Postbuckling analysis of thin plates on an elastic foundation by HT FE approach

Qing Hua Qin

Department of Engineering Mechanics, Tsinghua University, Beijing, People's Republic of China

The paper presents a hybrid-Trefftz (HT) element approach for the numerical solution of postbuckling analysis of thin plates on an elastic foundation. The approach is based on a modified variational principle and Trefftz functions. Exact solutions of the Lame-Navier equations are used for the in-plane intraelement displacement field, and an increment form of the basic equations is adopted. With the aid of the incremental form of these equations, all nonlinear terms may be taken as pseudo-loads. Moreover, some modifications have been made on the nonlinear boundary equations to simplify the ensuing derivation. As a result the in-plane and out-of-plane equations are uncoupled, and then the derivation for the HT finite element (FE) formulation becomes quite simple. The practical efficiency of the new element model has been assessed through two examples. © 1997 by Elsevier Science Inc.

Keywords: postbuckling analysis, plate, elastic foundation, Trefftz finite element

1. Introduction

Unlike most known finite element (FE) models, the hybrid-Trefftz (HT) FE model was initially presented in the plate-bending context in 1977 and is now very popular as a numerical method in computational mechanics. In fact the HT FE model has many advantages: high accuracy, fast p-convergence rate, enhanced insensitivity to mesh distortion, great liberty in element geometry, the possibility of accurately representing, without troublesome mesh adjustment, various local effects due to loading and/or geometry,... etc. Up to now the method has been widely used in plate elasticity, plate-bending problems, shells, axisymmetric solid mechanics, Poisson's equation, and other applications. As far as we know, however, there are very few results by the HT FE approach for nonlinear problems.

The purpose of this paper is to develop a simple HT FE model for postbuckling analysis of thin plates on an elastic foundation. By way of an incremental form of the basic equations and some modifications on nonlinear boundary relationships the in-plane and out-of-plane equations are uncoupled, and then the derivation for the HT FE formulation becomes very simple. Moreover an iteration scheme is suggested to evaluate the nonlinear terms. As illustrations, two numerical examples are considered and a comparison is made with existing results.

2. Governing equations and their Trefftz functions

2.1 Basic equations

Consider a thin isotropic plate of uniform thickness \( h \), occupying a two-dimensional arbitrarily shaped region \( \Omega \) bounded by its boundary \( \partial \Omega \) and resting on an elastic foundation. The plate is subjected to an in-plane uniform compressive load, \( P_0 \) per unit length at the boundary \( \partial \Omega \). Throughout this paper repeated indices \( i, j, \) and \( k \) take values in the range \( 1, 2 \). The postbuckling behavior of the plate is governed by the following equations:

\[
L_1 U_1 + L_2 U_2 = P_1 \\
L_2 U_1 + L_3 U_2 = P_2 \\
L_4 W = P_3
\]

with

\[
L_1( ) = ( ),_{xx} + d_4( ),_{xy}, \quad L_2( ) = d_5( ),_{xy}, \\
L_3( ) = ( ),_{yy} + d_4( ),_{xx}, \quad L_4( ) = D(\nabla^4 + \lambda^2 \nabla^2 + S)( ), \\
\lambda^2 = p_0/D, \quad d_1 = (1 - \mu)/2, \quad d_2 = (1 + \mu)/2, \quad S = k_0/D
\]
Postbuckling analysis of thin plates: Q. H. Qin

where \( U_1 \) and \( U_2 \) represent the disturbed in-plane displacement components, \( D = E h^3 / 12(1 - \mu^2) \), \( E \) is Young's modulus, and \( \mu \) is Poisson's ratio, \( k_0 \) is the foundation stiffness coefficient, \( \nabla^4 = \nabla^2 \nabla^2 \), \( \nabla^2 \) is \( ( \cdot )_{xx} + ( \cdot )_{yy} \), \( W \) is the lateral deflection of the plate, a comma followed by a subscript indicates partial differentiation with respect to that subscript, and \( P_1, P_2, \) and \( P_3 \) are components of a pseudo-distributed load defined by:

\[
\begin{align*}
P_1 &= -W_{,x}(W_{,xx} + d_1 W_{,yy}) - d_2 W_{,x}W_{,xy} \\
P_2 &= -W_{,y}(W_{,yy} + d_2 W_{,xx}) - d_2 W_{,x}W_{,xy} \\
P_3 &= J\left[U_{1,xx} + 0.5W_{,xx}^2 + \mu W_{,yy}ight] \\
&\quad + \left[U_{2,yy} + 0.5W_{,yy}^2 + \mu W_{,xx}\right] \\
&\quad + J(1 - \mu)(U_{1,xy} + U_{2,xy} + W_{,x}W_{,y})W_{,xy}
\end{align*}
\]

in which \( J = Eh^2 / (1 - \mu^2) \), and the boundary conditions are

\[
\begin{align*}
U_n &= U_1 n_1 + \overline{U}_n \text{ (on } C_{U_1}) \quad U_s = U_1 s_1 + \overline{U}_s \text{ (on } C_{U_1}) \\
W &= \overline{W} \text{ (on } C_W) \quad W_{,n} = W_1 n_1 + \overline{W}_{,n} \text{ (on } C_{W_1}) \\
N_n &= N_{1,n} n_1 + N_{2,n} n_2 \text{ (on } C_{N_1}) \\
N_s &= N_{1,s} n_1 + N_{2,s} n_2 \text{ (on } C_{N_2}) \\
R &= R_1 + R_2 - \overline{R} \text{ (on } C_R) \\
M_n &= M_{1,n} n_1 + M_{2,n} n_2 \text{ (on } C_{M_1})
\end{align*}
\]

with

\[
\begin{align*}
N_{11} &= N_{11} n_1 n_1 + N_{12} n_1 n_2 \quad N_{22} &= N_{22} n_2 n_2 \\
N_{11}^s &= N_{11}^s n_1 s_1 + N_{12}^s n_2 s_1 \quad N_{22}^s &= N_{22}^s n_2 s_2 \\
R_{11} &= M_{11} n_1 + M_{12} n_2 s_1 \quad R_{22} &= N_{11} W_{,n} + N_{22} W_{,x} \\
(\partial \Omega) &= C = C_{U_1} \cup C_{N_1} = C_{U_2} \cup C_{N_2} \\
&\quad = C_W \cup C_R = C_{W_1} \cup C_{M_1}
\end{align*}
\]

where \( n_i \) and \( s_i \) are components of the outward normal and tangent to the boundary \( \partial \Omega \). Finally the constitutive relationships are given by

\[
\begin{align*}
N_{ij} &= N_{ij} + N_{ij}^s \\
N_{ij}^s &= \frac{Eh}{2(1 + \mu)} \left[ U_{i,j} + U_{j,i} + \frac{2\mu}{1 - \mu} U_{k,k} \delta_{ij} \right] \\
N_{ij}^{n} &= \frac{Eh}{2(1 + \mu)} \left[ W_{,i,j} + \frac{\mu}{1 - \mu} W_{,k} W_{,k} \delta_{ij} \right] \\
M_{ij} &= -D(2dW_{,ij} + \mu W_{,kk} \delta_{ij})
\end{align*}
\]

Noting that equations (1)–(5) are not, in general, suitable for HT FE analysis, an incremental form of the equations must therefore be adopted to linearize these nonlinear equations. Denoting the incremental variable by the superimposed dot and omitting those infinitesimal terms resulting from the product of incremental variables one obtains

\[
\begin{align*}
L_1 \ddot{U}_1 + L_2 \ddot{U}_2 &= \ddot{P}_1 \\
L_3 \ddot{U}_1 + L_4 \ddot{U}_2 &= \ddot{P}_2 \\
L_5 \ddot{W} &= \ddot{P}_3
\end{align*}
\]

where

\[
\begin{align*}
\ddot{P}_1 &= -W_{,xx}(W_{,xx} + d_1 W_{,yy}) - W_{,x}W_{,xx} + \dot{W}_{,x} \left[ W_{,xx} + d_1 W_{,yy} \right] \\
&\quad - d_2 W_{,x}W_{,xy} - d_2 \dot{W}_{,x}W_{,xy} \\
\ddot{P}_2 &= -W_{,yy}(W_{,yy} + d_2 W_{,xx}) - W_{,y}W_{,yy} + \dot{W}_{,y} \left[ W_{,yy} + d_2 W_{,xx} \right] \\
&\quad - d_2 W_{,y}W_{,xy} - d_2 \dot{W}_{,y}W_{,xy} \\
\ddot{P}_3 &= J\left(U_{1,xx} + \dot{W}_{,x}W_{,x} + \dot{W}_{,y}W_{,y} + \mu W_{,yy} \right) \\
&\quad + J(1 - \mu)(U_{1,xy} + U_{2,xy} + W_{,x}W_{,y})W_{,xy}
\end{align*}
\]

The related boundary conditions become

\[
\begin{align*}
\ddot{U}_n &= \ddot{U}_1 n_1 + \ddot{U}_2 n_2 \text{ (on } C_{U_1}) \\
\ddot{U}_s &= \ddot{U}_1 s_1 + \ddot{U}_2 s_2 \text{ (on } C_{U_1}) \\
\ddot{W} &= \ddot{W}_1 \text{ (on } C_W) \quad \ddot{W}_{,n} = \ddot{W}_1 n_1 + \ddot{W}_{,n} \text{ (on } C_{W_1}) \\
\ddot{N}_n &= \ddot{N}_{1,n} n_1 + \ddot{N}_{2,n} n_2 \text{ (on } C_{N_1}) \\
\ddot{N}_s &= \ddot{N}_{1,s} n_1 + \ddot{N}_{2,s} n_2 \text{ (on } C_{N_2}) \\
\ddot{R} &= \ddot{M}_{1,1} n_1 + \ddot{M}_{1,2} n_2 s_1 \text{ (on } C_{M_1}) \\
\ddot{M}_n &= \ddot{M}_{1,n} n_1 + \ddot{M}_{2,n} n_2 \text{ (on } C_{M_1})
\end{align*}
\]

It should be pointed out that \( \ddot{N}_n \ll \ddot{N}_{ij} \), \( \ddot{R} \ll \ddot{R}_i \) in practical problems. So we may move these nonlinear terms to the right-hand side of the above boundary equations and take them as pseudo-loads. In this way the in-plane and out-of-plane boundary conditions are uncoupled. As a
result the ensuing derivation becomes quite simple, but an iterative approach is required to evaluate the nonlinear terms $\dot{P}_i$, $N_i^a$, $N_i^b$, and $R^a$.

2.2 Trefftz functions

The Trefftz functions play an important role in the derivation of HT finite-element formulation. In this subsection the construction of Trefftz functions for thin plates on elastic foundations will be discussed in detail. The Trefftz functions of lines 1 and 2 of equation (8) can be generated in a systematic way from Muskhelishvili's complex variable formulation.\(^3\) For the reader's convenience we list those results as follows\(^3\)

\[
\begin{align*}
U_j^* &= \left\{ \begin{array}{l}
\text{Re} Z_{1k}^* \\
\text{Im} Z_{1k}^*
\end{array} \right\}, \quad \text{with} \quad Z_{1k} = (3 - \mu)z^k + (1 + \mu)iz^{k-1} \\
U_{j+1}^* &= \left\{ \begin{array}{l}
\text{Re} Z_{2k}^* \\
\text{Im} Z_{2k}^*
\end{array} \right\}, \quad \text{with} \quad Z_{2k} = (3 - \mu)z^k - (1 + \mu)i\bar{z}^{k-1} \\
U_{j+2}^* &= \left\{ \begin{array}{l}
\text{Re} Z_{3k}^* \\
\text{Im} Z_{3k}^*
\end{array} \right\}, \quad \text{with} \quad Z_{3k} = (1 - \mu)\bar{z}^k \\
U_{j+3}^* &= \left\{ \begin{array}{l}
\text{Re} Z_{4k}^* \\
\text{Im} Z_{4k}^*
\end{array} \right\}, \quad \text{with} \quad Z_{4k} = -(1 + \mu)\bar{z}^k
\end{align*}
\]

where $z = x + iy$, $\bar{z} = x - iy$, $i = \sqrt{-1}$, Re($z$) and Im($z$) stand for the real part and the imaginary part of $z$, respectively, and where $U_j^*$ satisfies

\[ L_{in}U_j^* = 0 \quad \text{(13)} \]

with

\[ L_{in} = \begin{bmatrix} L_1 & L_2 \\ L_2 & L_3 \end{bmatrix}, \quad U_j^* = \begin{bmatrix} U_1^* \\ U_2^* \end{bmatrix}_j \quad \text{(14)} \]

What follows is the derivation of the Trefftz functions of Line 3 of equation (8). In deriving these functions, consider the related homogeneous equation

\[ L_3W = (\nabla^2 + \lambda^2 \nabla^2 + S)W = (\nabla^2 + b_1)(\nabla^2 + b_2)W = 0 \quad \text{(15)} \]

where

\[ b_1 = \left( \lambda^2 - \sqrt{\lambda^4 - 4S} \right), \quad b_2 = \left( \lambda^2 + \sqrt{\lambda^4 - 4S} \right) \quad \text{(16)} \]

To find the solution of equation (15), set

\[ (\nabla^2 + b_1)W = A, \quad (\nabla^2 + b_2)W = B \quad \text{(17)} \]

Substituting (17) into (15), one obtains

\[ (\nabla^2 + b_1)B = 0 \quad (\nabla^2 + b_2)A = 0 \quad \text{(18)} \]

Thus the Trefftz functions for equation (18) can be expressed by

\[ A_{2m-1} = J_m(r\sqrt{b_1}) \cos m\theta \]

\[ A_{2m} = J_m(r\sqrt{b_2}) \sin m\theta \]

\[ B_{2m-1} = J_m(r\sqrt{b_1}) \cos m\theta \]

\[ B_{2m} = J_m(r\sqrt{b_2}) \sin m\theta \quad \text{(19)} \]

where $r^2 = x^2 + y^2$, $\theta = \arctan(y/x)$, and $J_m(\cdot)$ is a Bessel function of the first kind with order $m$.

Subtracting line 1 of equation (17) from line 2 of equation (17), and using equation (19) the Trefftz functions of equation (15) can be given by

\[ g_j = f_m(r) \cos m\theta, \quad g_{j+1} = f_m(r) \sin m\theta \quad \text{(20)} \]

where $f_m(r) = J_m(r\sqrt{b_1}) - J_m(r\sqrt{b_2})$. As a consequence the complete system of equation (15) may be taken as

\[ T = (F_0, f_m(r) \cos m\theta, f_m(r) \sin m\theta) = \{ F \} \quad \text{(21)} \]

where $F_0 = f_0(r)$, $F_{2m-1} = f_m(r) \cos m\theta$, and $F_{2m} = f_m(r) \sin m\theta$.

2.3 Assumed fields

The main idea of the HT FE model is to establish a finite element formulation whereby the intrabracket continuity is enforced on a nonconforming internal displacement field chosen so as to satisfy a priori the governing differential equations. In other words an obvious alternative to Rayleigh-Ritz method as a basis for a finite-element formulation, the model, here, is based on the method of Trefftz.\(^1\) With this method the domain $\Omega_e$ is subdivided into elements, and over each element "e," the assumed intrabracket fields are

\[ u = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \hat{u} + \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}e_{in} = \hat{u} + N_{in}e_{in} \quad \text{(22)} \]

\[ \tilde{W} = \hat{W} + N_3e_{out} \]

where $e_{in}$ and $e_{out}$ are two undetermined coefficient vectors and $\hat{U} = \{ U_1, U_2 \}$, $\hat{W}$, $N_{in}$, and $N_3$ are known functions that satisfy

\[ L_{in}u = \begin{bmatrix} \hat{p}_1 \\ \hat{p}_2 \end{bmatrix}, \quad L_{in}N_{in} = 0, \]

\[ L_3\hat{W} = \hat{p}_3, \quad L_3N_3 = 0 \quad \text{(on $\Omega_e$)} \quad \text{(23)} \]

\[ c = \begin{bmatrix} e_{in} \\ e_{out} \end{bmatrix}, \quad U = \begin{bmatrix} U_1 & U_2 & \hat{W} \end{bmatrix}^T \]
and where \( \mathbf{N}_{in} \) and \( \mathbf{N}_3 \) are formed by a suitably truncated complete system of equations (12) and (21). As an illustration the number of dimension for \( \mathbf{N}_{in} \) and \( \mathbf{N}_3 \) (or \( \mathbf{c}_{in} \) and \( \mathbf{c}_{out} \)) are, respectively, taken to be 12 and 16 for a quadrilateral element. The choice will satisfy the rank condition given in Ref. 19. Furthermore, to enforce on \( \mathbf{U} \) the interelelement conformity, \( \mathbf{U}^e = \mathbf{U}^f \) on \( \partial \Omega_e \cap \partial \Omega_f \) (where "e" and "f" stand for any two neighboring elements), we will use an auxiliary interelement frame field \( \tilde{\mathbf{U}} \) approximated in terms of the same degrees of freedom (DOF), \( \mathbf{d} \), as in the conventional elements. The conforming frame field is assumed as

\[
\tilde{\mathbf{U}} = \tilde{\mathbf{N}} \mathbf{d}
\]  

where

\[
\tilde{\mathbf{U}} = \begin{bmatrix} \tilde{U}_1 \\ \tilde{U}_2 \end{bmatrix} \quad \tilde{\mathbf{N}} = \begin{bmatrix} \tilde{N}_1 \\ \tilde{N}_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -n_1 L_{AB} \frac{\tilde{N}_2}{2} \end{bmatrix}, \quad \mathbf{d}_{in} = \begin{bmatrix} n_1 L_{AB} \frac{\tilde{N}_2}{2} \\ 0 \\ n_1 \tilde{N}_5 \\ n_2 \tilde{N}_5 \end{bmatrix}, \quad \mathbf{d}_{out} = \begin{bmatrix} n_1 L_{AB} \frac{\tilde{N}_4}{2} \\ 0 \\ n_1 \tilde{N}_6 \\ n_2 \tilde{N}_6 \end{bmatrix}
\]

\[
\mathbf{d}_{in} = \begin{bmatrix} \mathbf{U}_{in} \\ \mathbf{U}_{in} \end{bmatrix}, \quad \mathbf{d}_{out} = \begin{bmatrix} \mathbf{U}_{out} \\ \mathbf{U}_{out} \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} \mathbf{d}_{in} \\ \mathbf{d}_{out} \end{bmatrix}
\]

(24)

(25)

and where \( \mathbf{d}_{in} \) and \( \mathbf{d}_{out} \) stand for vectors of in-plane and out-of-plane displacement nodal parameters, and \( \mathbf{N}_i \) (i = 1, 2, 3, 4) are the conventional FE interpolation functions. In the development of the present element model the following assumptions are adopted. First of all a quadrilateral element is chosen to be an element model with five DOF \( (\mathbf{U}_1, \mathbf{U}_2, \mathbf{W}_n, \mathbf{\tilde{W}}_n, \mathbf{\tilde{W}}) \) at each corner node (see Figure 1). Therefore each element has 20 DOF. Moreover the element model is a \( C^1 \)-continuity element because the normal derivative of \( \mathbf{W} \) on the side shared by two adjacent elements is continuous (see equation [36]). To illustrate, take the side A-B of a particular element as an example (see Figure 1). Along the side A-B a simple interpolation of the frame displacements may be given in the form

\[
\tilde{\mathbf{U}} = \begin{bmatrix} \mathbf{N}_{AB1} \\ \mathbf{N}_{AB2} \end{bmatrix} \mathbf{d}_{in} = \begin{bmatrix} \mathbf{N}_{AB3} \\ \mathbf{N}_{AB4} \end{bmatrix} \mathbf{d}_{out},
\]

where

\[
\begin{bmatrix} \mathbf{N}_{AB1} \\ \mathbf{N}_{AB2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -t & 0 & 1 & t & 0 \\ 0 & 1 & -t & 0 & 1 & t \end{bmatrix},
\]

\[
\mathbf{d}_{in} = \begin{bmatrix} \mathbf{U}_n \\ \mathbf{U}_s \\ \mathbf{U}_t \\ \mathbf{U}_s \\ \mathbf{U}_t \end{bmatrix}, \quad \mathbf{d}_{out} = \begin{bmatrix} \mathbf{U}_n \\ \mathbf{U}_s \\ \mathbf{U}_t \end{bmatrix}
\]

(26)

Figure 1. A quadrilateral HT element.

(30)

\[
\mathbf{Q}_i = \begin{bmatrix} Q_{1i} \\ Q_{2i} \end{bmatrix} \quad (i = 1, 2, 3, 4, 5)
\]

\[
\mathbf{N} = \begin{bmatrix} \mathbf{N}_1 \\ \mathbf{N}_2 \end{bmatrix}
\]

\[
\mathbf{M} = \begin{bmatrix} \mathbf{M}_x \\ \mathbf{M}_y \end{bmatrix}
\]

\[
\mathbf{\dot{U}} = \begin{bmatrix} \mathbf{U}_n \\ \mathbf{U}_s \end{bmatrix}, \quad \mathbf{\tilde{U}} = \begin{bmatrix} \mathbf{U}_n \\ \mathbf{U}_s \end{bmatrix}
\]

\[
\mathbf{\dot{W}} = \begin{bmatrix} \mathbf{W}_n \\ \mathbf{W}_s \end{bmatrix}, \quad \mathbf{\tilde{W}} = \begin{bmatrix} \mathbf{W}_n \\ \mathbf{W}_s \end{bmatrix}
\]

(31)
2.4 Particular solutions

The particular solutions of $\hat{u}$ and $\hat{W}$ can be obtained by means of their source functions. The source functions corresponding to equation (8) have been given in Ref. 16:

$$U_{ij}^\mu(p,q) = \frac{1 + \mu}{4\pi E} \left[ -(3-\mu) \delta_{ij} \ln r_{pq} + (1+\mu) r_{pq,i} r_{pq,j} \right]$$

$$W^\mu(p,q) = \frac{1}{4D(b_2-b_1)} \left[ N_0(r_{pq}\sqrt{b_1}) - N_0(r_{pq}\sqrt{b_2}) \right]$$

$$r_{pq} = (x_p-x_q)^2 + (y_p-y_q)^2$$

where $U_{ij}^\mu(p,q)$ designates $i$th component of in-plane displacements at field point $q$ of an infinite plate when a unit point force ($j=1,2$) is applied at the source point $p$, and $N_0(\ )$ is the Bessel function of zeroth order of the second kind. By way of these source functions the particular solutions of $\hat{u}$ and $\hat{W}$ can be expressed as

$$\hat{u} = \begin{bmatrix} \hat{U}_1 \\ \hat{U}_2 \end{bmatrix} = \iint_{\Omega} \hat{P}_j \begin{bmatrix} U_{ij}^\mu \\ W^\mu \end{bmatrix} d\Omega$$

$$\hat{W} = \iint_{\Omega} \hat{P}_j W^\mu d\Omega$$

(33)

The area integration in equation (33) will be performed by numerical quadrature using the Gauss-Legendre rule.

3. Variational principle and element matrix

3.1 Modified variational principle

The HT FE formulation for postbuckling analysis of thin plates on an elastic foundation can be obtained by means of a modified variational principle. The related functional used for deriving the HT FE formulation can be, in this case, given in the form: 19

$$\Gamma_{in} = \sum \left\{ \Gamma_{in}^c + \int_{\partial\Omega_1^c} \left( N_{n*}^c - N_n \right) \hat{U}_n d\sigma \right\}$$

$$+ \int_{\partial\Omega_2^c} \left( N_{n*}^c - N_n \right) \hat{U}_n d\sigma - \int_{\partial\Omega_2^c} \hat{F} \hat{v} d\sigma$$

(34)

$$\Gamma_{out}^c = \sum \left\{ \Gamma_{out}^c - \int_{\partial\Omega_2^e} \left( \Delta \tilde{M}_n - \tilde{M}_n \right) \hat{W}_n d\sigma \right\}$$

$$+ \int_{\partial\Omega_2^e} \left( R^* - R \right) \hat{W} d\sigma$$

$$+ \int_{\partial\Omega_2^e} \left( \tilde{M}_n \hat{W}_n - \tilde{R} \hat{W} \right) d\sigma \right\}$$

(35)

where

$$\Gamma_{in}^c = \iint_{\Omega} \hat{V}_n d\Omega - \iint_{\partial\Omega_1} \hat{U}_n d\sigma - \iint_{\partial\Omega_2^c} \hat{U}_n d\sigma$$

$$\Gamma_{out}^c = \iint_{\Omega} \hat{V}_n d\Omega + \iint_{\partial\Omega_2^c} \hat{M}_n d\sigma - \iint_{\partial\Omega_2^c} \hat{R} \hat{W} d\sigma$$

$$\hat{V}_n = \frac{1-2\mu}{6Eh} \hat{N}_{kz} \hat{N}_{kz} + \frac{1+\mu}{2Eh} \hat{N}_{ij} \hat{N}_{ij}$$

$$\hat{V}_{out} = \frac{1}{2D(1-\mu)} \left[ \left( \hat{M}_x + \hat{M}_y \right)^2 + 2(1+\mu) \left( \hat{M}_{xy} - \hat{M}_x \hat{M}_y \right)\right] + \frac{1}{2} k_0 \hat{W}^2$$

and where equation (8) is assumed to be satisfied, a priori. The boundary $\partial\Omega_1^c$ of a particular element consists of the following parts:

$$\partial\Omega_1^c = \partial\Omega_1 \cup \partial\Omega_2 \cup \partial\Omega_3^c = \partial\Omega_2 \cup \partial\Omega_3^c$$

$$= \partial\Omega_2 \cup \partial\Omega_3 \cup \partial\Omega_4$$

$$= \partial\Omega_2 \cup \partial\Omega_3 \cup \partial\Omega_4$$

$$= \partial\Omega_2 \cup \partial\Omega_3 \cup \partial\Omega_4$$

$$= \partial\Omega_2 \cup \partial\Omega_3 \cup \partial\Omega_4$$

and $\partial\Omega_4$ is the interelement boundary of the element. Consequently we will discuss some properties and their proofs on these two functionals. They are:

(i) Modified complementary principle

$$\delta \Gamma_{in}^c = 0 \Rightarrow (10)_{1,2} \text{ and} \Psi_n^c = \hat{U}_n \hat{U}_n, \hat{U}_n \hat{U}_n \text{ (on } \partial\Omega_1 \cup \partial\Omega_2 \text{)}$$

$$\delta \Gamma_{out}^c = 0 \Rightarrow (10)_{3,4} \text{ and} \Psi_n^c = \hat{W}_n \hat{W}_n, \hat{W}_n \hat{W}_n \text{ (on } \partial\Omega_3 \cup \partial\Omega_4 \text{)}$$

(36)

(ii) Theorems on the existence of extremum are:

(a) If the expression

$$\iint_{\Omega} \delta^2 \hat{V}_n d\Omega - \iint_{\Omega} \delta \hat{U}_n \delta \hat{U}_n d\Omega$$

$$- \int_{\partial\Omega_2} \delta \hat{U}_n \delta \hat{U}_n d\sigma - \sum \delta \hat{F} \delta \hat{v} d\sigma$$

(37)

is uniformly positive (or negative) at the neighborhood of $u_0(u_0 = U_{10} U_{20})$, where $u_0$ is such a value that $\Gamma_{in}(u_0) = (\Gamma_{in})_0$, and where $(\Gamma_{in})_0$ stands for the sta-
tionary value of $\Gamma^m$, we have
\begin{equation}
\Gamma^m \geq \left( \Gamma^m \right)_0 \text{ [or } \Gamma^m \leq \left( \Gamma^m \right)_0 \right]
\end{equation}
(38)

(b) If the expression
\begin{equation}
\int_\Omega \delta^2 V_{\text{out}} d\Omega + \int_{C_{\text{in}}} \delta M_n \delta W_n dc - \int_{C_{\text{out}}} \delta R \delta \tilde{W} dc 
+ \sum_{e} \int_{\delta \Omega^e_i} \left( \delta M_n \delta \tilde{W}_n - \delta R \delta \tilde{W} \right) dc
\end{equation}
(39)
is uniformly positive (or negative) at the neighborhood of $W_0$, where $W_0$ is such a value that $\Gamma^m(W_0) = \left( \Gamma^m \right)_0$, and where $\left( \Gamma^m \right)_0$ stands for the stationary value of $\Gamma^m$, we have
\begin{equation}
\Gamma^m \geq \left( \Gamma^m \right)_0 \text{ [or } \Gamma^m \leq \left( \Gamma^m \right)_0 \right]
\end{equation}
(40)
where "e" and "f" stand for any two neighboring elements and where $U^e = U^f$ is identical on $\delta \Omega_e \cap \delta \Omega_f$ due to the assumed frame field $U$ (see equation [24]).

Proof—From the first we derive the stationary conditions for functional $\Gamma^m$. To this end, taking the variation of $\Gamma^m$ and noting that lines 1 and 2 of equation (8) hold a priori by the previous assumption one obtains
\begin{equation}
\delta \Gamma^m \overset{(8), (2)}{=} \int_{C_{\text{in}}} \left( \hat{U}_n - \Delta \hat{U}_n \right) \delta N_n dc 
+ \int_{C_{\text{in}}} \left( \hat{U}_s - \Delta \hat{U}_s \right) \delta N_s dc 
- \int_{C_{\text{out}}} \left( \hat{N}_n - \hat{N}_n^* \right) \delta U_n dc 
- \int_{C_{\text{out}}} \left( \hat{N}_s - \hat{N}_s^* \right) \delta U_s dc 
- \sum_{e} \int_{\delta \Omega^e_i} \left[ \left( \hat{U}_n - \hat{U}_n \right) \delta N_n + \left( \hat{U}_s - \hat{U}_s \right) \delta N_s \right] dc
\end{equation}
(41)
where the constrained equality stands for the lines 1 and 2 of equation (8) being satisfied, a priori. Therefore the Euler equations for equation (41) are lines 1 and 2 of equations (10) and (11) and
\begin{equation}
\hat{U}_n = \hat{U}_n^e, \quad \hat{U}_s = \hat{U}_s^e \quad \text{(on } \delta \Omega_e \cap \delta \Omega_f) \end{equation}
(42)
The principle in line 1 of equation (36) has been, thus, proved.

As for the theorem on the existence of extremum we may prove it by means of the so-called second variational approach. In doing this, taking the variation of $\delta \Gamma^m$ and using the constrained conditions in lines 1 and 2 of equation (8) we see
\begin{equation}
\delta^2 \Gamma^m = \int_\Omega \delta^2 V_{\text{in}} d\Omega - \int_{C_{\text{in}}} \delta \hat{N}_n \delta U_n dc
\end{equation}
(43)
- \int_{C_{\text{out}}} \delta \hat{N}_n \delta U_n dc - \sum_{e} \int_{\delta \Omega^e_i} \delta F^T \delta \hat{v} dc
= \text{expression (37)}

So the theorem has been proved from the sufficient condition on the existence of a local extreme of a functional. In the same way as above one may easily prove the inequality (40), and we omit those details here.

3.2 Element matrix

The element matrix may be established by setting $\delta (\Gamma^m)_e = 0$ and $\delta (\Gamma^m)_e = 0$. To simplify the derivation, all domain integrals in equations (34) and (35) are transformed into boundary ones except those loading terms by use of solution properties of the intrapixel trial functions, for which the functions $\Gamma^m$ and $\Gamma^m$ are rewritten as
\begin{equation}
(\Gamma^m)_e = \frac{1}{2} \int_\Omega \hat{P}_n \hat{U}_n d\Omega - \int_{\delta \Omega^e_i} \hat{N}_n \Delta \hat{U}_n dc 
- \int_{\delta \Omega^e_i} \hat{N}_n \Delta \hat{U}_n dc - \int_{\delta \Omega^e_i} \left( \hat{N}_n - \hat{N}_n^* \right) \hat{U}_n dc 
- \int_{\delta \Omega^e_i} \left( \hat{N}_s - \hat{N}_s^* \right) \hat{U}_s dc 
+ \frac{1}{2} \int_{\delta \Omega^e_i} F^T v dc - \int_{\delta \Omega^e_i} F^T \hat{v} dc
\end{equation}
(44)
\begin{equation}
(\Gamma^m)_e = \frac{1}{2} \int_\Omega \hat{P}_n \hat{W}_n d\Omega + \int_{\delta \Omega^e_i} M_n \Delta \hat{W}_n dc 
- \int_{\delta \Omega^e_i} \left( \hat{R} \Delta \hat{W}_n \right) dc + \int_{\delta \Omega^e_i} \left( \hat{M}_n - \hat{M}_n^* \right) \hat{W}_n dc 
- \int_{\delta \Omega^e_i} \left( \hat{R} - \hat{R}^* \right) \hat{W}_n dc 
+ \frac{1}{2} \int_{\delta \Omega^e_i} \left( \hat{R} \hat{W}_n - \hat{M}_n \hat{W}_n \right) dc
\end{equation}
(45)
The substitution of equations (22), (24), and (30) into equations (44) and (45), one gets
\begin{equation}
(\Gamma^m)_e = -c_{in} H_{in} c_{in}/2 + c_{in} S_{in} d_{in} + c_{in} r_1 + d_{in} r_2 
+ \text{terms without } c_{in} \text{ or } d_{in}
\end{equation}
(46)
\begin{equation}
(\Gamma^m)_e = -c_{out} H_{out} c_{out}/2 + c_{out} S_{out} d_{out} + c_{out} r_3 
+ d_{out} r_4 + \text{terms without } c_{out} \text{ or } d_{out}
\end{equation}
(47)
where
\[
H_{in} = H_{in}^* + (H_{in}^*)^T
\]
\[
H_{in}^* = -\frac{1}{2} \int_{\Omega_{in}} Q_2^T Q_1 \, dc + \int_{\Omega_{in}} Q_2^T Q_{12} \, dc
\]
\[
S_{in} = -\int_{\Omega_{in}} Q_2^T Q_3 \, dc
\]
\[
r_1 = \frac{1}{2} \int_{\Omega_{in}} \left( N_{n1}^T \tilde{P}_1 + N_{n2}^T \tilde{P}_2 \right) d\Omega
\]
\[
+ \frac{1}{2} \int_{\Omega_{in}} \left( Q_2^T \tilde{N} + Q_3^T \tilde{u} \right) dc - \int_{\Omega_{in}} \Delta \tilde{u}_n Q^T_{12} \, dc
\]
\[
- \int_{\Omega_{in}} \Delta \tilde{u}_n Q_{22} \, dc
\]
\[
+ \int_{\Omega_{in}} N_{n1}^* Q_{11} \, dc + \int_{\Omega_{in}} N_{n2}^* Q_{12} \, dc - \int_{\Omega_{in}} \tilde{N}_n Q_{12} \, dc
\]
\[
- \int_{\Omega_{in}} \tilde{N}_n Q_{22} \, dc + \int_{\Omega_{in}} \tilde{u}_n Q_{11} \, dc + \int_{\Omega_{in}} \tilde{u}_n Q_{12} \, dc
\]
\[
r_2 = \int_{\Omega_{in}} Q_2^T \tilde{N} \, dc
\]
\[
H_{out} = H_{out}^* + (H_{out}^*)^T
\]
\[
H_{out}^* = -\frac{1}{2} \int_{\Omega_{out}} \left( Q_4^T Q_1 - Q_4^T Q_2 \right) \, dc
\]
\[
- \int_{\Omega_{out}} Q_2^T Q_{11} \, dc + \int_{\Omega_{out}} Q_2^T Q_{12} \, dc
\]
\[
S_{out} = \int_{\Omega_{out}} \left( Q_4^T \tilde{N}_4 - Q_4^T \tilde{N}_3 \right) \, dc
\]
\[
r_3 = \frac{1}{2} \int_{\Omega_{in}} N_{n1}^T \tilde{P}_3 \, d\Omega
\]
\[
+ \frac{1}{2} \int_{\Omega_{in}} \left( Q_{31}^T \tilde{R} - Q_{32}^T \tilde{M}_n + \tilde{W} Q_{11}^T - \tilde{W}_n Q_{12}^T \right) \, dc
\]
\[
+ \int_{\Omega_{in}} \Delta \tilde{W}_n Q_{22} \, dc
\]
\[
- \int_{\Omega_{in}} \Delta \tilde{W} Q_{12} \, dc - \int_{\Omega_{in}} \Delta M_n Q_{12} \, dc
\]
\[
+ \int_{\Omega_{in}} R^* Q_{31} \, dc
\]
\[
+ \int_{\Omega_{in}} \left( \tilde{M}_n Q_{32} + \tilde{W}_n Q_{32} \right) \, dc
\]
\[
- \int_{\Omega_{in}} \left( \tilde{R} Q_{31} + \tilde{W} Q_{41} \right) \, dc
\]
\[
r_4 = \int_{\Omega_{in}} \left( \tilde{N}_4^T \tilde{M}_n - \tilde{N}_3^T \tilde{R} \right) \, dc
\]

It should be pointed out that all terms not involving \(e\) and \(d\) are of no significance for an approximate solution and are therefore not listed explicitly.

To obtain the element matrices, taking the vanishing variation of equations (46) and (47) with respect to \(e\) at the element level, one sees
\[
\frac{\partial (\Gamma_{in}^m)}{\partial e_{in}} = -H_{in} e_{in} + S_{in} d_{in} + r_1
\]
\[
\frac{\partial (\Gamma_{out}^m)}{\partial e_{out}} = -H_{out} e_{out} + S_{out} d_{out} + r_3
\]
which lead to
\[
c_{in} = G_{in} d_{in} + g_{in}
\]
\[
c_{out} = G_{out} d_{out} + g_{out}
\]
where
\[
G_{in} = H_{in}^{-1} S_{in} \quad g_{in} = H_{in}^{-1} r_1
\]
\[
G_{out} = H_{out}^{-1} S_{out} \quad g_{out} = H_{out}^{-1} r_3
\]
As a consequence the functionals \((\Gamma_{in}^m)\) and \((\Gamma_{out}^m)\), can be expressed only in terms of \(d\) and other known matrices
\[
(\Gamma_{in}^m) = -d_{in}^T G_{in}^T H_{in} G_{in} d_{in}/2 + d_{in}^T (G_{in}^T H_{in} g_{in} + r_2) + \text{terms without }d_{in}
\]
\[
(\Gamma_{out}^m) = -d_{out}^T G_{out}^T H_{out} G_{out} d_{out}/2 + d_{out}^T (G_{out}^T H_{out} g_{out} + r_4) + \text{terms without }d_{out}
\]
Thus the customary force-displacement relationships are in the form
\[
K_{in} d_{in} = P_{in}
\]
\[
K_{out} d_{out} = P_{out}
\]
where
\[
K_{in} = G_{in}^T H_{in} G_{in} \quad K_{out} = G_{out}^T H_{out} G_{out}
\]
\[
P_{in} = G_{in}^T H_{in} g_{in} + r_2 \quad P_{out} = G_{out}^T H_{out} g_{out} + r_4
\]
and where \(K_{in}\) and \(K_{out}\) can be calculated in the usual way, while \(P_{in}\) and \(P_{out}\) contain the unknown variables \((\tilde{U}_1 \tilde{U}_2 \tilde{W})\), and therefore cannot be calculated directly.

### 3.3 Iterative scheme

Since \(P_{in}\) and \(P_{out}\) contain the unknown variables \((\tilde{U}_1 \tilde{U}_2 \tilde{W})\) an iterative procedure is, thus, required. Before describing the scheme, however, let us study some properties of \(P_{in}\).

It can be seen from the definition of \(P_{in}\) that not only depends upon \(\tilde{W}\). So only an initial value \(\tilde{W}_0\) is required.
As long as the value of \( \dot{W} \) in \( \Omega \) is known, we can calculate the pseudo-load \( \mathbf{P}_{in} \), and then all of the unknown variables in line 1 of equation (60) are in-plane displacements (\( \dot{U}_1, \dot{U}_2 \)). We may solve (60) for them. As a consequence \( \mathbf{P}_{out} \) can be calculated from the current values of \( \{\dot{U}_1, \dot{U}_2, \dot{W}\} \). An iterative scheme may be established according to the above analysis. Specifically, suppose that \( \dot{W}_{k+1}, \dot{W}_k \), and \( W_k \) stand for the \( k \)th approximations, which can be obtained from the proceeding cycle of iteration. The \((k+1)\)th solution may be evaluated as follows:

(i) Assume the initial value \( \dot{W}_0 \) and \( W_0 \) in \( \Omega \). If the current loading step is not the first one, but is the \((k+1)\)th step, \( \dot{W}_0 \) and \( W_0 \) may be taken as \( \dot{W}_k \) and \( W_k \), where \( \dot{W}_k \) and \( W_k \) stand for the incremental and the total deflection at the \( k \)th loading step, respectively.

(ii) Enter the iterative cycle for \( i = 1, 2, \ldots \). Calculate \( \mathbf{P}_{in} \) in line 1 of equation (60) by means of line 3 of equation (61), solve line 1 of equation (60) for the nodal displacement vector \( \mathbf{d}^{(i)} \), and then determine the values of \( \dot{U}^{(i)} \) and \( \dot{U}_2^{(i)} \) in \( \Omega \).

(iii) Calculate \( \mathbf{P}_{out} \) using the current values of \( U \), then solve line 2 of equation (60) for \( \mathbf{d}^{(i)} \) and determine the value of \( W_i \) in \( \Omega \).

(iv) If \( e_i = |(\mathbf{d}^{(i)})\mathbf{d}^{(i)} - (\mathbf{d}^{(i-1)})\mathbf{d}^{(i-1)}|/(\mathbf{d}^{(i-1)}\mathbf{d}^{(i-1)}) \leq \epsilon \) (\( \epsilon \) is the convergence tolerance), proceed to the next loading step and calculate

\[
\dot{U}_{k+1} = \dot{U}_k + \dot{U}_{(i)} \quad \dot{W}_{k+1} = \dot{W}_{(i)}
\]  

otherwise, set

\[
\dot{W}_0 = \dot{W}_{(i)} \quad W_0 = W_k + \dot{W}_{(i)}
\]

4. Numerical examples

As a numerical illustration of the proposed method, two simple examples are considered. In order to allow for comparisons with conventional FE model results, which appeared in Refs. 22 and 23, the obtained results are limited to a circular plate and a square plate subjected to a uniform in-plane compressive load \( p_0 \), at the boundary \( \partial \Omega \) and resting on Winkler foundations. We used the same space discretization as in Refs. 22 and 23. In all the calculations the value of Poisson's ratio is taken as 0.3, and the convergence tolerance is \( \epsilon = 0.0001 \).

4.1 Example 1: A circular plate

Consider a circular plate resting on a Winkler foundation with the following parameters

\[
k_0 = 10D/a^4 \quad h/a = 1/50 \quad E = 2 \times 10^6 \text{ kg/cm}^2
\]

where \( a \) is radius of the circular plate. The plate is subjected to a uniform in-plane compressive load \( p_0 \).

Owing to the axisymmetry of the problem only a narrow-sector of the plate is considered (see Figure 2), and the load step \( p_0 \) is taken to be 0.02 \( p_{cr} \) (\( p_{cr} \) is the linear buckling load of the plate, and is equal to 5.92 \( D/a^2 \) for the simply supported plate as well as to 16.05 \( D/a^2 \) for the clamped plate). It should be pointed out that only those nodes on line AB (see Figure 2) are independent in the sector due to the axisymmetry of the problem. In other words, Line AC, as shown in Figure 2, has the same deformation as line AB. Thus we need only analyze line AB. In this case, each node in our element model has three independent parameters, i.e., \( U_r, W \), and \( W_x \), and then each element has six independent parameters. Moreover, two meshes are used with four and eight elements, as shown in Figure 2, to make the convergence study. The elements, independent nodes, and free parameters adopted in the example are as follows (Table 1).

Table 2 gives the results of the maximum deflection, \( W_{max} \), occurring at the center versus with the compressive load coefficient \( \beta = p_0/p_{cr} \) (\( \beta \geq 1 \)) for the simply supported plate. Table 3 presents the corresponding results for the clamped plate. Tables 4 and 5 list the results of deflection along the radius for the above two plates. However we cannot find the corresponding results in the literature. It can be seen from Tables 2 and 3 that the present results are in good agreement with those by the conventional FE method. It is also observed from 2 to 5 tables that a good convergence was obtained along with refinement of the element meshes. In the course of computations, convergence was achieved with about nine iterations for the simply supported plate and seven iterations for the clamped plate at each load step.

4.2 Example 2: A square plate

Consider a square plate on a Winkler foundation subjected to a uniform in-plane compressive load \( p_0 \). The geometric and material parameters are

\[
k_0 = 5D \pi^4/L^4 \quad h/L = 1/50 \quad E = 2 \times 10^6 \text{ kg/cm}^2
\]

where \( L \) is the length of the square plate. In the analysis a \( 2 \times 2 \) mesh is used in a quarter of the plate to allow for comparisons with those in Ref. 23. The load step is, again, taken to be 0.02 \( p_{cr} \) (\( p_{cr} \) is the linear buckling load of the square plate, and is equal to 4.501 \( D \pi^2/L^2 \) for a simply supported plate, to 6.017 \( D \pi^2/L^2 \) for a clamped-simply supported plate, and to 7.229 \( D \pi^2/L^2 \) for a clamped plate, where the clamped-simply supported plate means

![Figure 2. Two element meshes in example 1.](image)

Table 1. Free parameters in two FE models

<table>
<thead>
<tr>
<th>No. of elements</th>
<th>Total free parameters</th>
<th>Independent nodes</th>
</tr>
</thead>
<tbody>
<tr>
<td>This paper</td>
<td>4</td>
<td>15</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>27</td>
</tr>
<tr>
<td>Ref. 22</td>
<td>4</td>
<td>20</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>36</td>
</tr>
</tbody>
</table>

Table 2. Maximum deflection $W_m/h$ for the simply supported circular plate

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>1.02</th>
<th>1.04</th>
<th>1.06</th>
<th>1.08</th>
<th>1.10</th>
</tr>
</thead>
<tbody>
<tr>
<td>This paper</td>
<td>0.3246</td>
<td>0.4593</td>
<td>0.5635</td>
<td>0.6516</td>
<td>0.7290</td>
</tr>
<tr>
<td>8 elements</td>
<td>0.3245</td>
<td>0.4590</td>
<td>0.5632</td>
<td>0.6511</td>
<td>0.7283</td>
</tr>
<tr>
<td>Ref. 22</td>
<td>0.3242</td>
<td>0.4589</td>
<td>0.5626</td>
<td>0.6503</td>
<td>0.7278</td>
</tr>
<tr>
<td>8 elements</td>
<td>0.3243</td>
<td>0.4589</td>
<td>0.5627</td>
<td>0.6503</td>
<td>0.7277</td>
</tr>
</tbody>
</table>

Table 3. Maximum deflection $W_m/h$ for the clamped circular plate

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>1.02</th>
<th>1.04</th>
<th>1.06</th>
<th>1.08</th>
<th>1.10</th>
</tr>
</thead>
<tbody>
<tr>
<td>This paper</td>
<td>0.3270</td>
<td>0.4632</td>
<td>0.5685</td>
<td>0.6690</td>
<td>0.7388</td>
</tr>
<tr>
<td>8 elements</td>
<td>0.3267</td>
<td>0.4628</td>
<td>0.5680</td>
<td>0.6682</td>
<td>0.7381</td>
</tr>
<tr>
<td>Ref. 22</td>
<td>0.3262</td>
<td>0.4620</td>
<td>0.5669</td>
<td>0.6656</td>
<td>0.7343</td>
</tr>
<tr>
<td>8 elements</td>
<td>0.3260</td>
<td>0.4618</td>
<td>0.5665</td>
<td>0.6652</td>
<td>0.7337</td>
</tr>
</tbody>
</table>

Table 4. Deflection $W/h$ along the radius in the simply supported circular plate ($\beta = 1.10$)

<table>
<thead>
<tr>
<th>$r/a$</th>
<th>0.1</th>
<th>0.3</th>
<th>0.5</th>
<th>0.7</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>This paper</td>
<td>0.7247</td>
<td>0.6902</td>
<td>0.6033</td>
<td>0.4438</td>
<td>0.1995</td>
</tr>
<tr>
<td>8 elements</td>
<td>0.7241</td>
<td>0.6897</td>
<td>0.6028</td>
<td>0.4432</td>
<td>0.1991</td>
</tr>
<tr>
<td>Ref. 22</td>
<td>Not available</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 5. Deflection $W/h$ along the radius in the clamped circular plate ($\beta = 1.10$)

<table>
<thead>
<tr>
<th>$r/a$</th>
<th>0.1</th>
<th>0.3</th>
<th>0.5</th>
<th>0.7</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>This paper</td>
<td>0.7241</td>
<td>0.6118</td>
<td>0.4156</td>
<td>0.1922</td>
<td>0.0267</td>
</tr>
<tr>
<td>8 elements</td>
<td>0.7234</td>
<td>0.6112</td>
<td>0.4152</td>
<td>0.1920</td>
<td>0.0266</td>
</tr>
<tr>
<td>Ref. 22</td>
<td>Not available</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 6. Maximum deflection $W_m/h$ for a simply supported square plate

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>1.02</th>
<th>1.04</th>
<th>1.06</th>
<th>1.08</th>
<th>1.10</th>
</tr>
</thead>
<tbody>
<tr>
<td>This paper</td>
<td>0.5033</td>
<td>0.7194</td>
<td>0.8846</td>
<td>1.0284</td>
<td>1.1495</td>
</tr>
<tr>
<td>4 x 4 mesh</td>
<td>0.5041</td>
<td>0.7199</td>
<td>0.8862</td>
<td>1.0304</td>
<td>1.1525</td>
</tr>
<tr>
<td>Ref. 23</td>
<td>0.5057</td>
<td>0.7205</td>
<td>0.8892</td>
<td>1.0350</td>
<td>1.1670</td>
</tr>
<tr>
<td>4 x 4 mesh</td>
<td>Not available</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 7. Maximum deflection $W_m/h$ for the clamped-simply supported square plate

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>1.02</th>
<th>1.04</th>
<th>1.06</th>
<th>1.08</th>
<th>1.10</th>
</tr>
</thead>
<tbody>
<tr>
<td>This paper</td>
<td>0.4413</td>
<td>0.6286</td>
<td>0.7747</td>
<td>0.8996</td>
<td>1.0185</td>
</tr>
<tr>
<td>4 x 4 mesh</td>
<td>0.4409</td>
<td>0.6280</td>
<td>0.7739</td>
<td>0.8987</td>
<td>1.0168</td>
</tr>
<tr>
<td>Ref. 23</td>
<td>0.4401</td>
<td>0.6264</td>
<td>0.7722</td>
<td>0.8978</td>
<td>1.0109</td>
</tr>
<tr>
<td>4 x 4 mesh</td>
<td>Not available</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 8. Maximum deflection $W_m/h$ for the clamped square plate

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>1.02</th>
<th>1.04</th>
<th>1.06</th>
<th>1.08</th>
<th>1.10</th>
</tr>
</thead>
<tbody>
<tr>
<td>This paper</td>
<td>0.3623</td>
<td>0.5145</td>
<td>0.6342</td>
<td>0.7325</td>
<td>0.8229</td>
</tr>
<tr>
<td>4 x 4 mesh</td>
<td>0.3621</td>
<td>0.5139</td>
<td>0.6335</td>
<td>0.7313</td>
<td>0.8218</td>
</tr>
<tr>
<td>Ref. 23</td>
<td>0.3615</td>
<td>0.5128</td>
<td>0.6300</td>
<td>0.7298</td>
<td>0.8186</td>
</tr>
<tr>
<td>4 x 4 mesh</td>
<td>Not available</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The results of $W_m/h$ for a square plate with all edges clamped are listed in Table 8.

It can be, again, found from Tables 6, 7, and 8 that the present results are in good agreement with the conventional FE solution, which shows that the present HT FE model is suitable for the analysis of postbuckling plates. In the calculations, convergence was achieved with about 14 iterations for the simply supported plate, 11 iterations for the clamped-simply supported plate, and 15 iterations for the clamped square plate at each loading step.

5. Conclusions

An HT FE model has been presented for the postbuckling analysis of thin plates subjected to uniform compressive load and resting on an elastic foundation. As far as we know, the most previous HT FE results gave only concerned the linear problems. To some extent this paper studies how to apply the HT FE approach to nonlinear problems. In the analysis we use the incremental form of the basic equations and have made some modifications on nonlinear boundary conditions (see equations [10] and the square plate with a pair of opposite edges clamped and the other pair of opposite edges simply supported). It should be pointed out that the results in Ref. 23 also were obtained with a $2 \times 2$ mesh. Table 6 gives the relationship between the central deflection $W_m/h$ and the compressive load coefficient $\beta = P_0/P_{cr}$ ($\beta > 1$) for the simply supported square plate. Table 7 gives the corresponding values for a clamped-simply supported plate. The results of $W_m/h$ for a square plate with all edges clamped are listed in Table 8.
Postbuckling analysis of thin plates: Q. H. Qin

[11]). The numerical results show that these modifications are practicable. We also see that these modifications greatly simplify the related derivation.

Acknowledgments

The paper was partly done at ICA, University of Stuttgart, and I wish to express my thanks to Professor J. Argyris for providing me the computer facilities in his institute. Also acknowledged is the support of the foundation of DAAD/K.C. Wang.

References
