Self-consistent boundary element solution for predicting overall properties of cracked bimaterial solid

Qing-Hua Qin\textsuperscript{a,b,*}

\textsuperscript{a}Department of Mechanics, Tianjin University, Tianjin, People’s Republic of China
\textsuperscript{b}School of AMME, University of Sydney, Sydney, NSW 2006, Australia

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Abstract

In this paper, a self-consistent boundary element method is introduced to predict overall properties of a cracked bimaterial plate. The boundary element (BE) formulation is developed based on Green’s function for displacement discontinuity of a bimaterial solid and potential variational principles. The self-consistent method is then introduced into the BE formulation to provide an effective means for estimating overall material constants of a bimaterial solid weakened by cracks. Numerical results for a bimaterial plate with cracks are presented to illustrate the application of the proposed self-consistent BE formulation.

Keywords: Self-consistent method; Boundary element method; Crack; Bimaterial

1. Introduction

The problem of reduction in stiffness of engineering materials due to the development or presence of many cracks is of scientific significance and engineering importance and has been the subject of many investigations. By relating this stiffness reduction to the state of microcracking, it may be possible to assess the integrity of a structure and the mechanism of failure in the materials [1–4]. Over the past decades, several approaches have been proposed to estimate the stiffness reduction of a cracked solid. Most typical are the dilute scheme [5], the self-consistent method [6], the generalized self-consistent method [7], the Mori–Tanaka method [8,9] and the differential method [10]. Common to each of these micromechanics theories is the use of the well-known stress and strain concentration factors obtained through the solution of a single crack embedded in an infinite medium. However, for a complex problem with complexity in the aspects of geometry and mechanical deformation, a combination of these approaches and numerical methods such as the finite element (FE) and the boundary element (BE) method presents a powerful computational tool for estimating effective material properties. It should be mentioned that the main disadvantage of the FE method is that domain discretization is required to perform the analysis. Moreover, in some cases it results in both an inaccurate and an expensive technique, especially in solving crack problems. On the other hand, the BE method involves discretization of the boundary of a structure only, because the governing differential equation is satisfied exactly inside the domain, leading to a relatively smaller system size with sufficient accuracy. In this paper, an analytical solution for edge dislocations in a bimaterial plate is introduced and used to derive BE formulations for multiple crack problems. The BE formulation is used together with a self-consistent model to estimate the effective material constants of a bimaterial solid weakened by cracks. Numerical results for a bimaterial plate with cracks are presented to illustrate the application of the proposed formulation.

2. Dislocation in bimaterial isotropic plate

Dislocations in bimaterial solids have received considerable attention in practical engineering. For isotropic half-spaces, the expression for the force was given by Head [11] for screw dislocations and by Dundurs and Sendeckyj [12] for edge dislocations. For isotropic inhomogeneous media, the stress field due to an edge dislocation was derived by Head [13], and further studied by Dundurs [14] and others.
In this section, a Green’s function for edge dislocations in a bimaterial plate is introduced as the basis of the boundary formulation given in Section 3.

Consider a bimaterial plate subjected to an edge dislocation \((b_x, b_y)\) located at a point \((x_0, y_0)\) shown in Fig. 1, for which the right half-plane \((x > 0)\) is occupied by material 1, and the left half-plane \((x < 0)\) is occupied by material 2. The materials are rigidly bonded together so that \((u_{x1}) = (u_{x2}), \quad (u_{y1}) = (u_{y2}), \quad (\sigma_{x1}) = (\sigma_{x2}), \quad (\sigma_{y1}) = (\sigma_{y2})\)

where \(u_i\) and \(\sigma_{ij}\) are displacements and stresses, and the subscripts ‘1’ and ‘2’ label the quantities relating to the materials 1 and 2, respectively. The elastic solution can be found from that for an infinite homogeneous plane by superposing appropriate image dislocations to satisfy the interface condition. The Airy stress function for an infinite plane subjected to an edge dislocation \((b_x, b_y)\) at point \((x_0, y_0)\) is expressed by [15]

\[
\chi = -4H(\kappa)G(y_1 b_x - x_1 b_y) \ln r_1
\]

where \(H(\kappa) = 1/2\pi(\kappa + 1), \ G\) is the shear modulus of the material, \(\kappa = 3 - 4\mu\) for plane strain and \(\kappa = (3 - \mu)/(1 + \mu)\) for plane stress, \(\mu\) being the Poisson’s ratio, \(x_1 = x - x_0, \ y_1 = y - y_0, \ r_1^2 = x_1^2 + y_1^2\). The related stresses and displacements can be evaluated by the following relations:

\[
\sigma_{xx} = \chi_{yy}, \quad \sigma_{yy} = -\chi_{xx}, \quad \sigma_{xy} = \chi_{xy}
\]

\[
\begin{align*}
2G\sigma_{xx} &= -\chi_{xx} + (\kappa + 1)\psi_y, \\
2G\sigma_{yy} &= -\chi_{yy} + (\kappa + 1)\psi_x
\end{align*}
\]

The function \(\psi\) is determined from the following two conditions

\[
\nabla^2 \psi = 0, \quad \psi_{xy} = \frac{1}{2} \nabla^2 \chi
\]

and a comma followed by an argument stands for differentiation, \(\partial^() = (\partial_{xx} + \partial_{yy})\).

Fig. 1. Dislocations in a bimaterial plate.

Using the superposing procedure presented in Ref. [16], the Airy stress functions for an edge dislocation \((b_x, b_y)\) applied at \((x_0, y_0)\) of the material on the right (material 1 in Fig. 1(a)) are

\[
\chi_1^{(1)} = F_1^{(1)} b_x + F_1^{(1)} b_y, \quad \chi_2^{(1)} = F_2^{(1)} b_x + F_2^{(1)} b_y
\]

where \(F_1^{(1)}\) and \(F_2^{(1)}\) are given in Appendix A, the superscript ‘(i)’ is used to distinguish the loading case, \(i = 1\) for dislocation applied at material 1, \(i = 2\) for dislocation applied at material 2 and \(i = 3\) for dislocation applied at the interface.

From Eqs. (3)–(5), we have the expressions for \(u_i\), due to an edge dislocation \((b_x, b_y)\) applied at \((x_0, y_0)\) of the material on the right in the form

\[
\begin{align*}
\begin{bmatrix}
\alpha_{1x}^{(1)} \\
\alpha_{1y}^{(1)}
\end{bmatrix}
&= 
\begin{bmatrix}
U_{xx1}^{(1)} & U_{xy1}^{(1)} \\
U_{yx1}^{(1)} & U_{yy1}^{(1)}
\end{bmatrix}
\begin{bmatrix}
b_x \\
b_y
\end{bmatrix}, \\
\begin{bmatrix}
\alpha_{2x}^{(1)} \\
\alpha_{2y}^{(1)}
\end{bmatrix}
&= 
\begin{bmatrix}
V_{xx2}^{(1)} & V_{xy2}^{(1)} \\
V_{yx2}^{(1)} & V_{yy2}^{(1)}
\end{bmatrix}
\begin{bmatrix}
b_x \\
b_y
\end{bmatrix}
\end{align*}
\]

where the functions

\[
\begin{align*}
U_{ij}^{(1)} &= U_{ij}^{(1)}(\kappa_1, \theta_1, \theta_2, x_1, x_2, y_1, A, B, \Gamma, x_0, y_0) \\
V_{ij}^{(1)} &= V_{ij}^{(1)}(G_1, \kappa_1, x_1, x_2, y_1, A, B, x_0, y_0)
\end{align*}
\]

are given in Appendix A, and

\[
A = \frac{1 - \Gamma}{1 + \Gamma \kappa_1}, \quad B = \frac{\kappa_2 - \Gamma \kappa_1}{\kappa_2 + \Gamma}, \quad \Gamma = G_2/G_1,
\]

\[x_2 = x + x_0\]

The stress and displacement solutions given in Eq. (7) are readily extended to the case of dislocation applied at a point in material 2 or on the interface. If the dislocation is applied in material 2 (Fig. 1(b)), the solution can be obtained by way of rotating the \(x-y\) system \(180^\circ\) in a counter-clockwise
Finally, when the dislocation is located at the interface, the related displacements and stresses are

\[
U_{ijm}^{(2)} = U_{ijm}^{(1)}(\kappa_2, \theta', -x_1, -x_2, -y_1, \mathbf{A}^*, \mathbf{B}^*, 1/\Gamma, -x_0, -y_0)
\]

(10)

\[
V_{ikm}^{(2)} = V_{ikm}^{(1)}(\kappa_2, -x_1, -x_2, -y_1, \mathbf{A}^*, \mathbf{B}^*, -x_0, -y_0)
\]

in which the two coordinate systems are related by

\[
\begin{align*}
\theta' &= \theta + \varphi, \\
x' &= -x, \\
y' &= -y
\end{align*}
\]

(11)

\[
\mathbf{A}^* = \frac{\Gamma - 1}{\Gamma + \kappa_2}, \quad \mathbf{B}^* = \frac{\Gamma_1 - \kappa_2}{\Gamma_1 + 1},
\]

Finally, when the dislocation is located at the interface, the Airy stress functions can be obtained by setting \(x_0 = 0\) and then letting \(r_1 = r_2 = r, \theta_1 = \theta_2 = \theta\) in Eqs. (A1)–(A4), where \(r^2 = x^2 + y^2\). They are

\[
\begin{align*}
\chi_1^{(3)} &= 2H(\kappa_1)G_1 [(2 - A - B)r[b_y \cos \theta - b_x \sin \theta] \ln x + (B - A)r[b_y \sin \theta + b_x \cos \theta]} \\
\chi_2^{(3)} &= 2H(\kappa_1)G_1 [(2 - A - B)r[b_y \cos \theta - b_x \sin \theta] \ln x - (B - A)r[b_y \sin \theta + b_x \cos \theta]
\end{align*}
\]

(12) (13)

Thus the related displacements and stresses are

\[
\begin{align*}
\{u_{a}^{(3)}\} &= \begin{bmatrix} U_{xaxa}^{(3)} & U_{xyax}^{(3)} \end{bmatrix} \begin{bmatrix} b_x \\ b_y \end{bmatrix}, \\
\{u_{i}^{(3)}\} &= \begin{bmatrix} U_{yaxi}^{(3)} & U_{yxiy}^{(3)} \end{bmatrix} \begin{bmatrix} b_x \\ b_y \end{bmatrix}, \\
\{\sigma_{xx}^{(3)}\} &= \begin{bmatrix} V_{xxax}^{(3)} & V_{xyax}^{(3)} \end{bmatrix} \begin{bmatrix} b_x \\ b_y \end{bmatrix}, \\
\{\sigma_{yi}^{(3)}\} &= \begin{bmatrix} V_{yaxi}^{(3)} & V_{yxiy}^{(3)} \end{bmatrix} \begin{bmatrix} b_x \\ b_y \end{bmatrix}, \\
\{\sigma_{yy}^{(3)}\} &= \begin{bmatrix} V_{yyxi}^{(3)} & V_{yyxiy}^{(3)} \end{bmatrix} \begin{bmatrix} b_x \\ b_y \end{bmatrix}
\end{align*}
\]

(14)

in which

\[
\begin{align*}
U_{ijk}^{(3)} &= U_{ijk}^{(1)}(\kappa_1, \theta, x, x, y_1, A, B, 0, 0) \\
V_{ikm}^{(3)} &= V_{ikm}^{(1)}(\kappa_1, x, x, y_1, A, B, 0, 0)
\end{align*}
\]

3. Boundary element formulation

Consider a two-dimensional isotropic bimaterial solid inside which there are a number of cracks of arbitrary orientation and size. We assume that all cracks are located in material 1, while material 2 has no cracks. This assumption is made only to simplify the ensuing writing; extension to the case of cracks in the whole plane is straightforward. Under this assumption the superscripts (1) or (2) for identifying materials 1 or 2 are unnecessary in most parts of the following sections, and we omit them when the distinction is unnecessary. In what follows, we begin by deriving the variational principle for isotropic bimaterials with many cracks.

Let us consider a finite region \(\Omega_2\) bounded by \(\Gamma = \Gamma_1 + \Gamma_u + L\), as shown in Fig. 2(a). For stationary behavior in the absence of body forces, the boundary value problem to be considered is stated as

\[
\sigma_{ij} = 0, \quad \text{on } \Omega_1
\]

(16)

\[
t_m = \sigma_{nj} = \mathbf{n} \cdot \mathbf{t}_m, \quad \text{on } \Gamma_1
\]

(17)

\[
u_i = \hat{u}_n, \quad \text{on } \Gamma_u
\]

(18)

\[
t_m |_{L^+} = t_m |_{L^-} = 0, \quad \text{on } L
\]

(19)

where \(n_i\) is the normal to the boundary \(\Gamma\), where \(\Gamma_1\) and \(\Gamma_u\) are the boundaries on which the prescribed values of stress \(t_m\) and displacement \(\hat{u}_n\) are imposed, respectively. For simplicity, we define \(b_i = u_i |_{L^+} - u_i |_{L^-}\) on \(L = L^+ + L^-\), where \(L\) is the union of all cracks, \(L^+\) and \(L^-\) are defined in Fig. 2(b).

In a manner similar to that in Ref. [17], the total generalized potential energy for the crack problem defined
As in the conventional BE method, the boundaries $L$ and $\Gamma$ are divided into a series of BEs for which the displacement discontinuity may be approximated by a linear function. To illustrate this, take a particular element $m$, which is a line connected by nodes $m$ and $m + 1$, as an example (Fig. 3).

The final form of the linear equations to be solved is

\[
\begin{align*}
F_m(s) &= \frac{l_m - s}{l_m} \\
F_{m+1}(s) &= \frac{s}{l_m}
\end{align*}
\]

\[\text{element } m\]

\[\text{Fig. 3. Definitions of } F_m(s) \text{ and } F_{m+1}(s).\]

above is given by

\[
\Pi(b) = \frac{1}{2} \int_{\Gamma} R(b) \cdot b_x \, ds - \int_{\Gamma} \mathbf{t}^T \mathbf{b} \, ds
\]

(20)

where

\[
\mathbf{b} = \begin{bmatrix} b_x \\ b_y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}^T; \\
\mathbf{R} = \begin{bmatrix} R_x \\ R_y \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}; \\
\mathbf{t} = \begin{bmatrix} t_{i_1} \\ t_{i_2} \end{bmatrix}
\]

(21)

with

\[
R_j = \int \sigma_{ij} n_j \, ds, \quad i, j = 1, 2
\]

(22)

As in the conventional BE method, the boundaries $L$ and $\Gamma$ are divided into a series of BEs for which the displacement discontinuity may be approximated by a linear function. To illustrate this, take a particular element $m$, which is a line connected by nodes $m$ and $m + 1$, as an example (Fig. 3).

\[
\mathbf{u}_m(s) = \mathbf{b}_m F_m(s) + \mathbf{b}_{m+1} F_{m+1}(s)
\]

(23)

With the approximation (23), the displacements can now be expressed in the form

\[
\mathbf{u}(x) = \sum_{m=1}^{M} \mathbf{U}_m \mathbf{b}_m
\]

(24)

where $M$ is the total number of nodes, $x = \{x_1, x_2\}$, and

\[
\mathbf{U}_m = \frac{1}{l_{m-1}} \int_{l_{m-1}} \begin{bmatrix} U_{xx} & U_{xy} \\ U_{yx} & U_{yy} \end{bmatrix} s \, ds
\]

\[
+ \frac{1}{l_m} \int_{l_m} \begin{bmatrix} U_{xx} & U_{xy} \\ U_{yx} & U_{yy} \end{bmatrix} (l_m - s) \, ds
\]

(25)

For the total potential energy given by Eq. (20), substitution of Eq. (23) into it, yields

\[
\Pi(b) = \sum_{i=1}^{M} \mathbf{b}_i^T \sum_{j=1}^{M} \mathbf{k}_{ij} \mathbf{b}_j \frac{l(2 - g_i)}{l}
\]

where

\[
\mathbf{k}_{ij} = \frac{1}{l_{j-1}} \int_{l_{j-1}} \begin{bmatrix} F_{11}^{ij} & F_{12}^{ij} \\ F_{12}^{ij} & F_{22}^{ij} \end{bmatrix} \, ds
\]

\[
- \frac{1}{l_j} \int_{l_j} \begin{bmatrix} F_{11}^{ij} & F_{12}^{ij} \\ F_{12}^{ij} & F_{22}^{ij} \end{bmatrix} \, ds
\]

(27)

\[
g_j = \frac{1}{l_{j-1}} \int_{l_{j-1}} \mathbf{t}^T \mathbf{s} \, ds + \frac{1}{l_j} \int_{l_j} \mathbf{t}^T (l_j - s) \, ds
\]

(28)

with

\[
F_{11}^{ij} = \frac{1}{l_{j-1}} \int_{l_{j-1}} \left[ V_{xx}(s, x) n_1 + V_{xy}(s, x) n_2 \right] x \, dx
\]

\[
+ \frac{1}{l_j} \int_{l_j} \left[ V_{xx}(s, x) n_1 + V_{xy}(s, x) n_2 \right] (l_j - x) \, dx
\]

(29)

\[
F_{12}^{ij} = \frac{1}{l_{j-1}} \int_{l_{j-1}} \left[ V_{xy}(s, x) n_1 + V_{yy}(s, x) n_2 \right] x \, dx
\]

\[
+ \frac{1}{l_j} \int_{l_j} \left[ V_{xy}(s, x) n_1 + V_{yy}(s, x) n_2 \right] (l_j - x) \, dx
\]

\[
F_{22}^{ij} = \frac{1}{l_{j-1}} \int_{l_{j-1}} \left[ V_{yy}(s, x) n_1 + V_{yy}(s, x) n_2 \right] x \, dx
\]

\[
+ \frac{1}{l_j} \int_{l_j} \left[ V_{yy}(s, x) n_1 + V_{yy}(s, x) n_2 \right] (l_j - x) \, dx
\]

The minimization of Eq. (26) leads to a set of linear equations

\[
\sum_{j=1}^{M} \mathbf{k}_{ij} \mathbf{b}_j = \mathbf{g}_i
\]

(30)

The final form of the linear equations to be solved is obtained by selecting either Eq. (24) or Eq. (30). Eq. (24) will be chosen for those nodes at which displacements are prescribed, and Eq. (30) will be chosen for the remaining nodes.

4. Self-consistent method for cracked solid

The self-consistent method for a cracked body has been
discussed elsewhere [4]. For the reader’s convenience we describe the method here briefly.

In the self-consistent method, the effect of crack interaction on material properties is taken into account approximately by embedding each crack directly in the effective medium, i.e. the medium having the as-yet-unknown material properties of the cracked matrix.

The effective elastic moduli of a cracked body are defined as [18]

\[
\langle \sigma \rangle = E^* \langle \epsilon \rangle \quad \text{or} \quad \langle \epsilon \rangle = F^* \langle \sigma \rangle
\]  

(31)

where \( \langle \cdot \rangle \) denotes the area average (for a two-dimensional problem) of the quantity, and the superscript ‘*’ stands for the effective value.

The self-consistent theory may be established based on some fundamental results in the theory of two-phase elastic media. In the case of two-phase materials, the area average of stress and strain tensors is defined by

\[
\langle \sigma \rangle = \nu^{(1)} \langle \sigma^{(1)} \rangle + \nu^{(2)} \langle \sigma^{(2)} \rangle,
\]

\[
\langle \epsilon \rangle = \nu^{(1)} \langle \epsilon^{(1)} \rangle + \nu^{(2)} \langle \epsilon^{(2)} \rangle
\]

(32)

where superscripts (1) and (2) denote the matrix and inclusion phases, \( \nu^{(1)} \) and \( \nu^{(2)} \) their area fractions. Substituting Eq. (32) into Eq. (31) and noting that \( \sigma^{(0)} = E^{(0)} \epsilon^{(0)} \), we have

\[
E^* = E^{(1)} + (E^{(2)} - E^{(1)}) A^{(2)} \nu^{(2)},
\]

\[
F^* = F^{(1)} + (F^{(2)} - F^{(1)}) B^{(2)} \nu^{(2)}
\]

(33)

in which the symmetric tensors \( A^{(2)} \) and \( B^{(2)} \) are defined by the linear relations

\[
\langle \epsilon^{(2)} \rangle = A^{(2)} \epsilon^{(0)}, \quad \langle \sigma^{(2)} \rangle = B^{(2)} \sigma^{(0)}
\]

(34)

with \( \epsilon^{0} \) and \( \sigma^{0} \) being remote strain and stress fields applied on the effective medium. The interpretation of \( \langle \epsilon^{(2)} \rangle \) in Eq. (34) follows from the average strain theorem [18]

\[
\langle \epsilon^{(2)} \rangle = \frac{1}{2 \Omega_2} \int_{\partial \Omega_2} (u_i n_j + u_j n_i) d\Omega
\]

(35)

where \( \Omega_2 \) and \( \partial \Omega_2 \) are the total area and boundary of the voids.

What follows is concerned with the process: inclusion → void → crack. First, we consider the case when inclusions become voids which are thought of as being filled with air. This implies that \( E^{(2)} \to 0 \), \( F^{(2)} \to \infty \). Thus we assume \( E^{(2)} = 0 \), where \( E^{(2)} \) stands for stiffness constants of the void-phase. Then Eq. (33) become

\[
E^* = E^{(1)} (I - A^{(2)} \nu^{(2)}), \quad F^* = F^{(1)} (I + B^{(0)} \nu^{(2)})
\]

(36)

where \( I \) is the unit tensor and \( B^{(0)} \) is defined by

\[
\langle \epsilon^{(2)} \rangle = F^{(1)} B^{(0)} \epsilon^{(0)}
\]

(37)

Finally, cracks are defined as very flat voids of vanishing height and thus also of vanishing area. Multiplying both sides of Eq. (35) by \( \nu^{(2)} \) and considering the limit of flattening out into cracks, i.e. \( \nu^{(2)} \to 0 \), one obtains

\[
\lim_{\nu^{(2)} \to 0} \langle \epsilon^{(2)} \rangle = \frac{1}{2 \Omega} \int_L (b_i n_j + b_j n_i) dl = X_{ij}
\]

(38)

where \( L = I_1 \cup I_2 \cup \ldots \cup I_N \), \( I_i \) is the length of \( i \)th crack, \( N \) the number of cracks within a representative area element (RAE) whose area is \( \Omega \). For convenience we define [10]

\[
P = \lim_{\nu^{(2)} \to 0} (A^{(2)} \nu^{(2)}), \quad Q = \lim_{\nu^{(2)} \to 0} (B^{(0)} \nu^{(2)})
\]

(39)
Hence Eq. (36) can be rewritten as

\[
E^* = E^{(1)}(I - P), \quad F^* = F^{(1)}(I + Q)
\]

(40)

with the relation

\[
X = Pe^0 = F^{(1)Qe^0}
\]

(41)

Thus, the estimation of integral (38) and then \(P\) (or \(Q\)) is the key to predicting the effective material constants \(E^*\) and \(F^*\). The unknown \(b_i\) in integral (38) can be calculated through use of the BE method described in Section 3 and the iterative scheme described below.

5. Iterative procedure for SCBEM

As explained in Section 4, in the self-consistent method each crack is considered to be embedded in the effective medium whose properties are unknown. In this case, a set of initial values of the effective properties is assumed and an iteration algorithm is required. The algorithm is described in detail below.

(a) Assume initial material properties \(E^{(1)}\);  
(b) Solve Eqs. (24) and (30) for \(b_{ij}\) using the values of \(E^{(1)}\), where \(E^{(1)}\) and \(b_{ij}\) stand for the solutions of the

\[
\mu^{(1)}/\mu^{(1)}
\]

Fig. 6. Normalized Poisson’s ratio \(\mu^{(1)}/\mu^{(1)}\) versus crack density \(\eta\).

6. Numerical example

As an illustration, the proposed self-consistent boundary element method (SCBEM) is applied in the numerical example below. In the calculation, the convergent tolerance \(\varepsilon\) is set at 0.0001. Consider a bimaterial whose material constants are as follows:

1. Material 1 (cracks in this material). Boron: Young’s modulus, \(E^{(1)} = 400\) GPa, Poisson’s ratio, \(\mu^{(1)} = 0.23\)
2. Material 2 (no cracks in this material). Aluminium: Young’s modulus, \(E_0 = 68.9\) GPa, Poisson’s ratio, \(\mu_0 = 0.35\).

Numerical results for this bimaterial solid are presented for the following three cases.

(a) In this case all cracks are assumed for simplicity to have the same length \(a\) and to lie in the horizontal direction. The cracks are divided into 50 elements, and the boundary of a representative area (RAE) is divided into 200 elements. The total nodes for this problem are 251. The results for a RAE (Fig. 4) containing \(N\) cracks are shown in Figs. 5 and 6. In the figures, the normalized \(E^{(1)}/E^{(1)}\) and \(\mu^{(1)}/\mu^{(1)}\) are shown versus the crack density parameter \(\eta\), where \(\eta = Na^2/\Omega\). For comparison, the well-known FE method is used to obtain corresponding results. In the finite element analysis, the configuration of the RAE used is shown in Fig. 4 for \(N = 1, 4, 6, 8\), where \(N\) is the number of cracks in a particular RAE. The area occupied by material 1 in the RAE and crack length are assumed to be 10 and 2, respectively. Thus the crack density equals \(N/25\). It is observed from Figs. 4 and 5 that the two methods can provide almost the same results for \(\eta \leq 0.1\), but the discrepancy increases with increasing crack density \(\eta\). Moreover, Fig. 5 shows that the effective Young’s modulus

\[
\text{Table 1}
\]

Convergent performance of BE and FE methods

<table>
<thead>
<tr>
<th>BE method</th>
<th>FE method</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of nodes</td>
<td>(E^{(1)}/E^{(1)})</td>
</tr>
<tr>
<td>51</td>
<td>0.9532</td>
</tr>
<tr>
<td>76</td>
<td>0.9481</td>
</tr>
<tr>
<td>101</td>
<td>0.9448</td>
</tr>
<tr>
<td>151</td>
<td>0.9423</td>
</tr>
<tr>
<td>201</td>
<td>0.9410</td>
</tr>
<tr>
<td>251</td>
<td>0.9407</td>
</tr>
</tbody>
</table>

\[
\text{Fig. 5. Normalized Young’s modulus \(E^{(1)}/E^{(1)}\) versus crack density } \eta.
\]

\[
\text{Fig. 6. Normalized Poisson’s ratio } \mu^{(1)}/\mu^{(1)} \text{ versus crack density } \eta.
\]
Table 2
Effect of crack angle on $E^{(1)}/E^{(1)}$

<table>
<thead>
<tr>
<th>Crack angle (°)</th>
<th>$E^{(1)}/E^{(1)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.9407</td>
</tr>
<tr>
<td>15</td>
<td>0.9465</td>
</tr>
<tr>
<td>30</td>
<td>0.9482</td>
</tr>
<tr>
<td>45</td>
<td>0.9501</td>
</tr>
<tr>
<td>60</td>
<td>0.9533</td>
</tr>
</tbody>
</table>

decreases along with the increase in crack density. In contrast, Fig. 5 reveals that the effective Poisson’s ratio increases with increase in crack density. It should be noted that the convergent FE method results in Figs. 5 and 6 are obtained with 50 × 50 mesh plus high mesh density near the crack tip. The total nodes for the FE method calculation are 2606. However, only 251 nodes are required with the BE method for achieving convergent results in this particular problem. This indicates that the BE method formulation is more efficient than the conventional FE method.

(b) To study the convergent performance of both the BE and FE methods, numerical results for different element meshes, identified by the total number of nodes of the problem, are presented in Table 1. It can be seen from the table that the BE method can reach convergent results with many fewer nodes than are required in the FE method.

(c) To investigate the effect of crack angle on the effective properties, cracks with different angles between crack line and horizontal line in a bimaterial are considered. Numerical results of $E^{(1)}/E^{(1)}$ for different crack angles are listed in Table 2. We find that the effective Young’s modulus increases along with increasing values of angle.

7. Conclusion

A self-consistent BE method and a computer program have been developed for estimating the effective properties of cracked bimaterial solids. A system of BE formulation is derived, based on the potential variational principle, the concept of dislocation and Green’s function, developed in this paper. The BE formulation is then introduced into the self-consistent model to calculate the overall properties of a cracked bimaterial plate. Numerical results are presented to compare predictions obtained by the proposed approach and by the FE method for a particular cracked material. This study shows that the two methods can provide almost the same results for $\eta \leq 0.1$, but the discrepancy increases with increasing crack density $\eta$. The discrepancy at $\eta = 0.7$ is less than 5% which is acceptable in practical engineering. The study also shows that the BE method formulation is more efficient than the conventional FE method as it requires many fewer nodes to achieve a convergent result than are needed in the FE method.

Appendix A

\[ F^{(1)}_{x1} = 2H(\kappa_1)G_1 \left\{ x_1 \ln r_1 \sin \theta_1 + (B + A)r_2 \right\} \]

\[ - 2Ax_0 \left[ \sin 2\theta_2 - \frac{2x_0 \cos \theta_2}{r_2} \right] \]  

\[ F^{(1)}_{y1} = 2H(\kappa_1)G_1 \left\{ r_2 \ln r_1 \cos \theta_1 \right\} \]

\[ - (B + A)r_2 \ln r_2 \cos \theta_2 + (B - A)\sqrt{r_2} \sin \theta_2 + 2Ax_0 \left[ 2 \ln r_2 - \cos 2\theta_2 + \frac{2x_0 \cos \theta_2}{r_2} \right] \]  

\[ U^{(1)}_{x1} = H(\kappa_1) \left\{ \left( \kappa_1 + 1 \right) \theta_1 + \frac{2x_1 y_1}{r_1} - \left( A\kappa_1 + B \right) \theta_2 \right\} \]

\[ + 8Ax_0 \left( x_2 - x_2 \right) \frac{y_1}{r_2} \]  

\[ U^{(1)}_{y1} = H(\kappa_1) \left\{ \left( 1 - A \right) \kappa_1 + B - 1 \right\} \ln r_1 \]

\[ + 2(A - 1) \frac{x_1^2}{r_1} - 2A \left( \kappa_1 - 1 \right) x_2 + 2x_0 \frac{x_1^2}{r_2} \]

\[ + 8Ax_0 \left( x_2 - x_2 \right) \frac{x_1^2}{r_2} \]  

\[ U^{(1)}_{x1} = H(\kappa_1) \left\{ \left( 1 - \kappa_1 \right) \ln r_1 - \frac{2x_1^2}{r_1^2} \right\} \]

\[ + \left( A\kappa_1 - B \right) \ln r_2 + 2A \left( x_1^2 \right) \]

\[ - (\kappa_1 + 3)x_0 x_2 + 2x_1^2 \frac{1}{r_1^2} - 8Ax_0 \left( x_2 - x_2 \right) \frac{x_1^2}{r_2^2} \]  

\[ \text{(A1)} \]

\[ \text{(A2)} \]

\[ \text{(A3)} \]

\[ \text{(A4)} \]

\[ \text{(A5)} \]

\[ \text{(A6)} \]

\[ \text{(A7)} \]
\[ U_{y_1}^{(l)} = H(\kappa_1) \left\{ (\kappa_1 + 1) \theta_1 - \frac{2x_1y_1}{r_1^2} \right. \]

\[ - (A\kappa_1 + B)\theta_2 + 2A[(\kappa_1 - 1)x_0 + x_2]\frac{y_1}{r_2^2} \]

\[ - 8Ax_0(x_0 - x_2)\frac{x_2y_1}{r_2^4} \left\} \right. \] (A8)

\[ U_{y_2}^{(l)} = H(\kappa_1) \left\{ [(1 + A)\kappa_1 + 1 + B]\ln r_1 \]

\[ - \frac{2}{T}(1 - B)x_1 + x_0(A - B)\frac{v_1}{r_1^2} \left\} \right. \] (A9)

\[ U_{x_2}^{(l)} = H(\kappa_1) \left\{ [(1 - A)\kappa_1 + 1 + B]\ln r_1 \]

\[ - \frac{2}{T}(1 - B)x_1 + x_0(A - B)\frac{v_1}{r_1^2} \left\} \right. \] (A10)

\[ U_{y_2}^{(l)} = H(\kappa_1) \left\{ [(1 + A)\kappa_1 + 1 + B]\ln r_1 \]

\[ - \frac{2}{T}(1 - B)x_1 + x_0(A - B)\frac{v_1}{r_1^2} \left\} \right. \] (A11)

\[ V_{x_1}^{(l)} = 2H(\kappa_1)G_1 \left\{ - 2\left( 1 - \frac{2x_1^2}{r_1^2} \right)\frac{y_1}{r_1^2} \right. \]

\[ + \left( B + A + \frac{4Ax_1^3}{r_2^2} \right)\frac{y_1}{r_2^2} - 2Ax_0(x_2 - x_0) \]

\[ \times \left\{ \frac{4y_1}{r_1^2} - \frac{16x_1^2y_1}{r_2^4} \right\} \right. \] (A12)

\[ V_{x_2}^{(l)} = 2H(\kappa_1)G_1 \left\{ - 2\left( 1 - \frac{2x_1^2}{r_1^2} \right)\frac{x_1}{r_1^2} \right. \]

\[ + \left( 3A - B - \frac{4Ax_1^3}{r_2^2} \right)\frac{x_2}{r_2^2} \]

\[ - 2Ax_0 \left\{ \frac{2}{r_2^2} - \frac{16x_1^2 - 12x_0x_2}{r_2^6} \right\} \right. \] (A13)

\[ V_{y_1}^{(l)} = 2H(\kappa_1)G_1 \left\{ - 2\left( 1 - \frac{2x_1^2}{r_1^2} \right)\frac{y_1}{r_1^2} \right. \]

\[ + \left( 3A - B - \frac{4Ax_1^3}{r_2^2} \right)\frac{x_2}{r_2^2} \]

\[ - 2Ax_0 \left\{ \frac{2}{r_2^2} - \frac{16x_1^2 - 12x_0x_2}{r_2^6} \right\} \right. \] (A14)

\[ V_{y_2}^{(l)} = 2H(\kappa_1)G_1 \left\{ - 2\left( 1 - \frac{2x_1^2}{r_1^2} \right)\frac{y_1}{r_1^2} \right. \]

\[ + \left( 3A - B - \frac{4Ax_1^3}{r_2^2} \right)\frac{x_2}{r_2^2} \]

\[ - 2Ax_0 \left\{ \frac{2}{r_2^2} - \frac{16x_1^2 - 12x_0x_2}{r_2^6} \right\} \right. \] (A15)

\[ V_{x_1}^{(l)} = 2H(\kappa_1)G_1 \left\{ - 2\left( 1 - \frac{2x_1^2}{r_1^2} \right)\frac{y_1}{r_1^2} \right. \]

\[ + \left( 3A - B - \frac{4Ax_1^3}{r_2^2} \right)\frac{y_1}{r_2^2} \]

\[ - 2Ax_0 \left\{ \frac{2}{r_2^2} - \frac{16x_1^2 - 12x_0x_2}{r_2^6} \right\} \right. \] (A16)

\[ V_{x_2}^{(l)} = 2H(\kappa_1)G_1 \left\{ - 2\left( 1 - \frac{2x_1^2}{r_1^2} \right)\frac{x_1}{r_1^2} \right. \]

\[ + \left( 3A - B - \frac{4Ax_1^3}{r_2^2} \right)\frac{x_2}{r_2^2} \]

\[ - 2Ax_0 \left\{ \frac{2}{r_2^2} - \frac{16x_1^2 - 12x_0x_2}{r_2^6} \right\} \right. \] (A17)

\[ V_{y_1}^{(l)} = 2H(\kappa_1)G_1 \left\{ - 2\left( 1 - \frac{2x_1^2}{r_1^2} \right)\frac{y_1}{r_1^2} \right. \]

\[ + \left( 3A - B - \frac{4Ax_1^3}{r_2^2} \right)\frac{x_2}{r_2^2} \]

\[ - 2Ax_0 \left\{ \frac{2}{r_2^2} - \frac{16x_1^2 - 12x_0x_2}{r_2^6} \right\} \right. \] (A18)

\[ V_{y_2}^{(l)} = 2H(\kappa_1)G_1 \left\{ - 2\left( 1 - \frac{2x_1^2}{r_1^2} \right)\frac{y_1}{r_1^2} \right. \]

\[ + \left( 3A - B - \frac{4Ax_1^3}{r_2^2} \right)\frac{x_2}{r_2^2} \]

\[ - 2Ax_0 \left\{ \frac{2}{r_2^2} - \frac{16x_1^2 - 12x_0x_2}{r_2^6} \right\} \right. \] (A19)

\[ V_{x_1}^{(l)} = 2H(\kappa_1)G_1 \left\{ - 2\left( 1 - \frac{2x_1^2}{r_1^2} \right)\frac{y_1}{r_1^2} \right. \]

\[ + \left( 3A - B - \frac{4Ax_1^3}{r_2^2} \right)\frac{x_2}{r_2^2} \]

\[ - 2Ax_0 \left\{ \frac{2}{r_2^2} - \frac{16x_1^2 - 12x_0x_2}{r_2^6} \right\} \right. \] (A20)

\[ V_{x_2}^{(l)} = 2H(\kappa_1)G_1 \left\{ - 2\left( 1 - \frac{2x_1^2}{r_1^2} \right)\frac{y_1}{r_1^2} \right. \]

\[ + \left( 3A - B - \frac{4Ax_1^3}{r_2^2} \right)\frac{x_2}{r_2^2} \]

\[ - 2Ax_0 \left\{ \frac{2}{r_2^2} - \frac{16x_1^2 - 12x_0x_2}{r_2^6} \right\} \right. \] (A21)
\[ V_{x y 2}^{(1)} = 2H(\kappa_1)G_1 \left\{ (A + B - 2) x_1 \frac{x_1}{r_1^2} + (B - A) \frac{2x_0}{r_1^2} \right\} - 4x_1^2 \frac{x_1}{r_1^2} \left[ (B - A)x_0 + (B - 1)x_1 \right] \left\} \right. \] (A22)

\[ V_{x y 2}^{(1)} = 2H(\kappa_1)G_1 \left\{ (A + B - 2) y_1 \frac{y_1}{r_1^2} - \frac{4y_1 y_1}{r_1^2} \right\} \times [(B - A)x_0 + (B - 1)x_1] \left\} \right. \] (A23)

\[ V_{y y 2}^{(1)} = 2H(\kappa_1)G_1 \left\{ (6 - A - 5B) \frac{y_1}{r_1^2} - \frac{2x_0}{r_1^2} \right\} + 4x_1^2 \frac{x_1}{r_1^2} \left[ (B - A)x_0 + (B - 1)x_1 \right] \left\} \right. \] (A24)

**Appendix B**

\[ K_{11} = -1 \begin{bmatrix} G_{x x 2}^{(2)} & G_{x y 2}^{(2)} \\ G_{x y 2}^{(2)} & G_{y y 2}^{(2)} \end{bmatrix} \sin \alpha_1 + \begin{bmatrix} G_{x x 2}^{(2)} & G_{x y 2}^{(2)} \\ G_{x y 2}^{(2)} & G_{y y 2}^{(2)} \end{bmatrix} \cos \alpha_1 \]

\[ x_0 = t_{1 m} \cos \alpha_1, \quad x_2 = (s_{1 k} + t_{1 m}) \cos \alpha_1, \]

\[ y_0 = (s_{1 k} - t_{1 m}) \sin \alpha_1, \quad r_2 = x_2^2 + y_2^2 \]

\[ G_{x x 2}^{(2)} = 2H(\kappa_2)G_2 \left\{ (B' + A') \frac{x_2}{r_2^2} - \frac{4A' x_2^3}{r_2^2} \right\} \]

\[ - 2A' x_0 (x_2 - x_0) \left[ \frac{4y_1}{r_2^2} - \frac{16x_2 y_1}{r_2^2} \right] \left\} \right. \] (A25)

\[ G_{x y 2}^{(2)} = 2H(\kappa_2)G_2 \left\{ (3A' - B') \frac{x_2}{r_2^2} - \frac{4A' x_2^3}{r_2^2} \right\} \]

\[ - 2A' x_0 \left[ \frac{2}{r_2^2} - \frac{16x_2^3 - 12x_0 x_2}{r_2^2} \right] - \frac{16x_2 (x_2 - x_0)}{r_2^2} \left\} \right. \] (A26)

\[ G_{y y 2}^{(2)} = 2H(\kappa_2)G_2 \left\{ (3A' - B') \frac{y_1}{r_1^2} - \frac{4A' x_2^3}{r_2^2} \right\} \]

\[ - 2A' x_0 y_1 \left[ \frac{4x_0 - 12x_2}{r_2^2} - \frac{16x_2^3 (x_2 - x_0)}{r_2^2} \right] \left\} \right. \] (A27)

\[ G_{x x 2}^{(2)} = 2H(\kappa_2)G_2 \left\{ (B' + A') \frac{x_2}{r_2^2} - \frac{4A' x_2^3}{r_2^2} \right\} + 2A' x_0 \left[ \frac{2}{r_2^2} + \frac{12x_0 x_2 - 8x_2^3}{r_2^2} + \frac{16x_2^3 (x_2 - x_0)}{r_2^2} \right] \left\} \right. \] (A28)

\[ G_{x y 2}^{(2)} = 2H(\kappa_2)G_2 \left\{ (B' + A') \frac{y_1}{r_1^2} - \frac{4A' x_2^3 y_1}{r_2^2} \right\} \]

\[ - 2A' x_0 (x_2 - x_0) \left[ \frac{4y_1}{r_2^2} - \frac{16x_2^3 y_1}{r_2^2} \right] \left\} \right. \] (A29)

\[ G_{y y 2}^{(2)} = 2H(\kappa_2)G_2 \left\{ (B' + 5A') \frac{x_2}{r_2^2} + \frac{4A' x_2^3}{r_2^2} \right\} + 2A' x_0 \left[ \frac{2}{r_2^2} + \frac{16x_2^3 - 12x_0 x_2}{r_2^2} \right] - \frac{16x_2^3 (x_2 - x_0)}{r_2^2} \left\} \right. \] (A30)

\[ K_{12} = -1 \begin{bmatrix} V_{x x 2}^{(1)} & V_{x y 2}^{(1)} \\ V_{x y 2}^{(1)} & V_{y y 2}^{(1)} \end{bmatrix} \sin \alpha_1 + \begin{bmatrix} V_{x x 2}^{(1)} & V_{x y 2}^{(1)} \\ V_{x y 2}^{(1)} & V_{y y 2}^{(1)} \end{bmatrix} \cos \alpha_1 \]

\[ x_0 = t_{2 m} \cos \alpha_1, \quad x_1 = -(s_{2 k} \cos \alpha_1 + t_{2 m} \cos \alpha_2), \]

\[ y_0 = -(s_{2 k} \sin \alpha_1 + t_{2 m} \sin \alpha_1), \quad r_2^2 = x_1^2 + y_1^2 \]

\[ K_{21} = -1 \begin{bmatrix} G_{x x 1}^{(2)} & G_{x y 1}^{(2)} \\ G_{x y 1}^{(2)} & G_{y y 1}^{(2)} \end{bmatrix} \sin \alpha_2 + \begin{bmatrix} G_{x x 1}^{(2)} & G_{x y 1}^{(2)} \\ G_{x y 1}^{(2)} & G_{y y 1}^{(2)} \end{bmatrix} \cos \alpha_2 \]

\[ x_0 = t_{1 m} \cos \alpha_1, \quad x_1 = -(s_{1 k} \cos \alpha_1 + t_{1 m} \cos \alpha_1), \]

\[ y_1 = -(s_{1 k} \sin \alpha_1 + t_{1 m} \sin \alpha_1), \quad r_2^2 = x_1^2 + y_1^2 \]

\[ G_{x x 1}^{(2)} = 2H(\kappa_2)G_2 \left\{ (A' + B' - 2) \frac{y_1}{r_1^2} \right\} \]

\[ + \frac{4x_1 y_1}{r_1^4} \left[ (B' + A')x_0 + (B' - 1)x_1 \right] \left\} \right. \] (A31)

\[ G_{x y 1}^{(2)} = 2H(\kappa_2)G_2 \left\{ (3B' - A' - 2) \frac{x_1}{r_1^2} \right\} \]

\[ + (B' - A') \frac{2x_0}{r_1^2} - \frac{4x_2^3}{r_1^2} \left[ (B' - A')x_0 \right. \right. \]

\[ + (B' - 1)x_1 \right\} \left\} \right. \] (A32)
\[
G_{xy1}^{(2)} = 2H(\kappa_2)G_2 \left\{ (3B' - A' - 2) \frac{y_1}{r_1} \right\} - \frac{4x_1y_1}{r_1} \left\{ [(B' - A')x_0 + (B' - 1)x_1] \right\} \quad (A33)
\]
\[
V_{xy1}^{(3)} = 2H(\kappa_1)G_1 \left\{ -2 \left( 1 - \frac{2x_1^2}{r_1^2} \right) \frac{x_1}{r_1} \right\} + \left( 3A - B - \frac{4A_2^3}{r_1^2} \right) \frac{x_2}{r_2^2} \quad (A39)
\]
\[
G_{xy2}^{(1)} = 2H(\kappa_2)G_2 \left\{ (A' + B' - 2) \frac{x_1}{r_1} \right\} + (B' - A') \frac{2x_0}{r_1} - \frac{4x_1^2}{r_1^2} \left\{ [(B' - A')x_0] \right\} + (B' - 1)x_1 \right\} \quad (A34)
\]
\[
V_{xy1}^{(3)} = 2H(\kappa_1)G_1 \left\{ -2 \left( 1 - \frac{2x_1^2}{r_1^2} \right) \frac{x_1}{r_1} \right\} \quad (A40)
\]
\[
G_{xy1}^{(2)} = 2H(\kappa_2)G_2 \left\{ (A' + B' - 2) \frac{y_1}{r_1} \right\} - \frac{4x_1y_1}{r_1} \left\{ [(B' - A')x_0 + (B' - 1)x_1] \right\} \quad (A35)
\]
\[
V_{xy1}^{(3)} = 2H(\kappa_1)G_1 \left\{ -2 \left( 1 - \frac{2x_1^2}{r_1^2} \right) \frac{y_1}{r_1} \right\} + (B + A) \frac{x_2}{r_2^2} - \frac{4Ax_2^3}{r_2^2} \quad (A41)
\]
\[
G_{xy1}^{(2)} = 2H(\kappa_2)G_2 \left\{ (6 - A' - 5B') \frac{x_1}{r_1} \right\} - (B' - A') \frac{2x_0}{r_1} + \frac{4x_1^2}{r_1^2} \left\{ [(B' - A')x_0] \right\} + (B' - 1)x_1 \right\} \quad (A36)
\]
\[
V_{xy1}^{(3)} = 2H(\kappa_1)G_1 \left\{ -2 \left( 3 - \frac{2x_1^2}{r_1^2} \right) \frac{x_1}{r_1} \right\} - (B + 5A) \frac{x_2}{r_2^2} + \frac{4Ax_2^3}{r_2^2} \quad (A42)
\]
\[
K_{22} = - \left[ \begin{array}{ccc}
V_{xx1}^{(1)} & V_{xy1}^{(1)} & V_{yy1}^{(1)} \\
V_{xy1}^{(1)} & V_{xx1}^{(1)} & V_{yy1}^{(1)} \\
V_{yy1}^{(1)} & V_{yy1}^{(1)} & V_{xx1}^{(1)}
\end{array} \right] \sin \alpha_2 + \left[ \begin{array}{ccc}
V_{xx1}^{(1)} & V_{xy1}^{(1)} & V_{yy1}^{(1)} \\
V_{xy1}^{(1)} & V_{xx1}^{(1)} & V_{yy1}^{(1)} \\
V_{yy1}^{(1)} & V_{yy1}^{(1)} & V_{xx1}^{(1)}
\end{array} \right] \cos \alpha_2
\]
\[
L_1(s_{1k}) = - \left[ \begin{array}{ccc}
V_{xx1}^{(3)} & V_{xy1}^{(3)} & V_{yy1}^{(3)} \\
V_{xy1}^{(3)} & V_{xx1}^{(3)} & V_{yy1}^{(3)} \\
V_{yy1}^{(3)} & V_{yy1}^{(3)} & V_{xx1}^{(3)}
\end{array} \right] \sin \alpha_1 + \left[ \begin{array}{ccc}
V_{xx1}^{(3)} & V_{xy1}^{(3)} & V_{yy1}^{(3)} \\
V_{xy1}^{(3)} & V_{xx1}^{(3)} & V_{yy1}^{(3)} \\
V_{yy1}^{(3)} & V_{yy1}^{(3)} & V_{xx1}^{(3)}
\end{array} \right] \cos \alpha_1
\]
\[
V_{xx1}^{(3)} = 2H(\kappa_1)G_1 \left\{ (A + B - 2) \frac{x_1}{r_1} + \frac{4x_1y_1}{r_1^2} (B - 1)x_1 \right\} \quad (A45)
\]
\[
V_{xy1}^{(3)} = 2H(\kappa_1)G_1 \left\{ (3B - A - 2) \frac{x_1}{r_1} - \frac{4x_1^2}{r_1^2} (B - 1)x_1 \right\} \quad (A46)
\]
\[
V_{xy1}^{(3)} = 2H(\kappa_1)G_1 \left\{ (A + B - 2) \frac{x_1}{r_1} - \frac{4x_1^2}{r_1^2} (B - 1)x_1 \right\} \quad (A47)
\]
\[
V_{xy1}^{(3)} = 2H(\kappa_1)G_1 \left\{ (A + B - 2) \frac{x_1}{r_1} - \frac{4x_1^2}{r_1^2} (B - 1)x_1 \right\} \quad (A48)
\]
\[ V_{3y2}^{(3)} = 2H(\kappa_1)G_1 \left\{ (A + B - 2) \frac{y_1}{r_1^2} - \frac{4x_1y_1}{r_1^4}(B - 1)x_1 \right\} \]  
(A49)

\[ V_{3y2}^{(3)} = 2H(\kappa_1)G_1 \left\{ (6 - A - 5B) \frac{x_1}{r_1^2} + \frac{4x_1^2}{r_1^4}(B - 1)x_1 \right\} \]  
(A50)

References