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VARIATIONAL PRINCIPLES AND HYBRID ELEMENT ON A SANDWICH PLATE OF LRUSAKOV–DU'S TYPE

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Abstract—The variational principles and bounded theorem on a sandwich plate of Lrusakov–Du's type are presented. A hybrid element then is developed based on Hellinger–Reissner principle and the numerical results demonstrate good performance of the present element.

NOMENCLATURE

A	$12E_c/(G_c h^2)$
D	$(1-\mu^2)^{-1}E_f(h+t)^2 t/2$
D_f	$(1-\mu^2)^{-1}E_f t^3/12$
D_Q	$G_c \gamma h$
D_0	$D + 2D_f$
D_σ	$8E_c/h$
E_c	Young's modulus of core material
E_f	Young's modulus of face–sheet material
G_c	core shear modulus
h	core thickness
M_{ij}	$M'_{ij} + M''_{ij}$, bending moment per unit width
M'_{ij}	bending moment caused by rotations ψ_i
M''_{ij}	bending moment caused by deflection w
q	transverse load
Q^c_i	core transverse shear force per unit width
t	face–sheet thickness
U	total potential energy
V	total complementary energy
V'_M	$b'_{ijmn}M'_{ij}M'_{mn}/2$
V''_M	$b''_{ijmn}M''_{ij}M''_{mn}/2$
V_Q	$D_Q^{-1}Q^c_i Q^c_i/2$
V_σ	$D_\sigma^{-1}\sigma^2/2$
b'_{1122}	$b'_{2211} = -\mu b'_{1111} = -\mu b'_{2222} = -\mu(1+\mu)^{-1}b'_{1212}/2 = -\mu(1-\mu^2)^{-1}D^{-1}$
b''_{1122}	$b''_{2211} = -\mu b''_{1111} = -\mu b''_{2222} = -\mu(1+\mu)^{-1}b''_{1212}/2 = -\mu(1-\mu^2)^{-1}D_f^{-1}/2$ (the others are zero)
w	face–sheet deflection
w_0	deflection on core middle surface
w'	$w - w_0$
x_i	Cartesian coordinate as usual
α	2/3
γ	$1 + t/h$
μ	Poisson's ratio of face–sheet material
σ^c_z	core compressive stress, $\sigma^c_z = \sigma x_3/h$
ψ_i	rotation about middle surface in i direction

1. INTRODUCTION

The widespread interest in the present and future possibilities of applying the principles of sandwich construction in the design of various structural elements, especially that of the aerospace industry, has recently been reaffirmed by the appearance of a series of extensive bibliographies devoted to this subject. In what follows we discuss the below variational principles for the sandwich plate of Lrusakov–Du's type. Here the facings are viewed as general thin plates, while the core is possessed of transverse compressive deformation besides shear resistance. The proof of Theorem 4 as well as some numerical results are given in the following.

2. FUNDAMENTAL MODEL AND VARIATIONAL PRINCIPLES

Theorem 1

The basic equations of sandwich plates of Lrusakov–Du's type [1, 2] exist following matrix modelling:

$$\Omega: \quad A_{12}X_\sigma + EX_\sigma + Q = 0, \tag{1}$$

$$X_\epsilon = -A_{21}X_u - \overset{*}{E}X_u, \tag{2}$$

$$X_\sigma^T = \partial U/\partial X_\epsilon, \tag{3}$$

$$X_\epsilon^T = \partial V/\partial X_\sigma, \tag{4}$$

$$X_\sigma^T X_\epsilon = U(X_\epsilon) + V(X_\sigma), \tag{5}$$

$$C_\sigma: \quad E(v)X_\sigma = \bar{X}_\sigma, \tag{6}$$

$$C_u: \quad X_u = \bar{X}_u; \quad w_{,n} = \bar{w}_n, \tag{7}$$

$$(\partial\Omega = C = C_\sigma \cup C_u),$$

where

$$X_u^T = (\Psi_i, -w, w') \quad X_\sigma^T = (M'_{ij}, M''_{ij}, Q^c_i, \sigma),$$

$$E = \begin{pmatrix} E_2 & 0 & 0 & 0 \\ 0 & E_1E_2 & \gamma E_1 & 0 \\ 0 & 0 & \alpha E_1 & 0 \end{pmatrix}, \quad E_1 = (\partial/\partial x_1, \partial/\partial x_2)$$

$$E_2 = \begin{pmatrix} \partial/\partial x_1 & 0 & \partial/\partial x_2 \\ 0 & \partial/\partial x_2 & \partial/\partial x_1 \end{pmatrix}$$

$$A_{12} = A_{21}^T = \begin{pmatrix} 0 & 0 & -\gamma & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix}, \quad Q^T = (0, q, 0),$$

$$V = -V'_M - V''_M - \gamma V_Q - \alpha V, \tag{8}$$

$$C = C_{M'_n} \cup C_{\psi_n} = C_{M'_s} \cup C_{\psi_s} = C_{M''_n} \cup C_{w,n}$$
$$= C_{Q_n} \cup C_w = C_{Q^c_n} \cup C_{w'}, \tag{9}$$

here $X_u, X_\sigma, X_\epsilon$ are the generalized variables of displacement, stress and deformation respectively. E^* is a conjugate operator of E; (1) the equilibrium equation; (2) the geometric equation; (3), (4) and (5) are three equivalent forms of constitutive equation; (6) and (7) are the boundary conditions, in which the boundary operator $E(v)$ is obtained by replacing $\partial/\partial x_i$ in E with n_i (see Note 1 below). Here n_i is a unit normal vector on the boundary C, but $(E_1 E_2)(v)$ in operator $E(v)$, associated with inner product of w and M'', need to be specifically defined as

$$w(E_1 E_2)(v)M'' = wE_1(v)E_2 M'' + (E^*{}_1 w)^T E_2(v) M'', \tag{10}$$

where $M = (M''_{11}, M''_{22}, M''_{12})^T$ and thus

$$X_u^T E(v) X_\sigma = \psi_i M'_{ij} n_j - w M''_{ij,j} n_i + w_{,i} M''_{ij} n_j$$
$$- w\gamma Q^c_n + w'\alpha Q^c_n$$
$$= \psi_n M'_n + \psi_s M'_{ns} + \alpha w' Q^c_n$$
$$- w Q_n + w_{,n} M''_n \tag{11}$$
$$Q_n = \gamma Q^c_n + M''_{ij,j} n_i + M''_{ns,s}.$$

Note. Some equilibrium equations in Ref. [3] may be rewritten as follows:

$$M_{ij,j} - Q_i = 0$$
$$Q_{i,i} + q = 0 \tag{12}$$

here we modify them equivalently as

$$M'_{ij,j} - \gamma Q^c_i = 0.$$
$$M''_{ij,ji} + \gamma Q^c_{i,i} + q = 0, \tag{13}$$

where $M_{ij} = M'_{ij} + M''_{ij}$, $Q_i = \gamma Q^c_i + M''_{ij,j}$ and the variables become into M'_{ij}, M''_{ij}, Q^c_i. In doing so, it is guaranteed for the existence of the conservative system as well as the Theorem 1 here and following Theorems 2–4 being tenable. We call them canonical Hamilton form (see Theorem 2 below).

Theorem 2

(i) Eqns (1) and (4) possess below generalized Hamilton form in the case of (2) being identical, *a priori*

$$\Omega: \quad \begin{pmatrix} 0 & E \\ \overset{*}{E} & 0 \end{pmatrix} X = \delta H/\delta X, \tag{14}$$

where

$$X^T = (X_u, X_\sigma), \quad H = -X_u^T A_{12} X_\sigma - V - X_u^T Q.$$

(ii) Complete field eqns (6), (7) and (14) are the stationary conditions of following mutual complementary functionals (generalized variational principle of Hellinger–Reissner type)

$$\Pi_2 = \int_\Omega (-X_\sigma^T \overset{*}{E} X_u + H)\,d\Omega - \int_{C_\sigma} X_u^T \bar{X}_\sigma\,dc$$
$$- \int_{C_u} (X_u^T - \bar{X}_u^T) E(v) X_\sigma\,dc, \tag{15}$$

$$\Gamma_2 = \int_\Omega (-X_u^T E X_\sigma + H)\,d\Omega$$
$$- \int_{C_\sigma} X_u^T (\bar{X}_\sigma - E(v) X_\sigma)\,dc$$
$$+ \int_{C_u} \bar{X}_u^T E(v) X_\sigma\,dc, \tag{16}$$

$$\Pi_2 = \Gamma_2. \tag{17}$$

Theorem 3

Complete field eqns (1), (2), (3), (6) and (7) are the stationary conditions of following mutual complementary functions (generalized variational principle of Hu–Washizu type)

$$\Pi_3 = \int_\Omega (U - X_\sigma^T X_\epsilon - X_\sigma^T \overset{*}{E} X_u - X_u^T A_{12} X_\sigma$$
$$- X_u^T Q)\,d\Omega - \int_{C_\sigma} X_u^T \bar{X}_\sigma\,dc$$
$$- \int_{C_u} (X_u^T - \bar{X}_u^T) E(v) X_\sigma\,dc, \tag{18}$$

$$\Gamma_3 = \int_\Omega (U - X_\sigma^T X_\epsilon - X_u^T E X_\sigma - X_u^T A_{12} X_\sigma$$
$$- X_u^T Q)\,d\Omega - \int_{C_\sigma} X_u^T (\bar{X}_\sigma - E(v) X_\sigma)\,dc$$
$$+ \int_{C_u} \bar{X}_u^T E(v) X_\sigma\,dc, \tag{19}$$

$$\Pi_3 = \Gamma_3. \tag{20}$$

Theorem 4

We have

$$\Pi_1 \geqslant \Pi_1|_0 = \Gamma_1|_0 \geqslant -\Gamma_1, \qquad (21)$$

where $(\ )|_0$ represents the value at the stationary point

$$\Pi_1 = \int_\Omega (U - X_u^T Q)\,\mathrm{d}\Omega - \int_{C_\sigma} X_u^T \bar{X}_\sigma\,\mathrm{d}c, \qquad (22)$$

$$\Gamma_1 = \int_\Omega V\,\mathrm{d}\Omega - \int_{C_u} \bar{X}_u^T E(v) X_\sigma\,\mathrm{d}c, \qquad (23)$$

here it requires that (2), (3) and (7) are identical in advance for (22) and (1), (2), (4) and (6) are identical, *a priori*, for (23).

Proof. Theorem 1 is only verified here. To the end, take the first variation of Π_1 and Γ_1, we have

$$\delta\Pi_1 \overset{(2)(3)(7)}{=\!=\!=} -\int_\Omega \delta X_u^T\{A_{12}X_\sigma + EX_\sigma\} + Q\}\,\mathrm{d}\Omega,$$

$$-\int_{C_\sigma} \delta X_u^T\{E(v)X_\sigma - \bar{X}_\sigma\}\,\mathrm{d}c = 0 \Rightarrow (1), (6),$$

$$\delta\Gamma_1 \overset{(1)(2)(4)(6)}{=\!=\!=} \int_{C_u} (X_u^T - \bar{X}_u^T)E(v)\delta X_\sigma\,\mathrm{d}c = 0 \Rightarrow (7).$$

Hence their stationary property has been proved. The minimum principles, finally, need to be proved. For this reason, take the variation of $\delta\Gamma_1$ again and by using integral theorem, we obtain

$$\delta^2\Pi_1 = \delta^2\Gamma_1 = -(1/2)\int_\Omega [\delta X_\sigma^T \delta(E^* X_u) + \delta X_u^T A_{12}\delta X_\sigma]\,\mathrm{d}\Omega, \qquad (24)$$

furthermore, substituting (2) and (3) into (24) and by means of the positive definite of constitutive Hesse matrix $(\partial^2 U/\partial X_\epsilon \partial X_\epsilon)$, we obtain

$$\delta^2\Pi_1 = \delta^2\Gamma_1 = (1/2)\int_\Omega \delta X_\sigma^T \delta X_\epsilon\,\mathrm{d}\Omega$$

$$= (1/2)\int_\Omega \delta X_\epsilon^T(\partial^2 U/\partial X_\epsilon \partial X_\epsilon)\delta X_\epsilon\,\mathrm{d}\Omega > 0. \qquad (25)$$

So the first equality of eqn (21) is proved.

Besides, eqn (24) can be transformed to (25) in the case of linear strain–displacement relation [see eqn (2)]. Therefore the bounded theorem (21) holds.

3. HYBRID ELEMENT AND NUMERICAL RESULTS

For a three-node thin plate element with nine-degree of freedom, cubic interpolation of w does not satisfy C^1 continuity requirement because the boundary normal derivative of w is not C^1 continuous across the interelement boundary. To enforce compatibility (in a variational sense), two additional boundary displacements, $\tilde{w}$ and $\tilde{w}_{,n}$, are assumed along the boundary [4]. Hence, eqns (15) and (16) modified for relaxed element compatibility condition can be given as

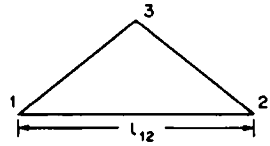

Fig. 1.

$$\Pi_2^m = \sum_e \left[\int_{A_e} (-X_u^T E X_\sigma + H)\,\mathrm{d}\Omega + \int_{A_e} \tilde{X}_u^T E(v) X_\sigma\,\mathrm{d}c\right] - \int_{C_\sigma} \tilde{X}_u^T \bar{X}_\sigma\,\mathrm{d}c$$

$$-\int_{C_u} (\tilde{X}_u^T - \bar{X}_u^T)E(v)X_\sigma\,\mathrm{d}c, \qquad (26)$$

where $\tilde{X}_u^T E(v) X_\sigma$ is obtained by replacing w in (11) with $\tilde{w}$.

Taking a three-node element with 18 degrees of freedom, consequently associated displacement and stress fields may be assumed in the following way. Rotations, ψ_i, and generalized deflection, w', are interpolated linearly as well as face–sheet deflection, w, is interpolated with a cubic function. Along element boundary, $\tilde{w}$ and $\tilde{w}_{,n}$ are interpolated quadratically and linearly respectively. Stress field may be assumed in an analogous way. In this way, one has

$$\begin{aligned}
X_u &= (\psi_i, -w, w')^T = \{F_1\, F_2\, F_3\} \times (\{\psi_i\}_{1\times 6}\{w, w_{,i}\}_{1\times 9}\{w'\}_{1\times 3})^T = Fu,\\
X_\sigma &= (M'_{ij}, M''_{ij}, Q_i^c, \sigma)^T = \{P_1\, P_2\, P_3\, P_4\} \times (\{\beta_1\}_{1\times 3}\{\beta_2\}_{1\times 9}\{\beta_3\}_{1\times 6}\{\beta_4\}_{1\times 3})^T = P\beta,\\
\tilde{X}_u &= (\psi_i, -\tilde{w}, w', -\tilde{w}_{,n})^T = \{Q_1\, Q_2\, Q_3\, Q_4\} \times (\{\psi_i\}_{1\times 6}\{\tilde{w}, \tilde{w}_{,s}\}_{1\times 4}\{w'\}_{1\times 3}\{\tilde{w}_{,n}\}_{1\times 2})^T = Q'u,
\end{aligned} \qquad (27)$$

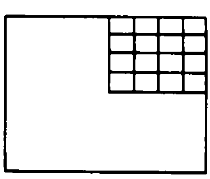
Fig. 2.

Table 1. Centre deflection factor $\pi^2 D_0 w/(4pa^2)$

	$h/t = 0$		$h/t = 20$	
A	Present	Ref. [3]	Present	Ref. [3]
60	6.741	6.603	23.297	23.310
80	5.403	5.404	18.556	18.950
100	4.450	4.584	15.501	15.950

here the interpolated matrix are diagonal one and we put it into symbol { }, and associated interpolation matrices are

$$F_1 = \begin{pmatrix} L_1 & 0 & L_2 & 0 & L_3 & 0 \\ 0 & L_1 & 0 & L_2 & 0 & L_3 \end{pmatrix}, \quad F_3 = (L_1\, L_2\, L_3),$$

$$F_2 = -(N_1\, N_{x1}\, N_{y1} \ldots N_3\, N_{x3}\, N_{y3}),$$

$$N_1 = L_1 + (L_3 L_1^2 - L_1 L_3^2) + (L_2 L_1^2 - L_1 L_2^2),$$

$$N_{x1} = c_3 L_1 L_2/2 - c_2 L_1 L_3/2$$
$$- c_3(L_1 L_2^2 - L_2 L_1^2)/2 - c_2(L_3 L_1^2 - L_1 L_3^2)/2,$$

$$N_{y1} = b_2 L_1 L_2/2 - b_3 L_1 L_2/2$$
$$+ b_3(L_1 L_2^2 - L_2 L_1^2)/2 + b_2(L_3 L_1^2 - L_1 L_3^2)/2.$$

here L_i is area coordinates. P_1 the 3×3 identity matrix, and

$$P_2 = \{F_3\, F_3\, F_3\}, \quad P_3 = \{F_3\, F_3\}, \quad P_4 = F_3,$$
$$Q_1 = F_1|_{\partial A_e}, \quad Q_3 = F_3|_{\partial A_e}.$$

As for Q_2 and Q_4, they may be given as follows (take sides 1 and 2 as example, see Fig. 1)

$$-\tilde{w} = L_1^2(2L_2 + 1)(w)_1 - L_2^2(2L_1 + 1)(w)_2$$
$$- l_{12} L_1^2 L_2 (w_{,s})_1 + l_{12} L_2^2 L_1 (w_{,s})_2$$

$$-\tilde{w}_{,n} = L_1(w_{,n})_1 - L_2(w_{,n})_2$$

$$b_3 = y_1 - y_2, \quad c_3 = x_2 - x_1$$

others may be obtained in the same way mentioned above.

Finally, substituting (27) into (26) and by taking variation of Π_2^m with respect to β, one can express β in terms of u. Therefore, the Π_2^m depend only on variable u. Again from $\partial \Pi_2^m/\partial u = 0$ we obtain

$$\sum_e K^e u = G, \tag{28}$$

where K^e is the element stiffness matrix and G equivalent nodal force.

As an example of a sandwich plate with simply supported edges and subjected to vertical concentrated load p at the centre. Some parameters are $\mu = 0.3$, $E_f = 1000$.

We study one-quarter of the plate with 4×4 mesh as shown in Fig. 2. The results are shown in Table 1 as well as compared with those obtained in Ref. [3]. Where a is the edge-length of the square plate.

The numerical results indicate that the present element model exist the potential capacity for calculating sandwich plate structures.

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