Axisymmetric bending analysis of functionally graded one-dimensional hexagonal piezoelectric quasi-crystal circular plate

Yang Li\(^1\), Yuan Li\(^3\), Qinghua Qin\(^4\), Lianzhi Yang\(^5\), Liangliang Zhang\(^2\) and Yang Gao\(^2\)

\(^{1}\)Department of Aerospace Science and Technology, Space Engineering University, Beijing 101416, People’s Republic of China
\(^{2}\)College of Science, China Agricultural University, Beijing 100083, People’s Republic of China
\(^{3}\)Henan Academy of Big Data, Zhengzhou University, Zhengzhou, Henan 450001, People’s Republic of China
\(^{4}\)College of Engineering and Computer Science, Australian National University, Canberra, ACT 2601, Australia
\(^{5}\)School of Civil and Resource Engineering, University of Science and Technology Beijing, Beijing 100083, People’s Republic of China

Within a framework of the state space method, an axisymmetric solution for functionally graded one-dimensional hexagonal piezoelectric quasi-crystal circular plate is presented in this paper. Applying the finite Hankel transform onto the state space vector, an ordinary differential equation with constant coefficients is obtained for the circular plate provided that the free boundary terms are zero and an exponential function distribution of material properties is assumed. The ordinary differential equation is then used to obtain the stress, displacement and electric components in the physical domain of the elastic simply supported circular plate through the use of the propagator matrix method and the inverse Hankel transform. The numerical studies are carried out to show the validity of the present solution and reveal the influence of material inhomogeneity on the axisymmetric
bending of the circular plate with different layers and loadings, which provides guidance for the design and manufacture of functionally graded one-dimensional hexagonal piezoelectric quasi-crystal circular plate.

1. Introduction

Piezoelectric quasi-crystal (QC) material is a structure that is ordered but not periodic. It shows unique physical and chemical properties that make them promising for technological applications. Therefore, smart and QC materials have drawn tremendous attention in recent years [1,2]. The ‘impossible’ atomic arrangements termed QCs were first discovered from a rapidly chilled molten mixture of Al-Mn under an electron microscope by Shechtman in the early 1980s [1]. By contrast with the piezoelectric ceramics reported in Qin et al. [3] and Yu et al. [4], QCs possess long-range order and no translational periodicity and due to their ordered patterns, many unusual and remarkable properties are found in QCs, such as low friction coefficient, high hardness, low thermal conductivity, in addition to the piezoelectric characteristic. Due to these interesting properties, QCs can be used as sensors, actuators, coatings, thermoelectric converters and so on [5]. As such, the investigation on QCs’ functions and behaviours has become an essential issue in the field of condensed matter physics, and many topics have been investigated, including defect problems [6–11], contact issues [12,13] and static as well as vibration analyses of layered QC structure [14–16]. In particular, Zhang et al. [17] investigated a spheroidal inclusion embedded within an infinite matrix of a one-dimensional (1D) piezoelectric QC and obtained the explicit expressions. Yang et al. [18] proposed a coupling method of symplectic expansion and IGA for the mode III fracture analysis of piezoelectric QCs with an electrically permeable/impermeable crack. Based on integral transform method and singular integral equations technique, Hu et al. [19] studied an interface crack between dissimilar piezoelectric QCs under anti-plane deformation and in-plane electric loading. By virtue of a Hamiltonian-based methodology, Zhou et al. [20] investigated the mode-III fracture behaviours of a V-shape interface notch in a finite-size 1D piezoelectric quasicrystalline bimaterial. Li et al. [21] investigated the interaction between a screw dislocation and an elliptical hole with two asymmetric cracks in a 1D piezoelectric QC in terms of the complex function method. Zhou and Li [22,23] studied a Yoffe-type moving crack and a penny-shaped dielectric crack in 1D piezoelectric QCs, respectively. By using the pseudo-Stroh formalism, Zhang et al. [24] and Sun et al. [25] studied the size effect and static deformation of 1D QC plate with piezoelectric effect, respectively.

On the other hand, another class of advanced materials: functionally graded (FG) materials concept originated in Japan in 1984, which are designed as the thermal barrier coatings to resist the temperature variation on the surface of the plane. FG materials are multi-functional materials with continuous and smoothly varying properties in one or more directions, which can alleviate and eliminate the abrupt changes at the interface. Thai and Kim [26] presented a comprehensive review of the development of various models for the modeling and analysis of FG plates and shells. Jha et al. [27] reviewed the stress, free vibration, and buckling analyses of FG plates. Swaminathan et al. [28] presented a review of the various methods employed to investigate the static, dynamic, and stability behaviour of FG plates. It is useful to idealize FG materials as continua with mechanical properties varying smoothly along the given direction. This idealization in the analytical models is usually achieved by assuming the material properties following the exponential law, power law, and arbitrary graded type. By using classical and first-order shear deformation plate theories, Kim et al. [29] presented a theoretical model for the bending, free vibration, and buckling analysis of FG porous micro-plates with the material constituents obeying a power-law distribution. Arshid et al. [30] investigated the asymmetric free vibration behaviour of shear deformable FG magneto-electro-thermo-elastic circular plates with material properties following exponential function through the thickness direction. Kashtalyan & Menshykova [31] studied the elastic deformation of an FG coating-substrate system of finite
thickness subjected to mechanical loading. Wang et al. [32] presented a meshless method for thermo-mechanical analysis of FG materials. Chen et al. [33] investigated the free vibration problem of a piezoelectric FG plate in terms of the state space method. Guo et al. [34] studied size-dependent behaviour of FG plates using modified couple-stress theory. Normally, QCs are used as coatings or films to cover the substrate materials, so the interface mechanical behaviour is a key issue in engineering. In addition to the inherent properties of QCs and piezoelectric materials, FG piezoelectric QCs with continuously varying material properties can also reduce or eliminate the interface problems, so as to enhance the durability and cut off the maintain cost of coatings and films. Therefore, it is necessary to study the mechanical behaviours of FG 1D piezoelectric QCs. However, to the best of the authors’ knowledge, there is no literature studying the axisymmetric bending of the circular plate made of FG 1D piezoelectric QCs.

Similar to conventional piezoelectric crystals, QCs are expected to be used as sensor and actuator intelligent structures and systems [35,36]. Different from piezoelectric crystals, piezoelectric effects in QCs are induced by both phonon and phason fields, which make the coupling relationships more complicated, especially for those of inhomogeneous piezoelectric QCs. Motivated by this, an axisymmetric analytical solution for FG 1D hexagonal piezoelectric QC circular plate is presented in this paper. We begin directly from the framework of the state space method, and then using the finite Hankel transform onto the state space vector yields an ordinary differential equation with variable coefficients. If the material properties being in an exponential function distribution along the thickness direction, the above ordinary differential equation can be transformed into an ordinary differential equation with constant coefficients. Finally, the exact solution of FG 1D hexagonal piezoelectric QC circular plate in the physical domain is obtained by way of the inverse Hankel transform. Numerical examples are considered to illustrate the influence of the exponential factor on the FG 1D hexagonal piezoelectric QC circular plate under different loading conditions.

2. Mathematical equations

Figure 1 shows a FG 1D piezoelectric QC circular plate in which \( a \) is the radius of the plate, \( h_j \) the \( j \)-th layer thickness and \( h \) the thickness of the plate. In the cylindrical coordinate system \((r, \theta, z)\), \( r, \theta \) and \( z \) stand for the radial, circumferential and axial coordinates, respectively. The \( z \)-axis pointing to the bottom surface from the top surface is along the symmetry axis of the circular plate, which is also the quasi-periodic and polarization directions of the 1D piezoelectric QCs. A number of simplifying assumptions about stress and displacement components are introduced in the classical thin, first-order and higher-order plate theories, which can cause the incompatible basic equation of elasticity. In order to overcome this problem and obtain the analytical solution for an arbitrary ratio, many scholars discarded those assumptions and verified the accuracy [37,38]. Following those studies, without any assumption about stress and displacement components, this paper starts from three-dimensional mechanical equations and sets up the state space equation. Due to the axisymmetry of the circular plate, all the physical quantities are independent of \( \theta \). In the absence of body forces and charge densities, the equilibrium equations for 1D hexagonal piezoelectric QCs are given by as follows [14,39]:

\[
\begin{align*}
\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta \theta}}{r} &= 0, \\
\frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} &= 0, \\
\frac{\partial H_{zz}}{\partial r} + \frac{\partial H_{rz}}{\partial z} + \frac{H_{rr}}{r} &= 0, \\
\frac{\partial D_r}{\partial r} + \frac{\partial D_z}{\partial z} + \frac{D_r}{r} &= 0,
\end{align*}
\]  

(2.1)

where \( \sigma_{rr}, \sigma_{\theta \theta}, \sigma_{zz} \) and \( \sigma_{rz} \) are phonon stresses; \( H_{zz} \) and \( H_{rz} \) are phason stresses; \( D_r \) and \( D_z \) are electric displacements. According to [36,39], the constitutive relations for 1D hexagonal
piezoelectric QCs in the cylindrical coordinate system are

\[
\begin{align*}
\sigma_{rr} &= C_{11}\varepsilon_{rr} + C_{12}\varepsilon_{\theta\theta} + C_{13}\varepsilon_{zz} + R_1 w_{zz} - e_{31} E_z, \\
\sigma_{\theta\theta} &= C_{12}\varepsilon_{rr} + C_{11}\varepsilon_{\theta\theta} + C_{13}\varepsilon_{zz} + R_1 w_{zz} - e_{31} E_z, \\
\sigma_{zz} &= C_{13}\varepsilon_{rr} + C_{13}\varepsilon_{\theta\theta} + C_{33}\varepsilon_{zz} + R_2 w_{zz} - e_{33} E_z, \\
\sigma_{rz} &= \sigma_{zr} = 2C_{44}\varepsilon_{zr} + R_3 w_{zr} - e_{15} E_r, \\
H_{zr} &= 2R_3 e_{zr} + K_2 w_{zr} - d_{15} E_r, \\
H_{zz} &= R_1 \varepsilon_{rr} + R_1 \varepsilon_{\theta\theta} + R_2 w_{zz} - d_{33} E_z, \\
D_r &= 2e_{15} e_{zr} + d_{15} w_{zr} + \xi_{11} E_r, \\
D_z &= e_{31}\varepsilon_{rr} + e_{31}\varepsilon_{\theta\theta} + e_{33}\varepsilon_{zz} + d_{33} w_{zz} + \xi_{33} E_z, \\
\end{align*}
\]

(2.2)

where \(C_{11}, C_{12}, C_{13}, C_{33}\) and \(C_{44}\) are phonon elastic constants; \(K_1\) and \(K_2\) refer to phason elastic constants; \(R_1, R_2\) and \(R_3\) denote phonon–phason coupling elastic constants; \(e_{31}, e_{33}\) and \(e_{15}\) stand for phonon piezoelectric constants; \(d_{33}\) and \(d_{15}\) are phason piezoelectric constants; \(\xi_{11}\) and \(\xi_{33}\) are dielectric constants; \(\varepsilon_{rr}, \varepsilon_{\theta\theta}, \varepsilon_{zz}\) and \(\varepsilon_{zr}\) are phonon strains; \(w_{zz}\) and \(w_{zr}\) are phason strains; \(E_z\) and \(E_r\) are electric field intensities.

The generalized strain–displacement relations of 1D piezoelectric QCs with respect to the cylindrical coordinates can be written as follows [14,39]:

\[
\begin{align*}
\varepsilon_{rr} &= \frac{\partial u_r}{\partial r}, \varepsilon_{\theta\theta} = \frac{u_r}{r}, \varepsilon_{zz} = \frac{\partial u_z}{\partial z}, \varepsilon_{zr} = \varepsilon_{zr} = 0.5 \left( \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \right), \\
and \quad w_{zz} &= \frac{\partial w_z}{\partial z}, w_{zr} = \frac{\partial w_z}{\partial r}, E_r = -\frac{\partial \phi}{\partial r}, E_z = -\frac{\partial \phi}{\partial z},
\end{align*}
\]

(2.3)

where \(u_r\) and \(u_z\) are phonon displacements; \(w_z\) is phason displacement; \(\phi\) denotes electric potential.
For simplicity of the solution process, introduce the following dimensionless quantities:

\[
\tilde{r} = \frac{r}{a}, \quad \tilde{z} = \frac{z}{h}, \quad \tilde{h}_j = \frac{h_j}{h}, \quad s = \frac{h}{a}, \quad \tilde{\bar{u}}_r = \frac{u_r}{h}, \quad \tilde{\bar{u}}_z = \frac{u_z}{h}, \quad \tilde{\bar{w}}_r = \frac{w_r}{h}, \quad \tilde{\bar{w}}_z = \frac{w_z}{h},
\]

\[
\tilde{\sigma}_{rr} = \frac{\sigma_{rr}}{C}, \quad \tilde{\sigma}_{\theta\theta} = \frac{\sigma_{\theta\theta}}{C}, \quad \tilde{\sigma}_{zz} = \frac{\sigma_{zz}}{C}, \quad \tilde{\sigma}_{rz} = \frac{\sigma_{rz}}{C}, \quad \tilde{H}_{zz} = \frac{H_{zz}}{C}, \quad \tilde{H}_{xz} = \frac{H_{xz}}{C},
\]

\[
\tilde{C}_{ij} = \frac{C_{ij}}{C}, \quad \tilde{R}_{ij} = \frac{R_{ij}}{C}, \quad \tilde{K}_{ij} = \frac{K_{ij}}{C}, \quad \tilde{\xi}_{ij} = \frac{\xi_{ij}}{\xi},
\]

and

\[
\tilde{e}_{ij} = \frac{e_{ij}}{\sqrt{C}\xi}, \quad \tilde{d}_{ij} = \frac{d_{ij}}{\sqrt{C}\xi}, \quad \tilde{D}_r = \frac{D_r}{C}, \quad \tilde{\phi} = \frac{\phi \sqrt{E/C}}{h},
\]

(2.4)

where \(C\) and \(\xi\) are constants with the dimensions of elastic and dielectric constant, respectively. They are taken as \(C_{11}^{(1)}\) and \(\xi_{33}^{(1)}\) in this paper, which refers to the phonon elastic constants and dielectric coefficient in the first layer, respectively; \(s\) is the thickness-to-span ratio of the circular plate. Then the basic equations for 1D hexagonal piezoelectric QCs can be rewritten as follows:

\[
\frac{\partial \tilde{\sigma}_{rr}}{\partial \tilde{r}} + \frac{1}{s} \frac{\partial \tilde{\sigma}_{rz}}{\partial \tilde{z}} + \tilde{\sigma}_{rr} - \tilde{\sigma}_{\theta\theta} = 0,
\]

\[
\frac{\partial \tilde{\sigma}_{rz}}{\partial \tilde{r}} + \frac{1}{s} \frac{\partial \tilde{\sigma}_{zz}}{\partial \tilde{z}} + \tilde{\sigma}_{rz} = 0,
\]

\[
\frac{\partial \tilde{H}_{xz}}{\partial \tilde{r}} + \frac{1}{s} \frac{\partial \tilde{H}_{zz}}{\partial \tilde{z}} + \tilde{H}_{xz} = 0,
\]

and

\[
\frac{\partial \tilde{D}_r}{\partial \tilde{r}} + \frac{1}{s} \frac{\partial \tilde{D}_z}{\partial \tilde{z}} + \tilde{D}_r = 0,
\]

(2.5)

and

\[
\tilde{\sigma}_{rr} = s \tilde{C}_{11} \frac{\partial \tilde{\bar{u}}_r}{\partial \tilde{r}} + s \tilde{C}_{12} \frac{\partial \tilde{\bar{u}}_r}{\partial \tilde{r}} + \tilde{C}_{13} \frac{\partial \tilde{\bar{u}}_z}{\partial \tilde{z}} + \tilde{R}_{1} \frac{\partial \tilde{\bar{w}}_z}{\partial \tilde{z}} + \tilde{\xi}_{31} \frac{\partial \tilde{\phi}}{\partial \tilde{r}},
\]

\[
\tilde{\sigma}_{\theta\theta} = s \tilde{C}_{12} \frac{\partial \tilde{\bar{u}}_r}{\partial \tilde{r}} + s \tilde{C}_{11} \frac{\partial \tilde{\bar{u}}_r}{\partial \tilde{r}} + \tilde{C}_{13} \frac{\partial \tilde{\bar{u}}_z}{\partial \tilde{z}} + \tilde{R}_{1} \frac{\partial \tilde{\bar{w}}_z}{\partial \tilde{z}} + \tilde{\xi}_{31} \frac{\partial \tilde{\phi}}{\partial \tilde{r}},
\]

\[
\tilde{\sigma}_{zz} = s \tilde{C}_{13} \frac{\partial \tilde{\bar{u}}_r}{\partial \tilde{r}} + s \tilde{C}_{13} \frac{\partial \tilde{\bar{u}}_r}{\partial \tilde{r}} + \tilde{C}_{33} \frac{\partial \tilde{\bar{u}}_z}{\partial \tilde{z}} + \tilde{R}_{2} \frac{\partial \tilde{\bar{w}}_z}{\partial \tilde{z}} + \tilde{\xi}_{33} \frac{\partial \tilde{\phi}}{\partial \tilde{r}},
\]

\[
\tilde{\sigma}_{rz} = \tilde{\sigma}_{zr} = \tilde{C}_{44} \left( s \frac{\partial \tilde{\bar{u}}_z}{\partial \tilde{r}} + \frac{\partial \tilde{\bar{u}}_r}{\partial \tilde{z}} \right) + s \tilde{R}_{3} \frac{\partial \tilde{\bar{w}}_z}{\partial \tilde{r}} + s \tilde{\xi}_{15} \frac{\partial \tilde{\phi}}{\partial \tilde{r}},
\]

\[
\tilde{H}_{xz} = s \tilde{R}_{1} \frac{\partial \tilde{\bar{u}}_r}{\partial \tilde{r}} + \tilde{R}_{1} \frac{\partial \tilde{\bar{u}}_r}{\partial \tilde{r}} + \tilde{K}_{1} \frac{\partial \tilde{\bar{w}}_z}{\partial \tilde{z}} + \tilde{\xi}_{31} \frac{\partial \tilde{\phi}}{\partial \tilde{r}},
\]

\[
\tilde{D}_r = \tilde{\xi}_{15} \left( s \frac{\partial \tilde{\bar{u}}_z}{\partial \tilde{r}} + \frac{\partial \tilde{\bar{u}}_r}{\partial \tilde{z}} \right) + s \tilde{R}_{1} \frac{\partial \tilde{\bar{w}}_z}{\partial \tilde{r}} - s \xi_{11} \frac{\partial \tilde{\phi}}{\partial \tilde{r}},
\]

and

\[
\tilde{D}_z = s \tilde{C}_{31} \frac{\partial \tilde{\bar{u}}_z}{\partial \tilde{r}} + \tilde{C}_{31} \frac{\partial \tilde{\bar{u}}_r}{\partial \tilde{z}} + \tilde{C}_{33} \frac{\partial \tilde{\bar{u}}_z}{\partial \tilde{z}} + s \tilde{\xi}_{31} \frac{\partial \tilde{\phi}}{\partial \tilde{r}}.
\]

(2.6)

3. State equations and Hankel transform

State space method converts three-dimensional elasticity equations into two independent state equations, i.e., primary variables and derived variables, which decouple and reduce the original equation. This method greatly simplifies the solution process, especially for the analysis of laminated structures. By setting primary variables \(\tilde{\bar{u}}_r, \tilde{\bar{u}}_z, \tilde{\bar{w}}_r, \tilde{\bar{w}}_z, \tilde{\bar{w}}_z, \tilde{\bar{w}}_z, \tilde{\bar{w}}_z\) and \(\tilde{\phi}\) as the state space vector for \(j\)-th layer, and utilizing Eqs. (2.5) and (2.6), the state space equation for \(j\)-th layer...
can be expressed as:

$$\frac{\partial \tilde{R}_j(\tilde{r}, \tilde{z})}{\partial \tilde{z}} = \begin{bmatrix} 0 & A_j \\ B_j & 0 \end{bmatrix} \tilde{R}_j(\tilde{r}, \tilde{z}),$$  \hspace{1cm} (3.1)

in which

$$\tilde{R}_j = [\tilde{u}_r \quad \tilde{\sigma}_{zz} \quad \tilde{H}_{zz} \quad \tilde{D}_z \quad \tilde{\sigma}_{rz} \quad \tilde{u}_z \quad \tilde{w}_z \quad \tilde{\phi}]^T,$$  \hspace{1cm} (3.2)

where ‘$T$’ is the transpose of matrix. The two $4 \times 4$ matrices $A_j$ and $B_j$ in equation (3.1) have the form of

$$A_j = \begin{bmatrix} \alpha_1 & -s & -\alpha_2 s & -\alpha_3 s \\ -s \left(\frac{1}{r} + \frac{\partial}{\partial r}\right) & 0 & 0 & 0 \\ -\alpha_2 \left(\frac{1}{r} + \frac{\partial}{\partial r}\right) & 0 & \alpha_4 \left(\frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2}\right) & \alpha_5 \left(\frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2}\right) \\ -\alpha_3 \left(\frac{1}{r} + \frac{\partial}{\partial r}\right) & 0 & \alpha_5 \left(\frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2}\right) & \alpha_6 \left(\frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2}\right) \end{bmatrix},$$  \hspace{1cm} (3.3)

and

$$B_j = \begin{bmatrix} \beta_1 \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2}\right) & \beta_2 \frac{\partial}{\partial r} & \beta_3 \frac{\partial}{\partial r} & \beta_4 \frac{\partial}{\partial r} \\ \beta_2 \left(\frac{1}{r} + \frac{\partial}{\partial r}\right) & \beta_5 & \beta_6 & \beta_7 \\ \beta_3 \left(\frac{1}{r} + \frac{\partial}{\partial r}\right) & \beta_6 & \beta_8 & \beta_9 \\ \beta_4 \left(\frac{1}{r} + \frac{\partial}{\partial r}\right) & \beta_7 & \beta_9 & \beta_{10} \end{bmatrix}.$$  \hspace{1cm} (3.4)

The other derived variables are obtained as follows:

$$\tilde{\sigma}_{rr}(\tilde{r}, \tilde{z}) = -\frac{1}{s} \left[ \beta_{11} \frac{\tilde{u}_r(\tilde{r}, \tilde{z})}{r} + \beta_1 \frac{\partial \tilde{u}_r(\tilde{r}, \tilde{z})}{\partial r} + \beta_2 \tilde{\sigma}_{zz}(\tilde{r}, \tilde{z}) + \beta_3 \tilde{H}_{zz}(\tilde{r}, \tilde{z}) + \beta_4 \tilde{D}_z(\tilde{r}, \tilde{z}) \right],$$

$$\tilde{\sigma}_{\theta\theta}(\tilde{r}, \tilde{z}) = -\frac{1}{s} \left[ \beta_1 \frac{\tilde{u}_r(\tilde{r}, \tilde{z})}{r} + \beta_{11} \frac{\partial \tilde{u}_r(\tilde{r}, \tilde{z})}{\partial r} + \beta_2 \tilde{\sigma}_{zz}(\tilde{r}, \tilde{z}) + \beta_3 \tilde{H}_{zz}(\tilde{r}, \tilde{z}) + \beta_4 \tilde{D}_z(\tilde{r}, \tilde{z}) \right],$$

$$\tilde{D}_r(\tilde{r}, \tilde{z}) = \frac{1}{s} \left[ \alpha_3 \tilde{\sigma}_{rz}(\tilde{r}, \tilde{z}) - \alpha_5 \frac{\partial \tilde{w}_z(\tilde{r}, \tilde{z})}{\partial r} - \alpha_6 \frac{\partial \tilde{\phi}(\tilde{r}, \tilde{z})}{\partial r} \right],$$

and

$$\tilde{H}_{rr}(\tilde{r}, \tilde{z}) = \frac{1}{s} \left[ \alpha_2 \tilde{\sigma}_{rz}(\tilde{r}, \tilde{z}) - \alpha_4 \frac{\partial \tilde{w}_z(\tilde{r}, \tilde{z})}{\partial r} - \alpha_5 \frac{\partial \tilde{\phi}(\tilde{r}, \tilde{z})}{\partial r} \right],$$  \hspace{1cm} (3.5)

where the parameters $\alpha_1, \alpha_2, \ldots, \alpha_6$ and $\beta_0, \beta_1, \ldots, \beta_{11}$ in equations (3.3)–(3.5) are given in Appendix A.

The finite Hankel transform is an effective method to handle the axisymmetric bending problems of a circular plate which is defined as follows [40]:

$$I_\mu[f(\tilde{r}, \tilde{z})] = \int_0^1 \tilde{f}(\tilde{r}, \tilde{z}) f_\mu(k\tilde{r}) d\tilde{r},$$  \hspace{1cm} (3.6)
where \( j_\mu(k\bar{r}) \) is the Bessel function of the first kind of order \( \mu \). Then, the state space vector related to the Hankel transform domain takes the form as follows:

\[
R_j(k, \bar{z}) = \begin{bmatrix}
U_r(k, \bar{z}) \\
S(k, \bar{z}) \\
H(k, \bar{z}) \\
D(k, \bar{z}) \\
T(k, \bar{z}) \\
U_z(k, \bar{z}) \\
W(k, \bar{z}) \\
F(k, \bar{z})
\end{bmatrix}_j = \begin{bmatrix}
J_1[\bar{u}_r(\bar{r}, \bar{z})] \\
J_0[\bar{\sigma}_{rz}(\bar{r}, \bar{z})] \\
J_0[\bar{H}_{zz}(\bar{r}, \bar{z})] \\
J_1[D_z(\bar{r}, \bar{z})] \\
J_1[\bar{\sigma}_{zz}(\bar{r}, \bar{z})] \\
J_0[\bar{u}_z(\bar{r}, \bar{z})] \\
J_0[\bar{u}_z(\bar{r}, \bar{z})] \\
J_0[\bar{\phi}(\bar{r}, \bar{z})]
\end{bmatrix}_j. \tag{3.7}
\]

Based on equations (3.6) and (3.7), the state space equation can be transformed into

\[
\frac{\partial R_j(k, \bar{z})}{\partial \bar{z}} = K_j(k)R_j(k, \bar{z}) + Q_j(k, \bar{z}), \tag{3.8}
\]

in which

\[
K_j(k) = \begin{bmatrix}
0 & 0 & 0 & 0 & \alpha_1 & \alpha_2k & \alpha_3k \\
0 & 0 & 0 & 0 & -sk & 0 & 0 \\
0 & 0 & 0 & 0 & -\alpha_2k & 0 & -\alpha_5k^2 & -\alpha_5k^2 \\
0 & 0 & 0 & 0 & -\alpha_3k & 0 & -\alpha_5k^2 & -\alpha_6k^2 \\
-\beta_1k^2 & -\beta_2k & -\beta_3k & -\beta_4k & 0 & 0 & 0 & 0 \\
\beta_2k & \beta_5 & \beta_6 & \beta_7 & 0 & 0 & 0 & 0 \\
\beta_3k & \beta_6 & \beta_8 & \beta_9 & 0 & 0 & 0 & 0 \\
\beta_4k & \beta_7 & \beta_9 & \beta_{10} & 0 & 0 & 0 & 0
\end{bmatrix}, \tag{3.9}
\]

and

\[
Q_j(k, \bar{z}) = \begin{bmatrix}
-s\bar{u}_z(1, \bar{z})J_1(k) - \alpha_2\bar{w}_z(1, \bar{z})J_1(k) - \alpha_3\phi(1, \bar{z})J_1(k) \\
-s\bar{\sigma}_{rz}(1, \bar{z})J_0(k) \\
-\alpha_2\bar{\sigma}_{rz}(1, \bar{z}) + \alpha_4 \frac{\partial \bar{w}_z(1, \bar{z})}{\partial \bar{r}} + \alpha_5 \frac{\partial \phi(1, \bar{z})}{\partial \bar{r}} \end{bmatrix} J_0(k) + [\alpha_4k\bar{w}_z(1, \bar{z}) + \alpha_5k\bar{\phi}(1, \bar{z})] J_1(k) \\
-\alpha_3\bar{\sigma}_{rz}(1, \bar{z}) + \alpha_5 \frac{\partial \bar{w}_z(1, \bar{z})}{\partial \bar{r}} + \alpha_6 \frac{\partial \bar{\phi}(1, \bar{z})}{\partial \bar{r}} \end{bmatrix} J_0(k) + [\alpha_5k\bar{w}_z(1, \bar{z}) + \alpha_6k\bar{\phi}(1, \bar{z})] J_1(k) \\
\left[ \beta_1 \frac{\partial \bar{u}_r(1, \bar{z})}{\partial \bar{r}} + \beta_1 \bar{u}_r(1, \bar{z}) + \beta_2\bar{\sigma}_{zz}(1, \bar{z}) + \beta_3\bar{H}_{zz}(1, \bar{z}) + \beta_4\bar{D}_z(1, \bar{z}) \right] J_1(k) - \beta_1k\bar{u}_r(1, \bar{z})J_0(k) \\
\beta_2\bar{u}_r(1, \bar{z})J_0(k) \\
\beta_3\bar{u}_r(1, \bar{z})J_0(k) \\
\beta_4\bar{u}_r(1, \bar{z})J_0(k)
\end{bmatrix} \tag{3.10}
\]

By setting \( \bar{r} = 1 \), three deduced variables in equation (3.5) can be derived as follows:

\[
s\bar{D}_r(1, \bar{z}) = \alpha_3\bar{\sigma}_{rz}(1, \bar{z}) - \alpha_5 \frac{\partial \bar{w}_z(1, \bar{z})}{\partial \bar{r}} - \alpha_6 \frac{\partial \bar{\phi}(1, \bar{z})}{\partial \bar{r}},
\]

\[
s\bar{\sigma}_{rr}(1, \bar{z}) = -\beta_1\bar{u}_r(1, \bar{z}) - \beta_1 \frac{\partial \bar{u}_r(1, \bar{z})}{\partial \bar{r}} - \beta_2\bar{\sigma}_{zz}(1, \bar{z}) - \beta_3\bar{H}_{zz}(1, \bar{z}) - \beta_4\bar{D}_z(1, \bar{z}),
\]

and

\[
\begin{align*}
s\bar{H}_{xr}(1, \bar{z}) &= \alpha_2\bar{\sigma}_{rz}(1, \bar{z}) - \alpha_4 \frac{\partial \bar{w}_z(1, \bar{z})}{\partial \bar{r}} - \alpha_5 \frac{\partial \bar{\phi}(1, \bar{z})}{\partial \bar{r}}. \tag{3.11}
\end{align*}
\]
By substituting equation (3.11) into equation (3.10), the matrix $Q_j(k, \tilde{z})$ in equation (3.10) can be reduced to:

$$Q_j(k, \tilde{z}) = \begin{bmatrix}
-\tilde{u}_z(1, \tilde{z})\tilde{f}_1(k) - \alpha_2\tilde{w}_z(1, \tilde{z})\tilde{f}_1(k) - \alpha_3\tilde{f}(1, \tilde{z})\tilde{f}_1(k) \\
-\tilde{s}\tilde{\sigma}_{rr}(1, \tilde{z})\tilde{f}_0(k) \\
-\tilde{s}\tilde{H}_{zz}(1, \tilde{z})\tilde{f}_0(k) + \alpha_4\tilde{k}\tilde{w}_z(1, \tilde{z})\tilde{f}_1(k) + \alpha_5\tilde{f}(1, \tilde{z})\tilde{f}_1(k) \\
-\tilde{s}\tilde{D}_y(1, \tilde{z})\tilde{f}_0(k) + \alpha_6\tilde{k}\tilde{w}_z(1, \tilde{z})\tilde{f}_1(k) + \alpha_7\tilde{f}(1, \tilde{z})\tilde{f}_1(k) \\
(\tilde{C}_{12} - \tilde{C}_{11})\tilde{s}^2\tilde{w}_z(1, \tilde{z})\tilde{f}_1(k) - \tilde{s}\tilde{\sigma}_{rr}(1, \tilde{z})\tilde{f}_1(k) - \tilde{\beta}_1\tilde{k}\tilde{w}_z(1, \tilde{z})\tilde{f}_0(k) \\
\tilde{\beta}_2\tilde{u}_r(1, \tilde{z})\tilde{f}_0(k) \\
\tilde{\beta}_3\tilde{u}_r(1, \tilde{z})\tilde{f}_0(k) \\
\tilde{\beta}_4\tilde{u}_r(1, \tilde{z})\tilde{f}_0(k)
\end{bmatrix}.$$  \hfill (3.12)

According to boundary conditions of circular plate presented by Ding et al. [40], the elastic simply supported condition can be extended to the 1D piezoelectric QC circular plate:

$$\tilde{u}_z(1, \tilde{z}) = 0, \tilde{w}_z(1, \tilde{z}) = 0, \tilde{\phi}(1, \tilde{z}) = 0, [(\tilde{C}_{11} - \tilde{C}_{12})\tilde{s}\tilde{u}_r(1, \tilde{z}) + \tilde{\sigma}_{rr}(1, \tilde{z})] = 0$$ and $\tilde{f}_0(k) = 0$. \hfill (3.13)

By virtue of the elastic simply supported condition in equation (3.13), the matrix $Q_j(k, \tilde{z})$ vanishes, and then equation (3.8) becomes a homogeneous equation as

$$\frac{\partial R_j(k, \tilde{z})}{\partial \tilde{z}} = K_j(k)R_j(k, \tilde{z}).$$ \hfill (3.14)

### 4. General solution for a functionally graded circular plate

Several approximations have been used to model the variation of properties in FG materials, and the exponential variation model has been found to be useful and convenient in solving the elasticity problems [41]. Therefore, it is assumed that the material properties vary exponentially along the thickness direction of the FG 1D piezoelectric QC circular plate, i.e.

$$C_{mnpq} = C_{mnpq}^0 e^{\eta z}, R_{mnpq} = R_{mnpq}^0 e^{\eta z}, K_{mnpq} = K_{mnpq}^0 e^{\eta z},$$

$$x_{mn} = x_{mn}^0 e^{\eta z}, \xi_{pmn} = \xi_{pmn}^0 e^{\eta z} d_{pmn} = d_{pmn}^0 e^{\eta z},$$ \hfill (4.1)

where all indices range over the values 1, 2 and 3, and the notation can be contracted into the short subscript fashion; $\eta$ is the exponential factor standing for the degree of the material gradient, and the material properties with superscript ‘0’ are the material properties at the top surface of each layer. Due to the matrix $K_j(k, \tilde{z})$ in equation (3.14) containing the variable material properties, it is not easy to obtain its solution in a straight way. To bypass this problem, equation (3.14) should be transferred into an ordinary differential equation with constant coefficients. To this end, following expressions are introduced

$$S(k, \tilde{z}) = \tilde{S}(k, \tilde{z}) e^{\eta \tilde{z}}, H(k, \tilde{z}) = \tilde{H}(k, \tilde{z}) e^{\eta \tilde{z}}$$

$$D(k, \tilde{z}) = \tilde{D}(k, \tilde{z}) e^{\eta \tilde{z}}, T(k, \tilde{z}) = \tilde{T}(k, \tilde{z}) e^{\eta \tilde{z}}.$$ \hfill (4.2)

Then we can establish a new set of state space vector for $j$-th layer in the Hankel domain as follows:

$$\hat{R}_j(k, \tilde{z}) = \begin{bmatrix}
\hat{U}_r(k, \tilde{z}) & \hat{S}(k, \tilde{z}) & \hat{H}(k, \tilde{z}) & \hat{D}(k, \tilde{z}) & \hat{T}(k, \tilde{z}) & \hat{U}_z(k, \tilde{z}) & \hat{W}(k, \tilde{z}) & \hat{F}(k, \tilde{z})
\end{bmatrix}^T.$$ \hfill (4.3)

Substituting equation (4.1) into equation (3.14), a homogeneous equation is derived as follows:

$$\frac{\partial \hat{R}_j(k, \tilde{z})}{\partial \tilde{z}} = \hat{K}_j(k)\hat{R}_j(k, \tilde{z}).$$ \hfill (4.4)
where the matrix $\hat{K}_j(k)$ is

$$
\hat{K}_j(k) = 
\begin{bmatrix}
0 & 0 & 0 & 0 & \alpha_1^0 & sk & \alpha_2^0 k & \alpha_3^0 k \\
0 & -\eta & 0 & 0 & -sk & 0 & 0 & 0 \\
0 & 0 & -\eta & 0 & -\alpha_2^0 k & 0 & -\alpha_4^0 k^2 & -\alpha_5^0 k^2 \\
0 & 0 & 0 & -\eta & -\alpha_3^0 k & 0 & -\alpha_5^0 k^2 & -\alpha_6^0 k^2 \\
\beta_1^0 k^2 & \beta_2^0 k & -\beta_3^0 k & \beta_4^0 k & -\beta_4^0 k & -\eta & 0 & 0 & 0 \\
\beta_2^0 k & \beta_3^0 & \beta_6^0 & \beta_7^0 & 0 & 0 & 0 & 0 \\
\beta_3^0 k & \beta_6^0 & \beta_8^0 & \beta_9^0 & 0 & 0 & 0 & 0 \\
\beta_4^0 k & \beta_7^0 & \beta_9^0 & \beta_{10}^0 & 0 & 0 & 0 & 0 \\
\end{bmatrix},
$$

and the parameters in equation (4.5) are listed in appendix B.

The solution for the ordinary differential equation (4.4) can be written as follows:

$$
\hat{R}_j(k, \bar{z}) = \hat{T}_j(k, \bar{z})\hat{R}_j(k, 0),
$$

where $\hat{R}_j(k, \bar{z})$ is the new set of state space vector, $\hat{T}_j(k, \bar{z})$ is the propagator matrix in the Hankel domain:

$$
\hat{T}_j(k, \bar{z}) = \exp[\hat{K}_j(k)\bar{z}],
$$

Similarly, by using the propagator matrix $T_j(k, \bar{z})$ corresponding to the original state space vector $R_j(k, \bar{z})$, equation (4.6) can be rewritten as follows:

$$
R_j(k, \bar{z}) = T_j(k, \bar{z})R_j(k, 0),
$$

in which $T_j(k, \bar{z}) = M\hat{T}_j(k, \bar{z})$, and

$$
M = 
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & e^{\theta \bar{z}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & e^{\theta \bar{z}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & e^{\theta \bar{z}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & e^{\theta \bar{z}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}.
$$

Assuming that the primary variables is continuous at the interface between $j$-th and $j+1$-th layer, then we have

$$
R_{j+1}(k, 0) = R_j(k, \bar{h}_j),
$$

and using the propagator matrix repeatedly, we can obtain the general solution for the FG 1D piezoelectric QC circular plate as follows:

$$
R_N(k, 1) = P(k)R_1(k, 0),
$$

where the propagator matrix $P(k)$ is

$$
P(k) = \prod_{j=1}^{N} T_j(k, \bar{h}_j).
$$
5. Loading condition of the functionally graded circular plate

It is assumed that the top and bottom surfaces of the FG circular plate are subjected to the stress and electric components. The elastic boundary conditions have the form as follows:

\[
\begin{align*}
\bar{\sigma}_{zz}(\bar{r}, 0) &= \sigma_0(\bar{r}), \quad \bar{\sigma}_{zz}(\bar{r}, 1) = \sigma_1(\bar{r}), \\
\bar{\sigma}_{rz}(\bar{r}, 0) &= \bar{\sigma}_{rz}(\bar{r}, 1) = 0, \\
\bar{H}_{zz}(\bar{r}, 0) &= \bar{H}_{zz}(\bar{r}, 1) = 0,
\end{align*}
\]

and the boundary conditions on the electric field in the case of the displacement vector are

\[
\bar{D}_z(\bar{r}, 0) = \bar{D}_z(\bar{r}, 1) = 0. \quad (5.2)
\]

Making use of equation (3.6), the boundary condition in equations (5.1) and (5.2) can be written as follows:

\[
S(k, 0) = \int_0^1 \bar{r} \sigma_0(\bar{r}) j_0(k \bar{r}) d\bar{r}, \quad S(k, 1) = \int_0^1 \bar{r} \sigma_1(\bar{r}) j_0(k \bar{r}) d\bar{r}. \quad (5.3)
\]

The general solution of equation (4.11) can be rewritten as follows:

\[
\begin{bmatrix}
U_t(k, 1) \\
S(k, 1) \\
H(k, 1) \\
D(k, 1) \\
T(k, 1) \\
U_z(k, 1) \\
W(k, 1) \\
F(k, 1)
\end{bmatrix} = \begin{bmatrix}
P_{11} & P_{12} & P_{13} & P_{14} & P_{15} & P_{16} & P_{17} & P_{18} \\
P_{21} & P_{22} & P_{23} & P_{24} & P_{25} & P_{26} & P_{27} & P_{28} \\
P_{31} & P_{32} & P_{33} & P_{34} & P_{35} & P_{36} & P_{37} & P_{38} \\
P_{41} & P_{42} & P_{43} & P_{44} & P_{45} & P_{46} & P_{47} & P_{48} \\
P_{51} & P_{52} & P_{53} & P_{54} & P_{55} & P_{56} & P_{57} & P_{58} \\
P_{61} & P_{62} & P_{63} & P_{64} & P_{65} & P_{66} & P_{67} & P_{68} \\
P_{71} & P_{72} & P_{73} & P_{74} & P_{75} & P_{76} & P_{77} & P_{78} \\
P_{81} & P_{82} & P_{83} & P_{84} & P_{85} & P_{86} & P_{87} & P_{88}
\end{bmatrix}^{-1}
\begin{bmatrix}
U_t(k, 0) \\
S(k, 0) \\
H(k, 0) \\
D(k, 0) \\
T(k, 0) \\
U_z(k, 0) \\
W(k, 0) \\
F(k, 0)
\end{bmatrix}, \quad (5.4)
\]

and then substituting equation (5.3) into equation (5.4), the unknowns on the top surface can be deduced as follows:

\[
\begin{bmatrix}
U_t(k, 0) \\
U_z(k, 0) \\
W(k, 0) \\
F(k, 0)
\end{bmatrix} = \begin{bmatrix}
P_{21} & P_{26} & P_{27} & P_{28} \\
P_{31} & P_{36} & P_{37} & P_{38} \\
P_{41} & P_{46} & P_{47} & P_{48} \\
P_{51} & P_{56} & P_{57} & P_{58}
\end{bmatrix}^{-1}
\begin{bmatrix}
S(k, 1) \\
H(k, 1) \\
D(k, 1) \\
T(k, 1)
\end{bmatrix}
\]

\[
- \begin{bmatrix}
P_{21} & P_{26} & P_{27} & P_{28} \\
P_{31} & P_{36} & P_{37} & P_{38} \\
P_{41} & P_{46} & P_{47} & P_{48} \\
P_{51} & P_{56} & P_{57} & P_{58}
\end{bmatrix}^{-1}
\begin{bmatrix}
P_{22} & P_{23} & P_{24} & P_{25} \\
P_{32} & P_{33} & P_{34} & P_{35} \\
P_{42} & P_{43} & P_{44} & P_{45} \\
P_{52} & P_{53} & P_{54} & P_{55}
\end{bmatrix} \begin{bmatrix}
S(k, 0) \\
H(k, 0) \\
D(k, 0) \\
T(k, 0)
\end{bmatrix}. \quad (5.5)
\]

With the known $R_1(k, 0)$ and using propagator matrix in equation (4.8) repeatedly, the state space vector at any given $\bar{z}$-level can be obtained. Then, the physical quantities in the physical domain corresponding to the state space vector $\bar{R}_1(\bar{r}, \bar{z})$ can be obtained by using the inverse Hankel transform as follows [42]:

\[
\bar{u}_r(\bar{r}, \bar{z}) = \sum_i U_t(k_i, \bar{z}) \frac{J_1(k_i \bar{r})}{[J_1(k_i)]^2}, \quad \bar{\sigma}_{zz}(\bar{r}, \bar{z}) = \sum_i S(k_i, \bar{z}) \frac{J_0(k \bar{r})}{[J_1(k_i)]^2},
\]

\[
\bar{H}_{zz}(\bar{r}, \bar{z}) = \sum_i H(k_i, \bar{z}) \frac{J_0(k \bar{r})}{[J_1(k_i)]^2}, \quad \bar{D}_z(\bar{r}, \bar{z}) = \sum_i D(k_i, \bar{z}) \frac{J_0(k \bar{r})}{[J_1(k_i)]^2},
\]

\[
\bar{\sigma}_{rz}(\bar{r}, \bar{z}) = \sum_i T(k_i, \bar{z}) \frac{J_1(k_i \bar{r})}{[J_1(k_i)]^2}, \quad \bar{u}_z(\bar{r}, \bar{z}) = \sum_i U_z(k_i, \bar{z}) \frac{J_0(k \bar{r})}{[J_1(k_i)]^2},
\]

\[
\bar{\omega}_z(\bar{r}, \bar{z}) = \sum_i W(k_i, \bar{z}) \frac{J_0(k \bar{r})}{[J_1(k_i)]^2}, \quad \bar{\phi}(\bar{r}, \bar{z}) = \sum_i F(k_i, \bar{z}) \frac{J_0(k \bar{r})}{[J_1(k_i)]^2},
\]

and

\[
\bar{\omega}_z(\bar{r}, \bar{z}) = \sum_i W(k_i, \bar{z}) \frac{J_0(k \bar{r})}{[J_1(k_i)]^2}, \quad \bar{\phi}(\bar{r}, \bar{z}) = \sum_i F(k_i, \bar{z}) \frac{J_0(k \bar{r})}{[J_1(k_i)]^2},
\]

\[
\bar{\omega}_z(\bar{r}, \bar{z}) = \sum_i W(k_i, \bar{z}) \frac{J_0(k \bar{r})}{[J_1(k_i)]^2}, \quad \bar{\phi}(\bar{r}, \bar{z}) = \sum_i F(k_i, \bar{z}) \frac{J_0(k \bar{r})}{[J_1(k_i)]^2},
\]
and the second variables are

$$
\sigma_{rr}(\vec{r}, \vec{z}) = e^{\text{ii} \alpha} \left[ \left(C_{0}^{0} - C_{11}^{0}\right)s \frac{\vec{u}(\vec{r}, \vec{z})}{\vec{r}} - \frac{2\rho_{0}^{0}}{s} \sum_{j} k_{j} U_{r}(k_{j}, \vec{z}) \frac{\bar{J}_{0}(k_{j})}{\bar{J}_{1}(k_{j})^2} \right] 
- \frac{\beta_{0}^{0}}{s} \bar{\sigma}_{zz}(\vec{r}, \vec{z}) - \frac{\rho_{3}^{0}}{s} \bar{H}_{zz}(\vec{r}, \vec{z}) - \frac{\beta_{4}^{0}}{s} \bar{D}_{z}(\vec{r}, \vec{z}),
$$

$$
\sigma_{\theta\theta}(\vec{r}, \vec{z}) = e^{\text{ii} \alpha} \left[ \left(C_{0}^{0} - C_{11}^{0}\right)s \frac{\vec{u}(\vec{r}, \vec{z})}{\vec{r}} - \frac{2\rho_{11}^{0}}{s} \sum_{j} k_{j} U_{r}(k_{j}, \vec{z}) \frac{\bar{J}_{0}(k_{j})}{\bar{J}_{1}(k_{j})^2} \right] 
- \frac{\beta_{0}^{0}}{s} \bar{\sigma}_{zz}(\vec{r}, \vec{z}) - \frac{\rho_{3}^{0}}{s} \bar{H}_{zz}(\vec{r}, \vec{z}) - \frac{\beta_{4}^{0}}{s} \bar{D}_{z}(\vec{r}, \vec{z}),
$$

$$
\bar{D}_{r}(\vec{r}, \vec{z}) = \frac{\alpha_{0}^{0}}{s} \bar{\sigma}_{zz}(\vec{r}, \vec{z}) + e^{\text{ii} \alpha} \left[ \frac{2\alpha_{0}^{0}}{s} \sum_{j} k_{j} W(k_{j}, \vec{z}) \frac{\bar{J}_{1}(k_{j})}{\bar{J}_{1}(k_{j})^2} + \frac{2\alpha_{6}^{0}}{s} \sum_{j} k_{j} F(k_{j}, \vec{z}) \frac{\bar{J}_{1}(k_{j})}{\bar{J}_{1}(k_{j})^2} \right],
$$

$$
\bar{H}_{zz}(\vec{r}, \vec{z}) = \frac{\alpha_{0}^{0}}{s} \bar{\sigma}_{zz}(\vec{r}, \vec{z}) + e^{\text{ii} \alpha} \left[ \frac{2\alpha_{0}^{0}}{s} \sum_{j} k_{j} W(k_{j}, \vec{z}) \frac{\bar{J}_{1}(k_{j})}{\bar{J}_{1}(k_{j})^2} + \frac{2\alpha_{6}^{0}}{s} \sum_{j} k_{j} F(k_{j}, \vec{z}) \frac{\bar{J}_{1}(k_{j})}{\bar{J}_{1}(k_{j})^2} \right].
$$

(5.7)

### 6. Numerical results

In this section, the validation study is first presented to verify the accuracy of the presented solution. Then numerical calculations are conducted to quantitatively study the influence of exponential factor on a single-layer FG plate under loading I, as well as the dependence of a double-layer FG plate under loading II on exponential factor.

(a) **Validation study**

In order to verify the present exact solution, considering a FG transversely isotropic circular plate with elastic simply supported boundary condition, and the material properties follow an exponential law distribution along the thickness direction. For the sake of comparison between the degenerated results in this paper with the results given by Wang *et al.* [43], the material properties, geometric sizes, exponential factor and loading conditions are the same as those in [43], i.e. (i) material properties: $C_{11}^{0} = 139$ GPa, $C_{12}^{0} = 77.8$ GPa, $C_{33}^{0} = 74.3$ GPa, $C_{44}^{0} = 115$ GPa, $C_{13}^{0} = 25.6$ GPa; (ii) thickness to span ratio: $s = 0.2$; (iii) exponential factor: $\eta = 5$ and (iv) top surface pressure $\sigma_{0}(\vec{r}) = -J_{0}(k_{1}\vec{r})$. The deflection factor $a = u_{z}(0, 0.5)s$ is presented by [43] is 0.2331, and the result using the present solution is 0.2323, which shows a good agreement.

(b) **Effect of exponential factor on single-layer circular plate under loading I**

In this subsection, we present the influence of exponential factor on the phonon, phason and electric fields of the axisymmetric FG 1D piezoelectric QC single-layer circular plate. The loading boundary conditions in equation (5.1) are assumed to take the form as (Loading I):

$$
\sigma_{0}(\vec{r}) = p_{0} J_{0}(k_{1}\vec{r}), \sigma_{1}(\vec{r}) = p_{1} J_{0}(k_{1}\vec{r}).
$$

(6.1)

If $k_{i}$ is the zero of the Bessel function $J_{\mu}(k_{i})$ or $J'_{\mu}(k_{i})$, the function system $J_{\mu}(k_{i}\vec{r})$ on the interval $[0, 1]$ satisfies the following orthogonal relation:

$$
\int_{0}^{1} \bar{r} J_{\mu}(k_{i}\bar{r}) J'_{\mu}(k_{g}\bar{r}) d\bar{r} = \left\{ \begin{array}{ll}
0 & i \neq g, \\
\bar{I}_{\bar{g}} & i = g,
\end{array} \right.
$$

and

$$
\bar{I}_{\bar{g}} = \begin{cases}
0.5[1 - (\mu/k_{i})^2] J_{\mu}(k_{i})^2 & J_{\mu}(k_{i}) = 0 \\
J_{\mu+1}(k_{i})^2 & J'_{\mu}(k_{i}) = 0.
\end{cases}
$$

(6.3)
Table 1. Material properties of 1D hexagonal piezoelectric QC.

<table>
<thead>
<tr>
<th>Material properties of 1D hexagonal piezoelectric QC</th>
</tr>
</thead>
<tbody>
<tr>
<td>phonon elastic (×10^9 N m⁻²): C^p_{ij}</td>
</tr>
<tr>
<td>phonon elastic (×10^9 N m⁻²): K^p</td>
</tr>
<tr>
<td>phonon–phason coupling (×10^9 N m⁻²): R^p_1 = −1.50, R^p_2 = 1.20, R^p_3 = 1.20</td>
</tr>
<tr>
<td>piezoelectric (C m⁻²): e^0_{ij}</td>
</tr>
<tr>
<td>dielectric (×10⁻12 F N⁻¹m⁻²): ⋆_ij</td>
</tr>
</tbody>
</table>

Considering the loading boundary conditions in equation (6.1) and the orthogonal relation in equation (6.2), there is only one non-zero term left in equations (5.6) and (5.7).

The material properties of 1D hexagonal piezoelectric QC are shown in Table 1 [36]. The thickness to span ratio of FG circular plate is s = 0.2. The mechanical load is p_0 = −1 and p_1 = 1, i.e. \( \sigma_0(\bar{r}) = −J_0(k_1 \bar{r}) \) and \( \sigma_1(\bar{r}) = J_0(k_1 \bar{r}) \), where \( k_1 \) is the first zero point of \( J_0(k) \) under the elastic simply supported boundary condition. Different exponential factors are chosen as \( \eta = −4, −2, 0, 2, 4 \) to unravel the impact of material inhomogeneity on FG circular plate.

Figure 2 shows the distributions of dimensionless phonon and phason stresses along the thickness direction with different exponential factors \( \eta \). The values of phonon stress \( \bar{\sigma}_{zz} \) in figure 2a and phason stress \( \bar{H}_{zz} \) in figure 2c at top and bottom surfaces are identical with the loading boundary conditions. It can be observed from figure 2a that the stress neutral layer of the FG circular plate moves up to the top surface by decreasing \( \eta \). Phonon stress \( \bar{\sigma}_{rr} \) in figure 2b is linear with related to homogeneous materials \( \eta = 0 \) and nonlinear corresponding to \( \eta = 0 \). Another observation is that the maximum value of \( \bar{\sigma}_{rr} \) appears at the top surface \( \bar{z} = 0 \) when \( \eta < 0 \) and at the bottom surface \( \bar{z} = 1 \) when \( \eta > 0 \). Compared with phonon stresses \( \bar{\sigma}_{zz} \) and \( \bar{\sigma}_{rr} \), the magnitudes of phason stresses \( \bar{H}_{zz} \) and \( \bar{H}_{rr} \) (figure 2d) are smaller than those phonon stresses. Besides, the maximum values of \( \bar{H}_{zz} \) and \( \bar{H}_{rr} \) occur at different \( z \)-levels with varied \( \eta \), which may influence the wear resistance of the FG 1D piezoelectric QC circular plate. Due to the symmetric boundary condition on both surfaces and the material distribution characteristics, phonon and phason stresses with \( \eta \) and \( −\eta \) are central symmetrical about the point \( (\bar{r}, \bar{z}) = (\bar{r}, 0.5) \).

Figure 3 depicts the dependence of dimensionless phonon and phason displacements on the exponential factor \( \eta \). Phonon displacement \( \bar{u}_{zz} \) in figure 3a increases with decreasing \( \eta \); the reason is that \( \eta > 0 \) corresponds to the graded stiff materials and \( \eta < 0 \) refers to the graded soft materials. The linear distribution of phonon displacement \( \bar{u}_{zz} \), in figure 3b follows a similar trend of \( \bar{u}_{zz} \) with related to \( \eta \). Phason displacement \( \bar{w}_{zz} \) reaches the maximum at both surfaces when \( \eta = −4 \).

Figure 4 presents the variations of dimensionless electric displacement and electric potential along the thickness direction with different exponential factors \( \eta \). Electric displacement \( \bar{D}_{rr} \) (figure 4a) and electric potential \( \bar{\phi} \) (figure 4b) with \( \eta = 0 \) are symmetrical about \( \bar{z} = 0.5 \), while \( \bar{D}_{rr} \) and \( \bar{\phi} \) with \( \eta = 0 \) don’t follow the same law because of the uneven distribution of material properties. \( \bar{\phi} \) increases with decreasing \( \eta \) due to the influence of the variable material properties on the phonon–phason–electric coupling phenomenon.

(c) Effect of exponential factor on two-layered circular plate under loading II

Consider a FG 1D piezoelectric QC two-layered circular plate with bottom layer being homogeneous QCs and top layer being inhomogeneous QCs. The thickness of substrate homogeneous layer is as twice as that of the top inhomogeneous layer. The thickness to span ratio of FG circular plate and the exponential factors in the top layer are same with those in Example 1, while exponential factor is zero in substrate layer.

A non-uniform loading \( \sigma_0(\bar{r}) = p_0(1 − \bar{r}^2) \) (Loading II) on the top surface is considered in this subsection and set \( p_0 = 1 \). According to the Fourier–Bessel expansion law [42], if \( k_1, k_2, k_3, \ldots \), are the positive roots of the Bessel function \( J_\mu(k_i) \) or \( J'_\mu(k_i) \), meantime function \( f(\bar{r}) \) are continuous
with a limited number of maxima and minima on the interval $[0, 1]$, we have [43]

$$f(\bar{r}) = \sum_{i=1}^{\infty} C_i J_{\mu}(k_i \bar{r}), \quad (6.4)$$

where

$$C_i = \begin{cases} 2 \left[ J_{\mu+1}(k_i) \right]^2 \int_{0}^{1} \bar{r} f(\bar{r}) J_{\mu}(k_i \bar{r}) d\bar{r} & J_{\mu}(k_i) = 0 \\ \frac{2}{[1 - (\mu/k_i)^2]} \left[ J_{\mu}(k_i) \right]^2 \int_{0}^{1} \bar{r} f(\bar{r}) J_{\mu}(k_i \bar{r}) d\bar{r} & J_{\mu}'(k_i) = 0 \end{cases} \quad (6.5)$$

Hence the non-uniform loading $\sigma_0(\bar{r}) = p_0(1 - \bar{r}^2)$ can be obtained through the superposition principle using equation (6.4), so do other general distribution loadings.

It is essential to truncate the Fourier–Bessel expansion in equation (6.4) after a finite number of terms in computation; therefore, the variation of the truncation error in $\sigma_{zz}$ and $u_z$ at $(\bar{z}, \bar{r}) = (0.5, 0)$ for different numbers of the series is presented in Table 2 to examine the convergence of the series, in which $s = 0.2$, $\eta = 2$ in the top layer and $\eta = 0$ in the bottom layer. It can be observed from Table 2 that the results converge rapidly and reach the almost convergent outcome with only the first fifteen items. In order to balance both the accuracy and computational cost, the first 20 items are chosen in this subsection.

Figure 5 illustrates the influence of exponential factor on the phonon and phason fields of the FG two-layered circular plate subjected to a top surface non-uniform loading. In-plane phonon stress $\bar{\sigma}_{rr}$ (figure 5a) and phason stress $\bar{H}_{zz}$ (figure 5b) are continuous at the interfaces when $\eta = 0$, while they are discontinuous when $\eta \neq 0$ due to the change of material constants of FG QCs.
Figure 3. Influence of the exponential factor on phonon and phason displacements of the FG single-layer circular plate. (Online version in colour.)

Besides, the discontinuous phenomena of $\bar{\sigma}_{rr}$ and $\bar{H}_{zr}$ alleviate by decreasing $\eta$. The phonon displacement $\bar{u}_z$ in figure 5c and phason displacement $\bar{w}_z$ in figure 5d are continuous at the interfaces, and their magnitudes vary with the variation of the degree of the material gradient.

Figure 6 displays the distribution of dimensionless electric displacement and electric potential along the thickness of the FG QC circular plate for different $\eta$. The interface discontinuity of electric displacement $\bar{D}_r$ (figure 6a) with $\eta = -4$ is apparently weaker than that with $\eta = 4$, which can enhance the durability of the FG QC circular plate. The continuous electric potential $\bar{\phi}$ with $\eta = -4$ in figure 6b has the maximum, which is useful for the design of the piezoelectric sensors and actuators.
Figure 5. Influence of the exponential factor on phonon and phason fields of the FG two-layered circular plate. (Online version in colour.)

Figure 6. Influence of the exponential factor on electric field of the FG two-layered circular plate. (Online version in colour.)

Table 2. The convergence study of the Fourier–Bessel.

<table>
<thead>
<tr>
<th>number of items</th>
<th>1</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_{zz}$</td>
<td>0.36467</td>
<td>0.36855</td>
<td>0.36853</td>
<td>0.36852</td>
<td>0.36852</td>
<td>0.36852</td>
<td>0.36852</td>
</tr>
</tbody>
</table>
7. Conclusion

In this paper, an axisymmetric bending analysis model of FG 1D hexagonal piezoelectric QC circular plate with elastic simply supported boundary condition is established using the state space method. By virtue of finite Hankel transform, the state space vector in the physical domain can be transformed into those in the Hankel domain. Assuming that the material properties follow an exponential function along the z-direction, we obtain an ordinary differential equation with constant coefficients. Applying the propagator matrix and the inverse Hankel transform formalism, the axisymmetric solution for FG 1D hexagonal piezoelectric QC circular plate is derived. The validation study shows that the present solution has a good agreement with the previous solution. The parametric study reveals some vital mechanisms, which are: (1) the neutral layer move up to the top surface by decreasing the exponential factor \( \eta \), which is induced by the variation of the material properties; (2) phonon stresses and electric displacement with \( \eta \) and those corresponding to \( -\eta \) are central symmetrical about the point \((\bar{r}, \bar{z}) = (\bar{r}, 0.5)\), phason stresses follow the similar law. Moreover, the maximum of phason stresses corresponding to \( \eta > 0 \) occur at the lower position of the circular plate, while those with \( \eta < 0 \) appear at the upper position; (3) phonon displacements, phason displacements and electric potential are symmetrical about \( \bar{z} = 0.5 \) when \( \eta = 0 \), while they are not symmetrical when \( \eta \neq 0 \). Furthermore, for any given \( \eta \), the z-direction phonon displacement of the circular plate is along one direction, but both positive and negative z-direction phason displacements exist; (4) the interface discontinuities of in-plane stresses and electric components can reduce along with decreasing \( \eta \), and the interface gap of phason stress with \( \eta \) is smaller than that with \( -\eta \), because \( \eta < 0 \) refers to the graded soft. In summary, the present results can not only accurately predict the axisymmetric bending behaviour of FG 1D piezoelectric QC circular plate so as to optimize their properties, but also provide the efficient method to minimize the mismatch of material properties in coating-substrate bonding.

Data accessibility. This work does not have any experimental data and the numerical calculation can be found in the electronic supplementary material.

Authors’ contributions. Y. Li conceived of the study, designed the study, coordinated the study, and drafted the manuscript; Y. Li participated in the design of the study and helped with numerical calculations; Q. Qin participated in the design of the study and critically revised the manuscript; L. Yang participated in numerical analysis and critically revised the manuscript; L. Zhang participated in numerical analysis and helped draft the manuscript; Y. Gao participated in the design of the study and critically revised the manuscript. All authors gave final approval for publication and agree to be held accountable for the work performed therein.

Competing interests. We declare we have no competing interests.

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Appendix A

\[
\begin{align*}
\alpha_1 &= 1/\bar{C}_{44}; \\
\alpha_2 &= \bar{s}\bar{R}_{23}/\bar{C}_{44}; \\
\alpha_3 &= \bar{s}\bar{e}_{15}/\bar{C}_{44}; \\
\alpha_4 &= s^2(-\bar{C}_{44}\bar{R}_2 + \bar{R}_2^3/\bar{C}_{44}); \\
\alpha_5 &= s^2(-\bar{C}_{44}\bar{d}_{15} + \bar{e}_{15}\bar{R}_3)/\bar{C}_{44}; \\
\alpha_6 &= s^2(\bar{e}_{15}^2 + \bar{C}_{44}\bar{e}_{111}/\bar{C}_{44}); \\
\beta_0 &= \bar{e}_{33}\bar{R}_1 + \bar{C}_{33}(\bar{d}_{33}^2 + \bar{R}_1\bar{e}_{33}) - \bar{R}_2(2\bar{d}_{33}\bar{e}_{33} + \bar{R}_2\bar{e}_{33}); \\
\beta_1 &= (-\bar{C}_{33}\bar{d}_{31}^2\bar{K}_1 + 2\bar{C}_{33}\bar{d}_{33}\bar{e}_{31}\bar{R}_1 + \bar{e}_{33}\bar{R}_2^2 - 2\bar{e}_{31}\bar{e}_{33}\bar{R}_1\bar{R}_2 + \bar{e}_{31}\bar{R}_2^2 + \bar{C}_{33}\bar{R}_1^2\bar{e}_{31} + \bar{C}_{13}\bar{R}_2^2 + \bar{C}_{13}\bar{R}_1\bar{e}_{31}) \\
&\quad + 2\bar{C}_{13}\bar{e}_{31}\bar{e}_{33}\bar{R}_1 - 2\bar{C}_{13}\bar{d}_{33}\bar{e}_{33}\bar{R}_1 - 2\bar{C}_{13}\bar{d}_{33}\bar{e}_{31}\bar{R}_2 - 2\bar{C}_{13}\bar{R}_1\bar{R}_2\bar{e}_{33} \\
&\quad - \bar{C}_{11}\bar{e}_{31}\bar{K}_1 - \bar{C}_{11}\bar{C}_{33}\bar{d}_{33}^2 - \bar{C}_{11}\bar{C}_{33}\bar{K}_1\bar{e}_{31} + 2\bar{C}_{11}\bar{R}_2\bar{d}_{33}\bar{e}_{33} + \bar{C}_{11}\bar{R}_2^2\bar{e}_{31})/\beta_0; \\
\beta_2 &= (-\bar{C}_{13}\bar{d}_{33}^2 - \bar{e}_{31}\bar{e}_{31}\bar{K}_1 + \bar{d}_{33}\bar{e}_{33}\bar{R}_1 + \bar{d}_{33}\bar{e}_{33}\bar{R}_2 - \bar{C}_{13}\bar{K}_1\bar{e}_{33} + \bar{R}_1\bar{R}_2\bar{e}_{33})s/\beta_0; \\
\beta_3 &= (\bar{C}_{13}\bar{d}_{33}\bar{e}_{33} - \bar{e}_{33}\bar{R}_1 + \bar{e}_{31}\bar{R}_2 + \bar{C}_{13}\bar{R}_2\bar{e}_{33} - \bar{C}_{33}\bar{d}_{33}\bar{e}_{31} - \bar{C}_{33}\bar{R}_2\bar{e}_{33})s/\beta_0; \\
\beta_4 &= (\bar{C}_{33}\bar{e}_{31}\bar{K}_1 - \bar{C}_{13}\bar{R}_2\bar{R}_1 - \bar{C}_{33}\bar{d}_{33}\bar{R}_1 + \bar{C}_{13}\bar{d}_{33}\bar{R}_2 + \bar{e}_{33}\bar{R}_1\bar{R}_2 - \bar{e}_{31}\bar{R}_2)/\beta_0;
\end{align*}
\]
Appendix B

\[ \alpha_0^1 = 1/C_{44}^0; \alpha_0^2 = s/C_{44}^0 \]

\[ \beta_0 = \frac{s^2(-C_{44}^0 e_1^0 + c_{15}^0 R_{13}^0)}{C_{44}^0}; \alpha_0^6 = s^2(e_1^0)^2 + C_{44}^0 e_{11}/C_{44}^0; \]

\[ \beta_0^0 = (e_{33}^0)^2 R_{13}^0 + (e_{33}^0)^2 K_{13}^0 - R_{13}^3 (2d_{33}^0 /\bar{\beta} + R_{13}^0 \bar{e}_{33}^0); \]

\[ \beta_0^1 = [-C_{33}^0 (e_{33}^0)^2 K_{13}^0 + 2c_{33}^0 d_{33}^0 e_{31}^0 R_{13}^0 + (e_{33}^0)^2 (R_{13}^0)^2 - 2c_{31}^0 d_{33}^0 R_{13}^0 R_{13}^0 + (e_{31}^0)^2 (R_{13}^0)^2 + C_{33}^0 (R_{13}^0)^2 \bar{e}_{33}^0 + C_{33}^0 (L_{13}^0)^2 C_{13}^0 - 2c_{31}^0 d_{33}^0 R_{13}^0 R_{13}^0 - C_{31}^0 (e_{31}^0)^2 K_{13}^0 - C_{11}^0 (e_{33}^0)^2 \bar{C}_{13}^0 + 2c_{11}^0 d_{33}^0 e_{31}^0 \bar{C}_{13}^0 + C_{11}^0 (2\bar{e}_{33}^0 + C_{33}^0 \bar{e}_{33}^0)^2 \bar{C}_{13}^0 /\beta_0^0; \]

\[ \beta_0^2 = [-C_{33}^0 (e_{33}^0)^2 - c_{31}^0 (e_{33}^0)^2 + e_{33}^0 \bar{R}_{13}^0 \bar{C}_{13}^0 R_{13}^0 + 2c_{33}^0 e_{31}^0 R_{13}^0 R_{13}^0 - C_{33}^0 (e_{31}^0)^2 \bar{C}_{13}^0 R_{13}^0 - C_{33}^0 (e_{33}^0)^2 \bar{e}_{33}^0 + 2c_{33}^0 e_{31}^0 R_{13}^0 R_{13}^0] /\beta_0^0; \]

\[ \beta_0^4 = [C_{33}^0 (e_{33}^0)^2 - C_{33}^0 (e_{31}^0)^2 R_{13}^0 + c_{33}^0 (e_{31}^0)^2 R_{13}^0 + C_{33}^0 (e_{33}^0)^2 \bar{R}_{13}^0 R_{13}^0 - C_{31}^0 (e_{31}^0)^2 \bar{C}_{13}^0 R_{13}^0 - C_{33}^0 (e_{33}^0)^2 \bar{e}_{33}^0 + 2c_{33}^0 e_{31}^0 R_{13}^0 R_{13}^0 - 2c_{33}^0 (R_{13}^0)^2 + C_{31}^0 (R_{13}^0)^2 + c_{31}^0 (R_{13}^0)^2 + 2c_{31}^0 (R_{13}^0)^2] /\beta_0^0; \]

\[ \beta_0^5 = [C_{33}^0 (e_{33}^0)^2 + C_{33}^0 (e_{31}^0)^2 /\beta_0^0; \]

\[ \beta_0^6 = [-C_{33}^0 (e_{33}^0)^2 - 2c_{33}^0 (e_{33}^0)^2 \bar{R}_{13}^0 R_{13}^0 + c_{33}^0 (e_{33}^0)^2 \bar{R}_{13}^0 R_{13}^0 + C_{33}^0 (e_{33}^0)^2 \bar{C}_{13}^0 R_{13}^0 - C_{31}^0 (e_{33}^0)^2 \bar{C}_{13}^0 R_{13}^0 - C_{33}^0 (e_{31}^0)^2 \bar{e}_{33}^0 + 2c_{33}^0 e_{31}^0 R_{13}^0 R_{13}^0 - C_{33}^0 (e_{33}^0)^2 \bar{e}_{33}^0 + 2c_{33}^0 e_{31}^0 R_{13}^0 R_{13}^0 - 2c_{33}^0 (R_{13}^0)^2 + C_{31}^0 (R_{13}^0)^2 + c_{31}^0 (R_{13}^0)^2 + 2c_{31}^0 (R_{13}^0)^2] /\beta_0^0; \]

References


