ELSEVIER

Theoretical and Applied Fracture Mechanics 29 (1998) 141–150

theoretical and applied fracture mechanics

Multiple cracks in thermoelectroelastic bimaterials

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Abstract

The formulation for thermal stress and electric displacement in an infinite thermopiezoelectric plate with an interface and multiple cracks is presented. Using Green's function approach and the principle of superposition, a system of singular integral equations for the unknown temperature discontinuity defined on each crack face is developed and solved numerically. The formulation can then be used to calculate some fracture parameters such as the stress–electric displacement and strain energy density factor. The direction of crack growth for many cracks in thermopiezoelectric bimaterials is predicted by way of the strain energy density theory. Numerical results for stress–electric displacement factors and crack growth direction at a particular crack tip in two crack system of bimaterials are presented to illustrate the application of the proposed formulation.

1. Introduction

The analysis of multiple cracks in an infinite thermopiezoelectric solid is of considerable importance in the field of fracture mechanics. Stress analysis of the multiple crack problems in isotropic materials has been done by many researchers, such as those in [1–3]. A historical review of this topic was given in [4].

For anisotropic materials, a solution for collinear cracks in an infinite plate was obtained [5]. Treated in [6] is the elastic interaction between a main crack and a parallel micro-crack in an orthotropic plate. Based on the strain energy density criterion, the direction of initial crack growth of two interacting cracks in an anisotropic solid was also studied [7]. Unlike the case of anisotropic elasticity, relatively little work has been done for the analysis of multiple crack problems in piezoelectric materials. This work is a continuation of our previous studies [8,9]. In this paper, Green's function for bimaterials and the principle of superposition are used to study thermoelectroelastic behaviour of multiple crack in an infinite bimaterial solid. The geometry of the problem is shown in Fig. 1. After introducing the extended Stroh formalism and the thermoelectroelastic Green's function for bimaterials, a system of singular integral equations for the unknown thermal analog of dislocation density defined on crack faces is derived by using the principle of superposition. The integral equations are solved numerically and used to calculate SED intensity factors and strain energy density factor. The direction of crack growth for many cracks in thermopiezoelectric bimaterials is then predicted by way of the strain energy density theory. One numerical example is considered to illustrate application of the proposed formulation.

2. Basic formulation

Summarized briefly are the governing equations of 2D piezoelectricity and some descriptions on the

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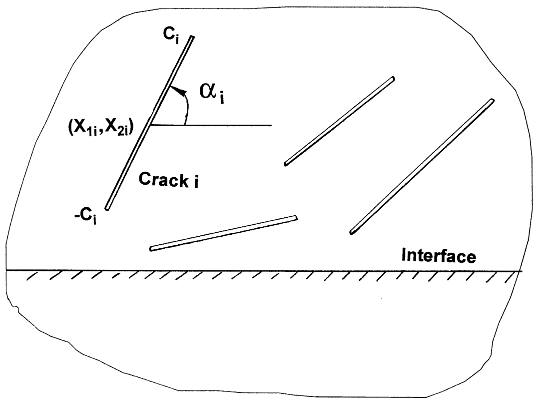

Fig. 1. Geometry of multiple cracks in bimaterial.

multiple crack problem. Throughout this paper the shorthand notation introduced in [10] and fixed Cartesian coordinate system (x_1, x_2, x_3) will be adopted. Lower case Latin subscripts will always range from 1 to 3, upper case Latin subscripts will range from 1 to 4 and the summation convention will be used for repeating subscripts unless it is otherwise indicated. In the stationary case when no free electric charge, body force and heat source are assumed to exist, the complete set of governing equations for uncoupled thermo-electroelastic problems are [11]

$$h_{i,i} = 0, \qquad \Pi_{iJ,i} = 0 \tag{1}$$

together with

$$h_i = -k_{ij}\theta_{,j}, \qquad \Pi_{iJ} = E_{iJKm}u_{K,m} - \chi_{iJ}\theta, \tag{2}$$

in which

$$\Pi_{iJ} = \begin{cases} \sigma_{ij}, & i,J = 1,2,3, \\ D_i, & J = 4, \ \ i = 1,2,3, \end{cases} \tag{3}$$

$$u_J = \begin{cases} u_k, & J = 1,2,3, \\ \vartheta, & J = 4, \end{cases}$$

$$\chi_{iJ} = \begin{cases} \gamma_{ij} & i,J = 1,2,3, \\ \chi_i & J = 4, \ \ i = 1,2,3, \end{cases}$$

$$E_{iJKm} = \begin{cases} C_{ijkm}, & i,J,K,m = 1,2,3, \\ e_{mij}, & K = 4, \ \ i,J,m = 1,2,3, \\ e_{ikm}, & J = 4, \ \ i,K,m = 1,2,3, \\ -\kappa_{im}, & J = K = 4, \ \ i,m = 1,2,3, \end{cases} \tag{4}$$

where θ and h_i are temperature change and heat flux, $u_i, \vartheta, \sigma_{ij}$ and D_i are elastic displacement, electric potential, stress and electric displacement, C_{ijkm}, e_{ijk} and κ_{ij} are elastic moduli, piezoelectric and dielectric constants, and k_{ij}, γ_{ij} and χ_i are the coefficients of heat conduction, thermal-stress constants and pyroelectric constants, respectively. A general solution to Eq. (1) can be expressed as [12]

$$\begin{aligned} T &= \mathrm{Im}[g'(z_t)], \\ \mathbf{u} &= \mathrm{Im}[\mathbf{A}\mathbf{f}(\mathbf{z})\mathbf{q} + \mathbf{c}g(z_t)] \end{aligned} \tag{5}$$

with

$$\begin{aligned} &\mathbf{A} = [\mathbf{A}_1 \mathbf{A}_2 \ \mathbf{A}_3 \ \mathbf{A}_4], \\ &\mathbf{f}(\mathbf{z}) = \mathrm{diag}[f(z_1) f(z_2) f(z_3) f(z_4)], \\ &\mathbf{q} = \{q_1 \ q_2 \ q_3 \ q_4\}^{\mathrm{T}}, \\ &z_t = x_1 + \tau x_2, \\ &z_i = x_1 + p_i x_2, \end{aligned}$$

in which "Im" stands for the imaginary part of a complex, the prime ($'$) denotes differentiation with the argument, g and $\mathbf{f}$ are arbitrary functions to be determined, p_i, τ, $\mathbf{A}$ and $\mathbf{c}$ are constants determined by

$$\begin{aligned} &k_{22}\tau^2 + (k_{12} + k_{21})\tau + k_{11} = 0, \\ &[\mathbf{Q} + (\mathbf{R} + \mathbf{R}^{\mathrm{T}})p_i + \mathbf{T}p_i^2]\mathbf{A}_i = 0, \\ &[\mathbf{Q} + (\mathbf{R} + \mathbf{R}^{\mathrm{T}})\tau + \mathbf{T}\tau^2]\mathbf{c} = \boldsymbol{\chi}_1 + \tau\boldsymbol{\chi}_2, \end{aligned} \tag{6}$$

in which superscript "T" denotes the transpose, $\boldsymbol{\chi}_i$ are 4×1 vectors, and $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{T}$ are 4×4 matrices defined by

$$\begin{aligned} &\boldsymbol{\chi}_i = \{\gamma_{i1} \ \gamma_{i2} \ \gamma_{i3} \ \chi_i\}^{\mathrm{T}}, \\ &(\mathbf{Q})_{IK} = E_{1IK1}, \quad (\mathbf{R})_{IK} = E_{1IK2}, \quad (\mathbf{T})_{IK} = E_{2IK2}. \end{aligned} \tag{7}$$

The heat flux, $\mathbf{h}$, and the stress–electric displacement $\boldsymbol{\Pi}$, obtained from Eq. (2) can be written as

$$h_i = -\mathrm{Im}[(k_{i1} + \tau k_{i2})g''(z_t)], \tag{8a}$$

$$\Pi_{1J} = -\phi_{J,2}, \quad \Pi_{2J} = \phi_{J,1} \tag{8b}$$

where $\boldsymbol{\phi}$ is the SED function given as

$$\boldsymbol{\phi} = \mathrm{Im}[\mathbf{B}\mathbf{f}(\mathbf{z}) + \mathbf{d}g(z_t)] \tag{9}$$

with

$$\mathbf{B} = \mathbf{R}^{\mathrm{T}}\mathbf{A} + \mathbf{TAP} = -(\mathbf{QA} + \mathbf{RAP})\mathbf{P}^{-1},$$
$$\mathbf{P} = \mathrm{diag}[p_1\ p_2\ p_3\ p_4],$$
$$\mathbf{d} = (\mathbf{R}^{\mathrm{T}} + \tau\mathbf{T})\mathbf{c} - \boldsymbol{\chi}_2 = -(\mathbf{Q} + \tau\mathbf{R})\mathbf{c}/\tau + \boldsymbol{\chi}_1/\tau. \tag{10}$$

Green's function used in this paper is described as follows. For a bimaterial plate, the basic solutions due to a discrete temperature discontinuity of magnitude θ_0 at a point in material 1, $\hat{z}_t^{(1)} = \hat{x}_1 + \tau^{(1)}\hat{x}_2$, are given by [8]

$$\theta^{(1)} = \theta_0 \mathrm{Im}(\ln y_1^{(1)} + b_1 \ln y_2^{(1)})/2\pi, \tag{11}$$

$$h_i^{(1)} = -\frac{\theta_0}{2\pi}\mathrm{Im}\Big[(k_{i1}^{(1)} + \tau^{(1)}k_{i2}^{(1)})\Big(1/y_1^{(1)} - b_1/y_2^{(1)}\Big)\Big], \tag{12}$$

$$\mathbf{U}^{(1)} = \frac{\theta_0}{2\pi}\mathrm{Im}\Big\{\mathbf{A}^{(1)}\mathbf{f}(\mathbf{z}^{(1)})\mathbf{q}_1 + \mathbf{c}^{(1)}\Big[y_1^{(1)}(\ln y_1^{(1)} - 1) + b_1 y_2^{(1)}\Big(\ln y_2^{(1)} - 1\Big)\Big]\Big\}, \tag{13}$$

$$\phi^{(1)} = \frac{\theta_0}{2\pi}\mathrm{Im}\{\mathbf{B}^{(1)}\mathbf{f}(\mathbf{z}^{(1)})\mathbf{q}_1 + \mathbf{d}^{(1)}[y_1^{(1)}(\ln y_1^{(1)} - 1) + b_1 y_2^{(1)}(\ln y_2^{(1)} - 1)]\} \tag{14}$$

for $\mathrm{Im}(z_t^{(1)}) > 0$, and

$$\theta^{(2)} = b_2\theta_0\, \mathrm{Im}\Big(\ln y_1^{(2)}\Big)/2\pi, \tag{15}$$

$$h_i^{(2)} = -\frac{b_2\theta_0}{2\pi}\mathrm{Im}\left[\frac{k_{i1}^{(2)} + \tau^{(2)}k_{i2}^{(2)}}{y_1^{(2)}}\right], \tag{16}$$

$$\mathbf{U}^{(2)} = \mathrm{Im}[\mathbf{A}^{(2)}\mathbf{f}(\mathbf{z}^{(2)})\mathbf{q}_2] + \frac{b_2\theta_0}{2\pi}\mathrm{Im}\{\mathbf{c}^{(2)}y_1^{(2)}(\ln y_1^{(2)} - 1)\}, \tag{17}$$

$$\phi^{(2)} = \mathrm{Im}[\mathbf{B}^{(2)}\mathbf{f}(\mathbf{z}^{(2)})\mathbf{q}_2] + \frac{b_2\theta_0}{2\pi}\mathrm{Im}\{\mathbf{d}^{(2)}y_1^{(2)}(\ln y_1^{(2)} - 1)\} \tag{18}$$

for $\mathrm{Im}(z_t^{(2)}) < 0$, where

$$\mathbf{f}(\mathbf{z}^{(i)}) = \mathrm{diag}[f(y_1^{*(i)}), f(y_2^{*(i)}), f(y_3^{*(i)}), f(y_4^{*(i)})] \quad (i = 1, 2) \tag{19}$$

$$\mathbf{q}_1 = [\mathbf{B}^{(1)} - \mathbf{B}^{(2)}\mathbf{A}^{(2)-1}\mathbf{A}^{(1)}]^{-1}\{[b_2\mathbf{d}^{(2)} + b_1\bar{\mathbf{d}}^{(1)} - \mathbf{d}^{(1)}] - \mathbf{B}^{(2)}\mathbf{A}^{(2)-1}[b_2\mathbf{c}^{(2)} + b_1\bar{\mathbf{c}}^{(2)} - \mathbf{c}^{(1)}]\}, \tag{20}$$

$$\mathbf{q}_2 = \frac{\theta_0}{2\pi}[\mathbf{B}^{(1)}\mathbf{A}^{(1)-1}\mathbf{A}^{(2)} - \mathbf{B}^{(2)}]^{-1} \times \{[b_2\mathbf{d}^{(2)} + b_1\bar{\mathbf{d}}^{(1)} - \mathbf{d}^{(1)}] - \mathbf{B}^{(1)}\mathbf{A}^{(1)-1}[b_2\mathbf{c}^{(2)} + [b_1\bar{\mathbf{c}}^{(1)} - \mathbf{c}^{(1)}]\} \tag{21}$$

together with

$$b_1 = \frac{k_2^{(2)} - k_2^{(1)}}{k_2^{(2)} + k_2^{(1)}},$$
$$b_2 = \frac{2k_2^{(1)}}{k_2^{(2)} + k_2^{(1)}}, \tag{22}$$

$$k_2^{(i)} = \sqrt{k_{11}^{(i)}k_{22}^{(i)} - (k_{12}^{(i)})^2}, \quad y_1^{(i)} = z_t^{(i)} - \hat{z}_t^{(i)}$$
$$y_2^{(i)} = z_t^{(i)} - \hat{\bar{z}}_t^{(i)}, \quad f(y) = y(\ln y - 1),$$

$$y_k^{*(i)} = z_k^{(i)} - \hat{z}_k^{(i)}, \quad \hat{z}_k^{(i)} = \hat{x}_1 + p_k^{(i)}\hat{x}_2 \quad (k = 1, 2, 3, 4, i = 1, 2), \tag{23}$$

where the superscripts (1) and (2) label the quantities relating to the materials 1 and 2. Similarly, Green's function due to a discrete temperature discontinuity applied at a point located in material 2 may be obtained.

In what follows, formulation will be derived for an infinite thermopiezoelectric plate of bimaterials with N arbitrary located cracks $2c_i (i = 1, 2, \ldots, N)$ in the plane (x_1, x_2) and subjected to remote heat flow h_0. The configuration of the crack system is shown in Fig. 1.

Assume that all cracks are located in material 1 and material 2 has no crack. The assumption is only for simplifying the ensuing writing and the extension to the case of cracks in the whole plane is straightforward. The central point of ith crack is denoted as (x_{1i}, x_{2i}) and the orientation angle is denoted as α_i (see Fig. 1). The cracks are initially assumed to remain open and hence be free of tractions and charges, and to prevent the transfer of heat between their faces. The corresponding boundaries are, then, as follows

On the faces of each crack i

$$\mathbf{t}_{ni}^{(1)} = -\Pi_1^{(1)} \sin\alpha_i + \Pi_2^{(1)} \cos\alpha_i = 0,$$
$$h_{ni}^{(1)} = -h_1^{(1)} \sin\alpha_i + h_2^{(1)} \cos\alpha_i = 0 \quad (i = 1, 2, \ldots, N). \tag{24}$$

At infinity

$$h_2^\infty = h_0, \quad \mathbf{\Pi}_1^\infty = \mathbf{\Pi}_2^\infty = h_1^\infty = 0, \tag{25}$$

where n stands for the normal direction to the lower face of a crack, $\mathbf{t}_{ni}$ is the surface traction and charge vector acting on the ith crack, $\mathbf{\Pi}_i = \{\sigma_{i1}\ \sigma_{i2}\ \sigma_{i3}\ D_i\}^{\mathrm{T}}$.

It is convenient to represent the solution as the sum of a uniform heat flux in an unflawed solid (which involves no thermal stress) and a corrective solution in which the boundary conditions are

On the faces of each crack i:

$$\mathbf{t}_{ni}^{(1)} = -\mathbf{\Pi}_1^{(1)} \sin \alpha_i + \mathbf{\Pi}_2^{(1)} \cos \alpha_i = 0, \tag{26a}$$

$$h_{ni}^{(1)} = -h_1^{(1)} \sin \alpha_i + h_2^{(1)} \cos \alpha_i = -h_0 \cos \alpha_i \quad (i = 1, 2, \ldots, N). \tag{26b}$$

At infinity

$$\mathbf{\Pi}_1^\infty = \mathbf{\Pi}_2^\infty = h_1^\infty = h_2^\infty = 0 \tag{27}$$

3. Singular integral equations for many cracks in bimaterials

Using the principle of superposition [2], the present problem shown in Fig. 1 is decomposed into N subproblems, each of which contains one single crack. The boundary conditions given by Eq. (26b), can be satisfied by redefining the discrete Green's functions θ_0 in Eq. (12) in terms of distributing Green's functions $\theta_0(\xi)$ defined along a given crack line, such as crack $i, z_{ti}^{(1)} = z_{ti}^{(0)} + \eta_i z_{ti}^*$, $\hat{z}_{ti}^{(1)} = z_{ti}^0 + \xi z_{ti}^*$, where $z_{ti}^0 = x_{1i} + \tau^{(1)} x_{2i}$, $z_{ti}^* = \cos \alpha_i + \tau^{(1)} \sin \alpha_i$, and α_i are shown in Fig. 1. Enforcing the satisfaction of the applied heat flux conditions on each crack face, a system of singular integral equations for Green's function is obtained as

$$\frac{1}{\pi}\mathrm{Re}\left[\int_{-c_i}^{c_i} \frac{\theta_{0i}(\xi_i)}{\eta_i - \xi_i} \mathrm{d}\xi_i + \int_{-c_j}^{c_j} \theta_{0j}(\xi_j) \left\{ \sum_{j=1, j\neq i}^{N} K_{0ij}(\eta_i, \xi_j) + \sum_{j=1}^{N} G_{0ij}(\eta_i, \xi_j) \right\} \mathrm{d}\xi_j = \frac{2h_0 \cos \alpha_i}{k_2^{(1)}} \right. \quad (i = 1, 2, \ldots, N) \tag{28}$$

in which the kernel function K_{0ij} and G_{0ij} are two regular known functions and given by

$$K_{0ij}(\eta_i, \xi_j) = \frac{z_{ti}^*}{z_{ti}^0 + \eta_i z_{ti}^* - z_{tj}^0 - \xi_j z_{tj}^*},$$
$$G_{0ij}(\eta_i, \xi_j) = \frac{b_{1j} z_{ti}^*}{z_{ti}^0 + \eta_i z_{ti}^* - \bar{z}_{tj}^0 - \xi_j \bar{z}_{tj}^*}, \tag{29}$$

where "Re" stands for the real part of a complex number.

In addition to Eq. (28), the single valuedness of the temperature around a closed contour surrounding the whole crack requires that

$$\int_{-c_i}^{c_i} \theta_{0i}(s)\, \mathrm{d}s = 0 \quad (i = 1, 2, \ldots, N). \tag{30}$$

For convenience, normalize the interval $(-c_i, c_i)$ by the change of variables;

$$\eta_i = c_i s_{0i}, \quad \xi_i = c_i s_i \quad (i = 1, 2, \ldots, N). \tag{31}$$

If we retain the same symbols for the new functions caused by the change of the variables, Eqs. (28) and (30) can be rewritten as

$$\frac{1}{\pi}\mathrm{Re}\left[\int_{-1}^{1} \frac{\theta_{0i}(s_i)}{s_{0i} - s_i} \mathrm{d}s_i + \int_{-1}^{1} \left\{ \sum_{j=1, j\neq i}^{N} K_{0ij}(s_{0i}, s_j) + \sum_{j=1}^{N} G_{0ij}(s_{0i}, s_j) \right\} \theta_{0j}(s_j)\, \mathrm{d}s_j \right] = \frac{2h_0 \cos \alpha_i}{k_2^{(1)}} \quad (i = 1, 2, \ldots, N), \tag{32}$$

$$\int_{-1}^{1} \theta_{0i}(s_i)\, \mathrm{d}s_i = 0 \quad (i = 1, 2, \ldots, N). \tag{33}$$

The coupled singular integral equations for the temperature dislocation density in Eq. (32) combined with Eq. (33) can be solved numerically [13]. Since the solution for the functions, $\theta_{0i}(\xi)$, have a square root singular at each crack tip, it is more efficient for the numerical calculations by letting

$$\theta_{0i}(\xi) = \frac{\Theta_i(\xi)}{\sqrt{c_i^2 - \xi^2}} = \frac{\sum_{j=1}^{m} B_{ij} T_j(t)}{c_i \sqrt{1 - t^2}}, \tag{34}$$

where $\Theta_i(t)$ is a regular function defined in the interval $|t| \leqslant 1$, B_{ij} are the real unknown coefficients, and $T_j(t)$ the Chebyshev polynomials of first kind. Thus the discretized form of Eqs. (32) and (33) may be written as [13]

$$\mathrm{Re}\sum_{k=1}^{m}\frac{1}{n}\left[\frac{\Theta_i(s_{ik})}{c_i(s_{0ir}-s_{ik})}+\left\{\sum_{j=1,j\neq i}^{N}K_{0ij}(s_{0ir},s_{jk}) +\sum_{l=1}^{N}G_{0ij}(s_{0ir},s_{jk})\right\}\frac{\Theta_j(s_{jk})}{c_j}\right]=\frac{2h_0\cos\alpha}{k_2^{(1)}},$$

$$\sum_{k=1}^{m}\Theta_i(s_{ik})=0. \tag{35}$$

where

$$s_{ik}=s_{jk}=\cos\left[\frac{(2k-1)\pi}{2m}\right]\quad(k=1,2,\ldots,m),$$
$$s_{0ir}=\cos(r\pi/m)\quad(r=1,2,\ldots,m-1). \tag{36}$$

Eq. (35) provides a system of $N\times m$ linear algebraic equations to determine the coefficients B_{ij}. Once the function $\Theta_i(t)$ has been found, the corresponding stress–electric displacements can be given from Eqs. (8b) and (14) as

$$\begin{aligned}\mathbf{\Pi}_1^{(1)}&=-\boldsymbol{\phi}_{,2}^{(1)}\\&=-\frac{1}{2\pi}\sum_{i=1}^{N}\mathrm{Im}\int_{-c_i}^{c_i}\Big[\mathbf{B}^{(1)}\mathbf{P}^{(1)}\langle\ln\mathbf{z}_i\rangle\mathbf{q}_{1i}\\&\qquad+\mathbf{d}^{(1)}\tau^{(1)}(\ln y_{1i}^{(1)}+b_{1i}\ln y_{2i}^{(1)}\Big]\theta_{0i}(\xi_i)\,\mathrm{d}\xi_i,\\\mathbf{\Pi}_2^{(1)}&=\boldsymbol{\phi}_{,1}^{(1)}=\frac{1}{2\pi}\sum_{i=1}^{N}\mathrm{Im}\int_{-c_i}^{c_i}\Big[\mathbf{B}^{(1)}\langle\ln\mathbf{z}_i\rangle\mathbf{q}_{1i}\\&\qquad+\mathbf{d}^{(1)}(\ln y_{1i}^{(1)}+b_{1i}\ln y_{2i}^{(1)})\Big]\theta_{0i}(\xi_i)\,\mathrm{d}\xi_i,\end{aligned} \tag{37}$$

where

$$\langle\ln\mathbf{z}_i\rangle=\mathrm{diag}\Big[\ln y_{1i}^{*(1)}\ \ln y_{2i}^{*(1)}\ \ln y_{3i}^{*(1)}\ \ln y_{4i}^{*(1)}\Big]. \tag{38}$$

Thus the traction-charge vector on the ith crack faces is of the form

$$\begin{aligned}\mathbf{t}_{ni}^0(\eta_i)&=-\mathbf{\Pi}_{1i}^{(1)}(\eta_i)\sin\alpha_i+\mathbf{\Pi}_{2i}^{(1)}(\eta_i)\cos\alpha_i\\&=\frac{1}{2\pi}\sum_{j=1}^{N}\mathrm{Im}\int_{-c_j}^{c_j}\Big[\mathbf{B}^{(1)}(\mathbf{I}\cos\alpha_i+\mathbf{P}^{(1)}\sin\alpha_i)\langle\ln\mathbf{z}_j\rangle\mathbf{q}_{1j}\\&\qquad+\mathbf{d}^{(1)}(\cos\alpha_i+\tau^{(1)}\sin\alpha_i)(\ln y_{1j}^{(1)}+b_{1j}y_{2j}^{(1)})\Big]\\&\qquad\times\theta_{0j}(\xi_j)\,\mathrm{d}\xi_j.\end{aligned} \tag{39}$$

Generally $\mathbf{t}_{ni}^0(\eta_i)\neq 0$ on the ith crack faces $|\eta_i|\leqslant c_i$. To satisfy the traction-charge free condition (26a), superpose a solution of the corresponding isothermal problem with a traction-charge vector equal and opposite to that of Eq. (39) in the range $|\eta_i|\leqslant c_i$. The elastic solution for a singular dislocation of strength $\mathbf{b}_0$ obtained in [14] is adopted. This solution can be straightforwardly extended to the case of electroelastic problem as

$$\begin{aligned}\mathbf{\Pi}_1^{(1)}=&-\frac{1}{\pi}\sum_{i=1}^{N}\mathrm{Im}\Big[\mathbf{B}^{(1)}\mathbf{P}^{(1)}\Big\langle(z_j^{(1)}-\hat{z}_{ji})^{-1}\Big\rangle\mathbf{B}^{(1)\mathrm{T}}\Big]\mathbf{b}_{0i}\\&-\frac{1}{\pi}\sum_{i=1}^{N}\sum_{\beta=1}^{4}\mathrm{Im}\Big[\mathbf{B}^{(1)}\mathbf{P}^{(1)}\Big\langle(z_j^{(1)}-\hat{\bar{z}}_{\beta i})^{-1}\Big\rangle\mathbf{B}^{*}\mathbf{I}_\beta\bar{\mathbf{B}}^{(1)\mathrm{T}}\mathbf{b}_{0i},\end{aligned} \tag{40}$$

$$\begin{aligned}\mathbf{\Pi}_2^{(1)}=&\frac{1}{\pi}\sum_{i=1}^{N}\mathrm{Im}\Big[\mathbf{B}^{(1)}\Big\langle(z_j^{(1)}-\hat{\bar{z}}_{ji})^{-1}\Big\rangle\mathbf{B}^{(1)\mathrm{T}}\Big]\mathbf{b}_{0i}\\&+\frac{1}{\pi}\sum_{i=1}^{N}\sum_{\beta=1}^{4}\mathrm{Im}\Big[\mathbf{B}^{(1)}\Big\langle(z_j^{(1)}-\hat{\bar{z}}_{\beta i})^{-1}\Big\rangle\mathbf{B}^{*}\mathbf{I}_\beta\bar{\mathbf{B}}^{(1)\mathrm{T}}\mathbf{b}_{0i},\end{aligned} \tag{41}$$

where $\mathbf{I}_\beta=\mathrm{diag}[\delta_{1\beta}\ \delta_{2\beta}\ \delta_{3\beta}\ \delta_{4\beta}]$, $\delta_{ij}=1$ for $i=j$; $\delta_{ij}=0$ for $i\neq j$, and

$$\hat{\bar{z}}_{\beta i}=x_{1i}+\bar{p}_\beta^{(1)}x_{2i}+\xi_i(\cos\alpha_i+\bar{p}_\beta^{(1)}\sin\alpha_i),$$
$$\langle(\ )\rangle=\mathrm{diag}[(\ )_1\ (\ )_2\ (\ )_3\ (\ )_4],$$
$$\mathbf{B}^{*}=\mathbf{B}^{(1)-1}[\mathbf{I}-2(\mathbf{M}_1^{-1}+\mathbf{M}_2^{-1})^{-1}\mathbf{L}^{-1}]$$

with

$$\mathbf{M}_j=-i\mathbf{B}^{(j)}\mathbf{A}^{(j)-1}\quad(j=1,2),$$
$$\mathbf{L}=-2i\mathbf{B}^{(1)}\mathbf{B}^{(1)\mathrm{T}}.$$

Therefore the boundary condition (26a) will be satisfied if

$$\frac{\mathbf{L}}{2\pi}\int_{-c_i}^{c_i}\frac{\mathbf{b}_{0i}(\xi_i)}{\eta_i-\xi_i}\,\mathrm{d}\xi_i+\int_{-c_j}^{c_j}\theta_{0j}(\xi_j)\left\{\sum_{j,j\neq i}^{N}\mathbf{K}_{0ij}(\eta_i,\xi_j)+\sum_{j=1}^{N}\mathbf{G}_{0ij}(\eta_i,\xi_j)\right\}\mathrm{d}\xi_j=-\mathbf{t}_{ni}^{0}(\eta_i)$$
$$(i=1,2,\ldots,N), \tag{42}$$

where

$$\mathbf{K}_{0ij}(\eta_i,\xi_j)=\frac{1}{\pi}\mathbf{B}^{(1)}\left\langle z_{ki}^{*}(z_{ki}^{0}+\eta_i z_{ki}^{*}-z_{kj}^{0}-\xi_j z_{kj}^{*})^{-1}\right\rangle\mathbf{B}^{(1)\mathrm{T}},$$
$$\mathbf{G}_{0ij}(\eta_i,\xi_j)=\frac{1}{\pi}\sum_{\beta=1}^{4}\mathbf{B}^{(1)}\left\langle z_{ki}^{*}(z_{ki}^{0}+\eta_i z_{ki}^{*}-\bar{z}_{\beta j}^{0}+\xi_j\bar{z}_{\beta j}^{*})^{-1}\right\rangle\mathbf{B}^{*}\mathbf{I}_{\beta}\bar{\mathbf{B}}^{(1)\mathrm{T}} \tag{43}$$

with

$$z_{ki}^{*}=\cos\alpha_i+p_k^{(1)}\sin\alpha_i,\quad z_{ki}^{0}=x_{1i}+p_k^{(1)}x_{2i}.$$

For single valued displacements and electric potential around a closed contour surrounding the whole ith crack, the following conditions have also to be satisfied:

$$\int_{-c_i}^{c_i}\mathbf{b}_{0i}(\xi_i)\,\mathrm{d}\xi_i=0. \tag{44}$$

As was done previously, let $\eta_i=c_i s_{0i}$, $\xi_i=c_i s_i$, and

$$\mathbf{b}_{0i}(\xi)=\frac{\mathbf{\Theta}_i(\xi)}{\sqrt{c_i^2-\xi^2}}\approx\frac{\sum_{k=1}^{n}\mathbf{E}_{ik}T_k(t)}{c_i\sqrt{1-t^2}}, \tag{45}$$

where $\mathbf{E}_{ik}=\{E_{ik1},\ E_{ik2},E_{ik3},\ E_{ik4}\}^{\mathrm{T}}$. Thus, from Eqs. (42) and (44), we obtain

$$\sum_{k=1}^{m}\frac{1}{n}\left[\frac{\mathbf{L}\mathbf{\Theta}_i(s_{ik})}{2c_i(s_{0ir}-s_{ik})}+\left\{\sum_{j=1,j\neq i}^{N}\mathbf{K}_{0ij}(s_{0ir},s_{jk})+\sum_{j=1}^{N}\mathbf{G}_{0ij}(s_{0ir},s_{jk})\right\}\frac{\mathbf{\Theta}_j(s_{jk})}{c_j}\right]=-\mathbf{t}_{ni}^{0}(s_{0ir}), \tag{46}$$
$$\sum_{k=1}^{m}\mathbf{\Theta}_i(s_{ik})=0.$$

Eq. (46) provide a system of $4N\times m$ linear algebraic equations to determine $\mathbf{\Theta}(ct_k)$ and then $\mathbf{E}_k$. Once the function $\mathbf{\Theta}_i(s_{ik})$ has been found from Eq. (46), the stresses and electric displacements, $\mathbf{\Pi}_{ni}^{(1)}(\eta_i)$, in a coordinate local to the crack line can be expressed in the form

$$\mathbf{\Pi}_{ni}^{(1)}(\eta_i)=\mathbf{\Omega}(\alpha_i)\left[\frac{\mathbf{L}}{2\pi}\int_{-c_i}^{c_i}\frac{\mathbf{b}_{0i}(\xi_i)}{\eta_i-\xi_i}\,\mathrm{d}\xi_i+\int_{-c_j}^{c_j}\mathbf{b}_{0j}(\xi_j)\left\{\sum_{j=1,j\neq i}^{N}\mathbf{K}_{0ij}(\eta_i,\xi_j)+\sum_{j=1}^{N}\mathbf{G}_{0ij}(\eta_i,\xi_j)\right\}\mathrm{d}\xi_j+\mathbf{t}_{ni}^{0}(\eta_i)\right]$$
$$(i=1,2,\ldots,N), \tag{47}$$

where the 4×4 matrix $\Omega(\alpha_i)$ whose components are the cosine of the angle between the local coordinates and the global coordinates is in the form

$$\Omega(\alpha_i)=\begin{bmatrix}\cos\alpha_i & \sin\alpha_i & 0 & 0\\ -\sin\alpha_i & \cos\alpha_i & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}. \tag{48}$$

The formulae for the stress and electric displacement intensity factors, $\mathbf{K}^{*}=(K_{\mathrm{II}},K_{\mathrm{I}},K_{\mathrm{III}},K_{\mathrm{D}}\}^{\mathrm{T}}$, at a given crack tip, e.g., at the right tip of ith crack ($\eta_i=c_i$) can be evaluated by first considering the traction and surface charge on the crack line and very near the crack tip ($\eta_i\rightarrow c_i$) which is given, from Eq. (47)

$$\mathbf{\Pi}_{ni}^{(1)}(\eta_i)=\mathbf{\Omega}(\alpha_i)\mathbf{t}_{in}(\eta_i)\approx\mathbf{\Omega}(\alpha_i)\mathbf{L}\frac{\Theta_i(c_i)}{\sqrt{8c_i(\eta_i-c_i)}}. \tag{49}$$

Using Eq. (49), the stress–displacement intensity factors may be evaluated by the following definition

$$\mathbf{K}(c_i)=\{K_{\mathrm{II}}K_{\mathrm{I}}K_{\mathrm{III}}K_{\mathrm{D}}\}^{\mathrm{T}}(c_i)=\lim_{\eta_i\rightarrow c_i}\sqrt{2\pi(\eta_i-c_i)}\mathbf{\Pi}_{ni}^{(1)}(\eta_i). \tag{50}$$

Combined with the results of Eq. (49), one then leads to

$$\mathbf{K}(c_i) \approx \sqrt{\frac{\pi}{4c_i}}\boldsymbol{\Omega}(\alpha_i)\mathbf{L}\boldsymbol{\Theta}_i(c_i). \tag{51}$$

Thus the solution of the singular integral equation enables the direct determination of the stress intensity factors.

4. Direction of crack initiation

The strain energy density criterion [15] will be used to predict the direction of crack initiation in the thermopiezoelectric bimaterials. To make the derivation tractable, the crack tip fields are first studied. In doing this, a polar coordinate system (r, ω) centered at a given crack tip, say the right tip of ith crack $(x_1, x_2) = (x_{1i} + c_i \cos \alpha_i, x_{2i} + c_i \sin \alpha_i)$ and $\omega = 0$ along the crack line is used and then the variable z_k becomes

$$z_k = z_{ki}^0 + c_i z_{ki}^* + r[\cos(\alpha_i + \omega) + p_k^{(1)} \sin(\alpha_i + \omega)]. \tag{52}$$

With this coordinate system, stress–electric displacement helds near the crack tip can be evaluated by taking the asymptotic limit of Eq. (47) and using expressions (40) and (41) as $r \to 0$

$$\begin{aligned}\boldsymbol{\Pi}_{1i}(r,\omega) &\approx \sqrt{\frac{1}{2c_i r}}\mathrm{Im}\left[\mathbf{B}^{(1)}\left\langle \frac{p_k^{(1)}}{\sqrt{z_k^*(\alpha_i)z_k^*(\alpha_i+\omega)}}\right\rangle\mathbf{B}^{(1)\mathrm{T}}\right]\boldsymbol{\Theta}_i(c_i)\\ &= \sqrt{\frac{1}{2c_i r}}\mathbf{V}_{1i}(\omega),\end{aligned} \tag{53}$$

$$\begin{aligned}\boldsymbol{\Pi}_{2i}(r,\omega) &\approx -\sqrt{\frac{1}{2c_i r}}\mathrm{Im}\left[\mathbf{B}^{(1)}\left\langle \frac{1}{\sqrt{z_k^*(\alpha_i)z_k^*(\alpha_i+\omega)}}\right\rangle\mathbf{B}^{(1)\mathrm{T}}\right]\boldsymbol{\Theta}_i(c_i)\\ &= \sqrt{\frac{1}{2c_i r}}\mathbf{V}_{2i}(\omega),\end{aligned} \tag{54}$$

where $z_k^*(x) = \cos x + p_k^{(1)} \sin x$.

For a thermopiezoelectric material, the strain energy density factor $S(\omega)$ can be calculated by considering the related thermoelectroelastic potential energy W. The relationship between the two functions is as follows:

$$S(\omega) = rW = r\{\Pi_1\Pi_2\}^{\mathrm{T}}\mathbf{F}\{\Pi_1\Pi_2\}/2, \tag{55}$$

where the matrix $\mathbf{F}$ is the inverse of stiffness matrix $\mathbf{E}$. The substitution of Eqs. (53) and (54) into Eq. (55), leads to

$$S_i(\omega) = \{\mathbf{V}_{1i}\mathbf{V}_{2i}\}^{\mathrm{T}}\mathbf{F}\{\mathbf{V}_{1i}\mathbf{V}_{2i}\}/4c_i. \tag{56}$$

Referring to the works in [16] the strain energy density criterion states that the direction of crack initiation coincides with the direction of the strain energy density factor $S_{\min}$, i.e., a necessary and sufficient condition of crack growth in the angle ω_0 is that

$$\left.\frac{\partial S}{\partial \omega}\right|_{\omega=\omega_0} = 0, \tag{57a}$$

and

$$\left.\frac{\partial^2 S}{\partial \omega^2}\right|_{\omega=\omega_0} > 0. \tag{57b}$$

Substituting Eq. (56) into Eq. (57a), yields

$$\{\mathbf{V}_{1i}\mathbf{V}_{2i}\}^{\mathrm{T}}\mathbf{F}\partial\{\mathbf{V}_{1i}\mathbf{V}_{2i}\}^{\mathrm{T}}/\partial\omega = 0. \tag{58}$$

Solving Eq. (58), several roots of ω may be obtained. The angle ω_0 will be the one satisfying Eq. (57b).

5. Numerical example

As numerical illustration of the proposed formulation, we consider a thermopiezoelectric bimaterial plate with two cracks of the same length $2c$

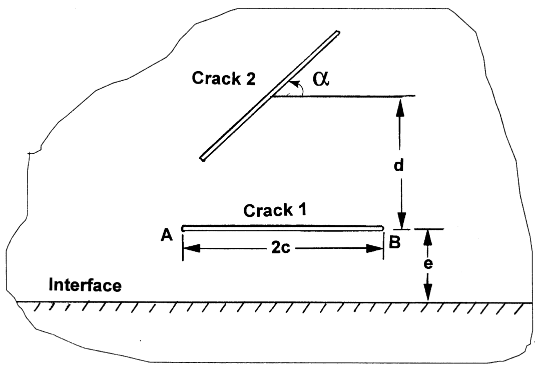

Fig. 2. Geometry of bimaterial with two cracks for $d/c = 2$ and $e = c$.

and its interface coinsided with x_1-axis shown in Fig. 2. The configuration of the crack system is given in Fig. 2. The upper and lower materials are assumed to be $BaTiO_3$ [17] and cadmium selenide [18], respectively. The material constants for the two mateials are as follows:(1) Material properties for $BaTiO_3$ [17]

$$c_{11} = 150\ \text{GPa}, \quad c_{12} = 66\ \text{GPa}, \quad c_{13} = 66\ \text{GPa},$$
$$c_{33} = 146\ \text{GPa}, \quad c_{44} = 44\ \text{GPa},$$
$$\alpha_{11} = 8.53 \times 10^{-6}\ \text{K}^{-1}, \quad \alpha_{33} = 1.99 \times 10^{-6}\ \text{K}^{-1},$$
$$\lambda_3 = 0.133 \times 10^5\ \text{N/CK},$$
$$e_{31} = -4.35\ \text{C/m}^2, \quad e_{33} = 17.5\ \text{C/m}^2,$$
$$e_{15} = 11.4\ \text{C/m}^2, \quad \kappa_{11} = 1115\kappa_0,$$
$$\kappa_{33} = 1260\kappa_0, \quad \kappa_0 = 8.85 \times 10^{-12}\ \text{C}^2/\text{Nm}^2. \tag{59}$$

(2) Material properties for Cadmium Selenide [18]

$$c_{11} = 74.1\ \text{GPa}, \quad c_{12} = 45.2\ \text{GPa},$$
$$c_{13} = 39.3\ \text{GPa}, \quad c_{33} = 83.6\ \text{GPa},$$
$$c_{44} = 13.2\ \text{GPa},$$
$$\gamma_{11} = 0.621 \times 10^6\ \text{N/Km}^2,$$
$$\gamma_{33} = 0.551 \times 10^6\ \text{N/Km}^2, \quad \chi_3 = -0.294\ \text{C/Km}^2,$$
$$e_{31} = -0.160\ \text{C/m}^2, \quad e_{33} = 0.347\ \text{C/m}^2,$$
$$e_{15} = 0.138\ \text{C/m}^2, \kappa_{11} = 82.6 \times 10^{-12}\text{C}^2/\text{Nm}^2,$$
$$\kappa_{33} = 90.3 \times 10^{-12}\ \text{C}^2/\text{Nm}^2. \tag{60}$$

Since the values of the coefficient of heat conduction both for $BaTiO_3$ and Cadmium Selenide could not be found in the literature, the value $k_{33}^{(1)}/k_{11}^{(1)} = 1.5, k_{33}^{(2)}/k_{11}^{(2)} = 2$ and $k_{13}^{(1)} = k_{13}^{(2)} = 0$ are assumed.

In our analysis, the plane strain deformation is considered and the crack line is assumed to be in the x_1–x_3 plane, i.e., $D_2 = u_2 = 0$. Therefore the stress intensity factor vector $\mathbf{K}^*$ has now only three components $(K_\text{I}, K_\text{II}, K_\text{D})$. Fig. 3 shows the numerical results for the coefficients of stress–electric intensity factors β_i versus the crack orientation α, where β_i are defined by

$$K_\text{I}(B) = h_0 c\sqrt{\pi c}\gamma_{11}^{(1)}\beta_1(\alpha)/k_2^{(1)},$$
$$K_\text{III}(B) = h_0 c\sqrt{\pi c}\gamma_{33}^{(1)}\beta_2(\alpha)/k_2^{(1)},$$
$$K_\text{D}(B) = h_0 c\sqrt{\pi c}\chi_3^{(1)}\beta_\text{D}(\alpha)/k_2^{(1)}. \tag{61}$$

It can be seen from Fig. 3 that the coefficient β_1 will decrease slowly along with the increase of the crack orientation α, while β_D will increase weakly

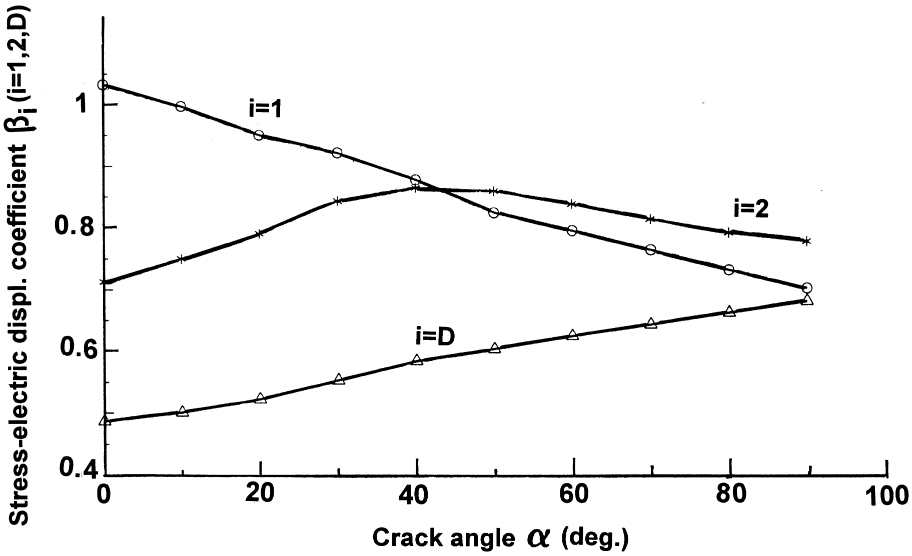

Fig. 3. Stress–electric displacement intensity factors versus crack angle.

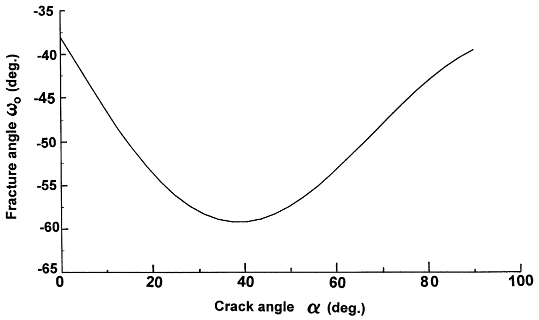

Fig. 4. Fracture angle versus crack angle.

along with the increase of the crack orientation α. Unlike the previous two parameters, β_2 reaches its peak value at about $\alpha = 43°$ and is insensitive to changes of the orientation α. Fig. 4 shows the variation of fracture angle ω_0 with the crack orientation α. It is found from the figure that the fracture angle ω_0 varies between $-59°$ and $-38°$ and reach its peak value at about $\alpha = 38°$.

6. Conclusion

This investigation presented a formulation for the problem of a thermopiezoelectric bimaterial plate with multiple cracks. A system of singular integral equations is developed with the aid of Green's function approach and the principle of superposition. Solutions for the thermal, electric and elastic fields are then obtained for multiple cracks in an infinite themopiezoelectric bimaterial plate subjected to far heat flux disturbances. For an infinite plate with two cracks (one is in horizontal direction and another is in arbitrary orientation) in a bimaterial plate, the numerical results show that the crack orientation α has a weak effect on the values of β_2. While both β_1 and β_D varies quasi-linearly with the crack orientation α. The numerical results also show that the fracture angle ω_0 varies between $-59°$ and $-38°$ and reaches its peak value at about $\alpha = 38°$.

Acknowledgements

The work was performed with the support of Australian Research Council Foundation. This support is gratefully acknowledged.

References

[1] Y.Z. Chen, General case of multiple crack problems in an infinite plate, Engrg. Frac. Mech. 20 (1984) 591–597.
[2] H. Horri, S. Nemat-Nasser, Elastic fields of interacting in homogeneities, Int. J. Solids Struct. 21 (1985) 731–745.
[3] H. Horri, S. Nemat-Nasser, Interacting micro-cracks near the tip in the process zone of a macro-crack, J. Mech. Phys. Solids 35 (1987) 601–629.
[4] M. Kachanov, Elastic solids with many cracks and related problems, Adv. Appl. Mech. 30 (1993) 259–428.
[5] C. Hwu, Collinear cracks in anisotropic bodies, Int. J. Frac. 52 (1991) 239–256.
[6] Y.H. Chen, N. Hasebe, Interaction between a main-crack and a parallel micro-crack in an orthotropic plane elastic solid, Int. J. Solids Struct. 31 (1994) 1877–1890.
[7] H. Ma, L.G. Zhao, Y.H. Chen, Crack initiation angles for systems of two interacting cracks in media with rectlinear preferred directions, Theoret. Appl. Frac. Mech. 24 (1996) 253–261.
[8] Q.H. Qin, Y.W. Mai, Thermoelectroelastic Green's function and application for bimaterial of piezoelectric materials, Arch. Appl. Mech., to appear.
[9] Q.H. Qin, Y.W. Mai, Crack growth prediction of an inclined crack in a half-plane thermopiezoelectric solid, Theoret. Appl. Frac. Mech. 26 (1997) 185–191.
[10] D.M. Barnett, J. Lothe, Dislocations and line charges in anisotropic piezoelectric insulators, Phys. Stat. Sol.(b) 67 (1975) 105–111.
[11] R.D. Mindlin, Equations of high frequency vibrations of thermopiezoelectric crystal plates, Int. J. Solids Struct. 10 (1974) 625–637.
[12] S.W. Yu, Q.H. Qin, Damage analysis of thermopiezoelectric properties: Part I-crack tip singularities, Theore. Appl. Frac. Mech. 25 (1996) 263–277.
[13] F. Erdogan, G.D. Gupta, On the numerical solution of singular integral equations, Q. Appl. Math. 32 (1972) 525–534.
[14] T.C.T. Ting, Image singularities of Green's functions for anisotropic elastic half-space and bimaterials, Q. J. Mech. Math. 45 (1992) 119–139.
[15] G.C. Sih, A special theory of crack propagation, Methods of analysis and solutions of crack problems, in: G.C. Sih (Ed.), Mechanics of Fracture I, Noordhoff, Leiden, 1973, pp. 21–45.

[16] G.C. Sih, Prediction of crack growth under mixed mode conditions, in: G.C. Sih, P.S. Theocaris (Eds.), USA–Greece Symposium on Mixed Mode Crack Propagation, Sijthoff & Noordhoff, Alphen a/d Rijn, (1981), pp. 3–19.

[17] M.L. Dunn, Micromechanics of coupled electroelastic compisites: effective thermal expanison and pyroelectric coefficients, J. Appl. Phys. 73 (1993) 5131–5140.

[18] F. Ashida, Th.R. Tauchert, N. Noda, A general solution technique for piezothermoelasticity of Hexogonal solids of class 6 mm in Cartesian coordinates, Z. Angew. Math Mech. 74 (1994) 87–95.