}_i u_i\,\mathrm{d}\Omega + \frac{1}{2}\int_{\Gamma_e} t_i u_i\,\mathrm{d}s + \int_{\Gamma_{te}} (\bar{t}_i - t_i)\tilde{u}_i\,\mathrm{d}s - \int_{\Gamma_{Ie}} t_i\tilde{u}_i\,\mathrm{d}s - \int_{\Gamma_{ue}} t_i\bar{u}_i\,\mathrm{d}s \tag{23}$$

Substituting the expressions given in Eqs. (9), (12) into (13) it produces

$$\Pi_{me} = -\frac{1}{2}\mathbf{c}^{\mathrm{T}}\mathbf{H}\mathbf{c} + \mathbf{c}^{\mathrm{T}}\mathbf{S}\mathbf{d} + \mathbf{c}^{\mathrm{T}}\mathbf{r}_1 + \mathbf{d}^{\mathrm{T}}\mathbf{r}_2 + \text{terms without}\mathbf{c}\text{or}\mathbf{d} \tag{24}$$

in which the matrices $\mathbf{H}$, $\mathbf{S}$ and the vectors $\mathbf{r}_1$, $\mathbf{r}_2$ are as follows:

$$
\begin{aligned}
\mathbf{H} &= -\int_{\Gamma_e} \mathbf{Q}^{\mathrm{T}} \mathbf{N} \, \mathrm{d}s \\
\mathbf{S} &= -\int_{\Gamma_{te}} \mathbf{Q}^{\mathrm{T}} \widetilde{\mathbf{N}} \, \mathrm{d}s - \int_{\Gamma_{Ie}} \mathbf{Q}^{\mathrm{T}} \widetilde{\mathbf{N}} \, \mathrm{d}s \\
\mathbf{r}_1 &= \frac{1}{2} \int_{\Omega_e} \mathbf{N}^{\mathrm{T}} \bar{\mathbf{b}} \, \mathrm{d}\Omega + \frac{1}{2} \int_{\Gamma_e} (\mathbf{Q}^{\mathrm{T}} \breve{\mathbf{u}} + \mathbf{N}^{\mathrm{T}} \breve{\mathbf{T}}) \, \mathrm{d}s - \int_{\Gamma_{ue}} \mathbf{Q}^{\mathrm{T}} \bar{\mathbf{u}} \, \mathrm{d}s \\
\mathbf{r}_2 &= \int_{\Gamma_{te}} \widetilde{\mathbf{N}}^{\mathrm{T}} (\overline{\mathbf{T}} - \breve{\mathbf{T}}) \, \mathrm{d}s - \int_{\Gamma_{Ie}} \widetilde{\mathbf{N}}^{\mathrm{T}} \breve{\mathbf{T}} \, \mathrm{d}s
\end{aligned}
\tag{25}
$$

where $\overline{\mathbf{T}} = \{\bar{t}_1, \bar{t}_2, \bar{t}_3\}^{\mathrm{T}}$ is the prescribed traction vector.

To enforce inter-element continuity on the common element boundary, the unknown vector $\mathbf{c}$ should be expressed in terms of nodal DOF $\mathbf{d}$. An optional relationship between $\mathbf{c}$ and $\mathbf{d}$ in the sense of variation can be obtained from

$$\frac{\partial \Pi_{me}}{\partial \mathbf{c}^{\mathrm{T}}} = -\mathbf{Hc} + \mathbf{Sd} + \mathbf{r}_1 = 0. \tag{26}$$

This leads to

$$\mathbf{c} = \mathbf{Gd} + \mathbf{g}, \tag{27}$$

where $\mathbf{G} = \mathbf{H}^{-1}\mathbf{S}$ and $\mathbf{g} = \mathbf{H}^{-1}\mathbf{r}_1$, and then straightforwardly yields the expression of Π_{me} only in terms of $\mathbf{d}$ and other known matrices

$$\Pi_{me} = -\frac{1}{2}\mathbf{d}^{\mathrm{T}}\mathbf{G}^{\mathrm{T}}\mathbf{HGd} + \mathbf{d}^{\mathrm{T}}(\mathbf{G}^{\mathrm{T}}\mathbf{Hg} + \mathbf{r}_2) + \text{terms without } \mathbf{d} \tag{28}$$

Therefore, the element stiffness matrix equation can be obtained by taking the vanishing variation of the functional Π_{me} as

$$\frac{\partial \Pi_{me}}{\partial \mathbf{d}^{\mathrm{T}}} = 0 \Rightarrow \mathbf{Kd} = \mathbf{P} \tag{29}$$

where $\mathbf{K} = \mathbf{G}^{\mathrm{T}}\mathbf{HG}$ and $\mathbf{P} = \mathbf{G}^{\mathrm{T}}\mathbf{Hg} + \mathbf{r}_2$ are, respectively, the element stiffness matrix and the equivalent nodal flow vector. The expression (29) is the elemental stiffness-matrix equation for Trefftz finite element analysis.

5. Numerical examples

As numerical illustrations of the proposed element formulation, two benchmark problems are considered. In order to allow for comparisons with those results appeared in the references (Timoshenko and Woinowsky-Krieger, 1959; Zhang, 1984), the obtained numerical results are limited to a square plate of side-length a subjected to uniformly transverse load q. To study the convergence properties of the proposed element model, three meshes of finite element are used in the analysis. In all calculations the Poisson's ratio is taken to be 0.3. The two examples are described as below.

Example 1. A square plate with two opposite edges simply supported and the remaining two free (Fig. 1) under a uniformly lateral load q. The boundary conditions are

$$x = 0; a, \quad w = 0, \quad M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2}\right) = 0 \tag{30}$$

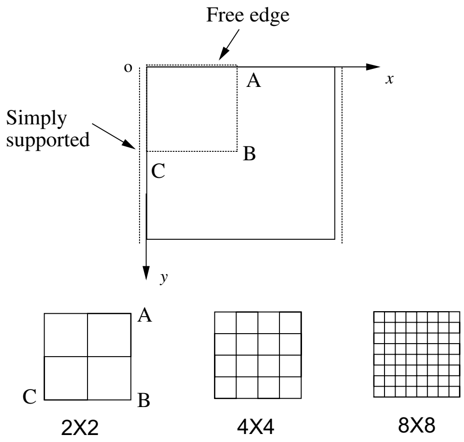

Fig. 1. The square plate and finite element meshes in Example 1.

$$y = 0; a, \quad M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + v\frac{\partial^2 w}{\partial x^2}\right) = 0, \quad V_y = -D\left(\frac{\partial^3 w}{\partial y^3} + (2 - v)\frac{\partial^3 w}{\partial x^2 \partial y}\right) = 0 \tag{31}$$

where w stands for a deflection of the plate, v Poisson's ratio, $D = Eh^3/12(1 - v^2)$, E Young's modulus and h thickness of the plate. Treffz functions and the two independent assumed fields (non-conforming intra-element field and conforming frame field) for thin plate problem can be found in (Qin, 2000) and we omit those details here for conciseness.

Owing to the symmetry of the problem only one quarter of the plate (see Fig. 1) is modelled by three meshes of finite element (2×2, 4×4, 8×8). It is noted from Fig. 1 that node B contains unknown deflection but it does not connect with any inter-element boundary. Therefore The variational functional of Qin (2000) cannot be directly used to solve this problem unless a triangular element connecting to the node is used or some special treatment is made. This limitation can, however, be removed by using the proposed formulation. Table 1 shows the results of deflection ($w_c = \alpha \times qa^4/D$) at central point B ($a/2, a/2$) and bending moment ($M_x = \beta \times qa^2$) at point A ($a/2, 0$) and compares the related analytical results (at p219 of Timoshenko and Woinowsky-Krieger, 1959).

Example 2. A uniformly loaded cantilever square plate with one clamped edge and the remaining edges are free (Fig. 2). The boundary conditions of this problem are

$$y = 0, \quad w = \frac{\partial w}{\partial y} = 0 \tag{32}$$

Table 1
Central deflection ($w_c = \alpha \times qa^4/D$) and moment ($M_x = \beta \times qa^2$) at point ($a/2, 0$) in Example 1

	α	β
Proposed FEM		
2×2	0.01300	0.1334
4×4	0.01304	0.1327
8×8	0.01305	0.1325
Analytical solution (Timoshenko and Woinowsky-Krieger, 1959)	0.01309	0.1318

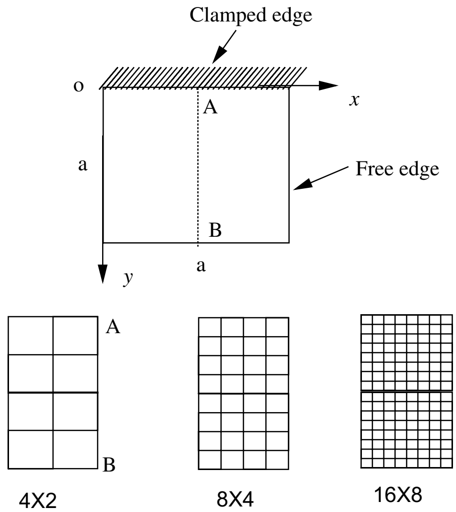

Fig. 2. The square plate and finite element meshes in Example 2.

$$y = a, \quad M_y = V_y = 0 \tag{33}$$

$$x = 0; a \;\; M_x = 0, \quad V_x = -D\left(\frac{\partial^3 w}{\partial x^3} + (2 - v)\frac{\partial^3 w}{\partial x \partial y^2}\right) = 0 \tag{34}$$

In addition two more conditions at corner points $(0, a)$ and (a, a) required that

$$R|_{x=0;y=a} = -2M_{xy} = -2D(1 - v)\frac{\partial^2 w}{\partial x \partial y} = 0 \tag{35}$$

$$R|_{x=a;y=a} = 2M_{xy} = 2D(1 - v)\frac{\partial^2 w}{\partial x \partial y} = 0 \tag{36}$$

A half of the cantilever plate (Fig. 2) is modelled by three meshes of finite element (4×2, 8×4 16×8). Some results obtained by the proposed element model are listed in Tables 2 and 3, and comparison is made with the analytical ones (Zhang, 1984).

It can be seen from the three tables that the results are in good agreement with analytical solutions. As expected for all examples, it is found from the numerical results that both deflection and bending moment converges gradually to the analytical one along with refinement of the element meshes.

Table 2
Deflections $(w = \alpha \times qa^4/D)|_{x=a}$ along $x = a$ in Example 2

y	0	$0.25a$	$0.5a$	$0.75a$	a
Proposed FEM					
4×2	0	0.01179	0.04351	0.08422	0.1274
8×4	0	0.01186	0.04360	0.08433	0.1284
16×8	0	0.01188	0.04363	0.08437	0.1286
Analytical solution (Zhang, 1984)	0	0.011949	0.044327	0.085046	0.12933

Table 3
Moment $(M_y = \beta \times qa^2)|_{y=0}$ along the clamped edge $y = 0$ in Example 2

x	$0.0625a$	$0.125a$	$0.25a$	$0.375a$	$0.5a$
Proposed FEM					
4×2	−0.4583	−0.5043	−0.5283	−0.5306	−0.5308
8×4	−0.4599	−0.5067	−0.5295	−0.5314	−0.5316
16×8	−0.4605	−0.5081	−0.5301	−0.5319	−0.5322
Analytical solution (Zhang, 1984)	−0.47314	−0.51270	−0.53353	−0.53550	−0.53560

6. Conclusion

Two complementary modified variational formulations are developed for Trefftz finite element analysis. With this formulation, the limitation(nodes containing unknown displacements must connect with at least one inter-element boundary) occurred in the previous variational functional (Qin, 2000) has been removed. Based on the assumed intraelement and frame fields as well as the newly constructed dual variational functional, an element stiffness matrix equation is obtained which can easily be implemented into computer programs for numerical analysis with Trefftz finite element method. Two numerical examples have been considered and the numerical results of both deflection and bending moment are observed to converge gradually to the analytical one along with refinement of the element meshes.

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