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ON THE PLANE PIEZOELECTRIC PROBLEM OF A LOADED CRACK TERMINATING AT A MATERIAL INTERFACE

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ABSTRACT The plane problem of a crack terminating at the interface of a bimaterial piezoelectric, and loaded on its faces, is treated. The emphasis is placed on how to transform this problem into a non-homogeneous Hilbert problem. To make the derivation tractable, the concept of the axial conjugate is introduced and related to the complex conjugate. The angle between the crack line and the interface may be arbitrary. Numerical results are given to illustrate the stress singularity at crack tip.

KEY WORDS piezoelectricity, crack, axial conjugate, interface, singular integral equation

I. INTRODUCTION

The crack problem of piezoelectric material is of paramount importance for many electroelastic micromechanics models and numerical fracture mechanics, because a piezoceramic material often contains many visible cracks prior to their employment. The existence of these defects greatly affects the electric, dielectric, piezoelectric, elastic and mechanical properties of piezoceramics[1]. But the study in this area has received relatively little attention in the literature. This is probably due to the complexity of the constituent equations for the materials, which are inherently anisotropic and involve a large number of material constants. Parton[2] has considered the problem of a finite crack at the interface between two piezoelectric materials subjected to a far field uniform tension. Sosa and Pak[3] developed a three-dimensional solution for a semi-infinite crack in a piezoelectric material. More recently, Pak[4] investigated the electroelastic fields and the energy-release rate for a finite crack by the method of distributed dislocations and electric dipoles. Kuo and Barnett[5] and Suo et al. [6] solved the boundary value problems of electroelastic with interface cracks. Most of the above studies concentrated on the singularities at the tips of an interface crack. However the problem of a crack terminating at, and at an arbitrary angle to the interface between two piezoelectric materials dose not seem to have been studied. In the following sections, the explicit solutions of an arbitrarily-oriented crack terminating at the interface be-

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tween dissimilar piezoelectric materials will be derived by using the extended Stroh formalism[7] and the concept of axial conjugate. The present problem will be transformed into a special case of the Hilbert problem and then its solution may easily be written out. Some numerical results are given for illustrating the stress singularity at crack tips.

Ⅱ. THE EXTENDED STROH FORMALISM

In this section, the extended Stroh formalism[7] used to treat crack problems in dissimilar piezoelectric materials is reviewed. Throughout this paper, the shorthand notation introduced by Barnett and Lothe[7] and the rectangular Cartesian coordinates are used. The summation convention will be used for repeating subscripts unless it is otherwise indicated.

Consider a 2-D electroelastic problem in which all fields depend only on in-plane coordinates. In the absence of body forces and free charges, the basic equations used are as follows[7]:

$$H_{ij,i} = 0 \tag{1}$$

$$H_{ij} = E_{ijkm} U_{k,m} \tag{2}$$

where

$$U_k = \begin{cases} u_k & k = 1,2,3 \\ \varphi & k = 4 \end{cases} \tag{3}$$

$$H_{ij} = \begin{cases} \sigma_{ij} & i,j = 1,2,3 \\ D_i & i = 1,2,3;\ j = 4 \end{cases} \tag{4}$$

$$E_{ijkm} = \begin{cases} C_{ijkm} & i,j,k,m = 1,2,3 \\ e_{mij} & i,j,m = 1,2,3;\ k = 4 \\ e_{ikm} & i,k,m = 1,2,3;\ j = 4 \\ -\varepsilon_{im} & i,m = 1,2,3;\ j = k = 4 \end{cases} \tag{5}$$

and where u_k, φ, σ_{ij} and D_i are, respectively, elastic displacements, electric potential, stresses and electric displacements, and C_{ijkm}, ε_{ij} and e_{ijk} are, respectively, elastic moduli, dielectric constants and piezoelectric constants, the solutions to equation (1) may be expressed as[4]:

$$U = af(z) \tag{6}$$

$$z = x_1 + px_2 \tag{7}$$

where $U = \{u_1\ u_2\ u_3\ \varphi\}^T$, f is an arbitrary function vector to be determined, and a and p are obtained from

$$[Q + p(R + R^T) + p^2 T]a = 0 \tag{8}$$

where Q, R and T are 4×4 matrices

$$Q_{ik} = E_{1ik1}, \quad R_{ik} = E_{1ik2}, \quad T_{ik} = E_{2ik2} \tag{9}$$

This is an eigenvalue problem consisting of four equations, a nontrivial a exists if p is a root of the determinant

$$\| Q + p(R + R^T) + p^2 T \| = 0 \tag{10}$$

in which $\| \cdot \|$ stands for the determinant.

Since (10) admits no real root, the eight roots form four conjugate pairs[6]. Let $\mathrm{Im}(p_I) > 0 (I \leqslant 4)$, a_I be the associated vectors, and define

$$f(z) = \{f_1(z_1) \quad f_2(z_2) \quad f_3(z_3) \quad f_4(z_4)\}^T \tag{11a}$$

$$A = [a_1 \quad a_2 \quad a_3 \quad a_4] \tag{11b}$$

The solution for U and σ_i is then given by

$$U = Af(z) + \overline{Af}(\bar{z}) \tag{12}$$

$$\sigma_i = B_i g(z) + \bar{B}_i \bar{g}(\bar{z}) \tag{13}$$

where $\sigma_i = \{\sigma_{i1} \quad \sigma_{i2} \quad \sigma_{i3} \quad D_i\}^T$, $g(z) = \mathrm{d}f(z)/\mathrm{d}z$, and the overbar denotes the complex conjugate, while B_i in (13) is a 4×4 matrix:

$$(B_i)_{km} = (E_{ikj1} + p_m E_{ikj2}) A_{jm} \qquad (m \text{ not summed}) \tag{14}$$

Ⅲ. AN ARBITRARILY-ORIENTED CRACK ENDING AT THE INTERFACE

Consider an inclined crack of length $2a$ terminating at the interface between two dissimilar anisotropic piezoelectric materials (Fig. 1). The bonded interface coincides with x_1-axis, and one of the crack tips touches the interface at $x_1 = 0$. The material constants of the inhomogeneous composite are expressed as

$$E_{ijkm} = \begin{cases} E^U_{ijkm} & x_2 > 0 \\ E^L_{ijkm} & x_2 < 0 \end{cases} \tag{15}$$

where the superscripts "U" and "L" stand for the upper material and the lower material, respectively. Let $t(s)$ denote the traction and surface charges acting over the crack faces. The boundary conditions for the problem with continuous U and σ_2 across the bimaterial interface are

$$U^L(x_1, 0) = U^U(x_1, 0) \tag{16}$$

$$\sigma^L_2(x_1, 0) = \sigma^U_2(x_1, 0) \tag{17}$$

$$U(x_1, x_2) \to 0 \quad |x_1^2 + x_2^2| \to \infty \tag{18}$$

$$\sigma_i^U(s,a^+)n_i = \sigma_i^U(s,a^-)n_i = -t(s) \qquad |s| < a \tag{19}$$

where $(n_1\ n_2) = (-\sin\alpha \quad \cos\alpha)$, and a and s are coordinates local to the crack, shown in Fig. 1.

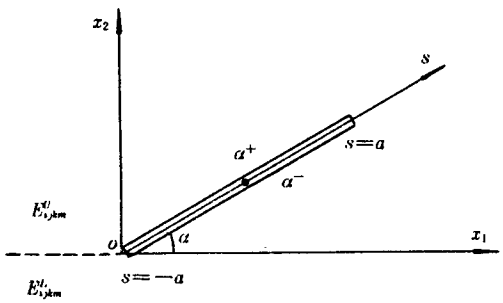

Fig. 1 Geometry for an inclined crack.

By inserting (12) into (16), we have

$$A^U f^U(x_1) - \overline{A}^L \overline{f}^L(x_1) = A^L f^L(x_1) - \overline{A}^U \overline{f}^U(x_1) \qquad x_2 = 0 \tag{20}$$

It is obvious that the left-hand side of (20) is a function, analytic in the upper half-plane and the right-hand side is another function, analytic in the lower half-plane. Hence, if one puts

$$A^U f^U(z) - \overline{A}^L \overline{f}^L(z) = \psi(z) \qquad z \in U \tag{21}$$

$$A^L f^L(z) - \overline{A}^U \overline{f}^U(z) = \psi(z) \qquad z \in L \tag{22}$$

where the function $\psi(z)$ is analytic in the whole plane, then both sides of (20) become satisfied identically. Similarly, from (13), (17) and (18), one has

$$\mathbf{B}_2^U \mathbf{g}^U(z) = \overline{\mathbf{B}}_2^L \overline{\mathbf{g}}^L(\bar{z}) \qquad z \in U \tag{23}$$

$$\mathbf{B}_2^L \mathbf{g}^L(z) = \overline{\mathbf{B}}_2^U \overline{\mathbf{g}}^U(\bar{z}) \qquad z \in L \tag{24}$$

If the eigenvalues p are all distinct, $\mathbf{B}_i^j (i=1,2,\ j=L,U)$ are non-singular. In this case, equations (23) and (24) may be solved for $\mathbf{g}^i(z)(i=L,U)$, which yields

$$\mathbf{g}^U(z) = (\mathbf{B}_2^U)^{-1} \overline{\mathbf{B}}_2^L \overline{\mathbf{g}}^L(\bar{z}) \qquad z \in U \tag{25}$$

$$\mathbf{g}^L(z) = (\mathbf{B}_2^L)^{-1} \overline{\mathbf{B}}_2^U \overline{\mathbf{g}}^U(\bar{z}) \qquad z \in L \tag{26}$$

Differentiating (21) and using (25), one obtains

$$\mathbf{C}\mathbf{g}^U(z) = \psi'(z) \qquad z \in U \tag{27}$$

where the prime denotes the derivative with respect to the associated argument, and

$$C = A^U - \overline{A}^L(\overline{B}_2^L)^{-1}B_2^U \tag{28}$$

If C is non-singular then

$$g^U(z) = C^{-1}\psi'(z) \qquad z \in U \tag{29}$$

What follows is to transform the present problem into a non-homogeneous Hilbert problem. To make the derivation tractable, the concept of axial conjugate is introduced in the following way. Let $\tilde{z}$ denote the axial conjugate of z about the x^*-axis (Fig. 2). It can be seen from Fig. 2 that

$$\tilde{z} = re^{i(\alpha-\beta)} = re^{-i(\alpha+\beta)+2i\alpha} = e^{2i\alpha}\bar{z} \tag{30}$$

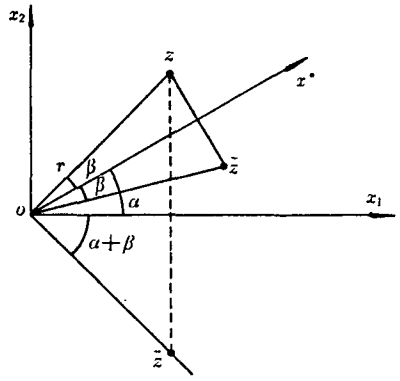

Fig. 2 The definitions of $\bar{z}$ and $\tilde{z}$.

which is the relation between axial and complex conjugates. As a consequence (13) may be rewritten in the form

$$\sigma_i = B_i g(z) + e^{-2i\alpha}\overline{B}_i\,\tilde{g}(\tilde{z}) \tag{31}$$

The substitution of (31) into (19) and noting (20), (21) and (29), yields

$$G\psi'^{+}(s) + \overline{G}\psi'^{-}(s) = -e^{i\alpha}t(s), \qquad |s| \leqslant a \tag{32}$$

where $s = \dfrac{z}{\cos\alpha + p\sin\alpha} - a$, and

$$G = (B_2^U\cos\alpha - B_1^U\sin\alpha)e^{i\alpha}C^{-1} \tag{33}$$

Following the technique of Clements[8], the problem can be transformed into

$$S\psi'^{+} + \lambda^{*} S\psi'^{-}(s) = -\mathrm{e}^{\iota\alpha} N t(s) \qquad |s| \leqslant a \tag{34}$$

in which $\lambda^{*} = \mathrm{diag}[\lambda_1 \quad \lambda_2 \quad \lambda_3 \quad \lambda_4]$, and N is obtained from

$$(\bar{G} - \lambda G)N = 0 \tag{35}$$

with

$$\| \bar{G} - \lambda G \| = 0 \tag{36}$$

while the matrix S is evaluated from

$$S = NG \tag{37}$$

Now turn our attention to equation (32). The appropriate solution to (32) may be given as (see [8])

$$S\psi'(z) = \frac{-\mathrm{e}^{\iota\alpha}}{2\pi i} X(s(z)) \int_{-a}^{a} X^{*}(s(z),\xi) N t(\xi) \mathrm{d}\xi \tag{38}$$

where

$$X(s) = \mathrm{diag}[X_1(s) \quad X_2(s) \quad X_3(s) \quad X_4(s)] \tag{39a}$$

$$X^{*}(s,\xi) = \mathrm{diag}[y_1(s,\xi) \quad y_2(s,\xi) \quad y_3(s,\xi) \quad y_4(s,\xi)] \tag{39b}$$

$$X_k(s) = (s+a)^{-m_k}(s-a)^{m_k - 1} \tag{39c}$$

$$m_k = \frac{1}{2\pi i}\log\lambda_k \tag{39d}$$

$$y_k(s,\xi) = \frac{1}{X_k^{+}(\xi)(\xi - s)} \tag{39e}$$

If $\| S \| \neq 0$ then

$$\psi'(z) = \frac{-\mathrm{e}^{\iota\alpha}}{2\pi i} S^{-1} X(s(z)) \int_{-a}^{a} X^{*}(s(z),\xi) N t(\xi) \mathrm{d}\xi \tag{40}$$

By inserting (40) into (31) and noting (27), we have

$$\sigma_i = B_i C^{-1} \psi'(z) + \mathrm{e}^{-2\iota\alpha} \bar{B}_i \bar{C}^{-1} \tilde{\psi}'(\tilde{z}) \tag{41}$$

From equations (39) and (41), we can see that the complex m_k represents the stress singularity at crack tips. To obtain the complex consider an inclined crack terminating at the interface between transversely isotropic materials. The upper and lower materials are assumed to be the PZT-5H and the PZT-5, respectively. The material constants for the two materials are taken from [4] and [9] as listed below.

(1) Material properties for PET-5H[4]:

$c_{22} = c_{33} = 126$ GPa, $c_{23} = 55$ GPa, $c_{12} = c_{13} = 53$ GPa, $c_{11} = 117$ GPa,

$c_{55}=c_{66}=35.3$ GPa, $c_{44}=(c_{33}-c_{23})/2=35.5$ GPa,

$e_{12}=e_{13}=-6.5$ C/m^2, $e_{11}=23.3$ C/m^2, $e_{35}=e_{26}=17$ C/m^2,

$\varepsilon_{33}=\varepsilon_{22}=151\times10^{-10}$ C/Vm, $\varepsilon_{11}=130\times10^{-10}$ C/Vm

(2) Material properties for PZT-5[9]:

$c_{22}=c_{33}=121$ GPa, $c_{23}=75.4$ GPa, $c_{12}=c_{13}=75.2$ GPa, $c_{11}=111$ GPa,

$c_{55}=c_{66}=21.1$ GPa, $c_{44}=(c_{33}-c_{23})/2=22.8$ GPa,

$e_{12}=e_{13}=-5.4$ C/m^2, $e_{11}=15.8$ C/m^2, $e_{35}=e_{26}=12.3$ C/m^2,

$\varepsilon_{33}=\varepsilon_{22}=81.07\times10^{-10}$ C/Vm, $\varepsilon_{11}=73.46\times10^{-10}$ C/Vm

It should be noted that the indices in Appendix of [4] and in Table 1 of [9] have been changed owing to the different coordinate system used. In our study the x_1-axis is chosen to be the poling direction, and the crack line is in the x_1-x_2 plane. Some numerical results for the variations of complex m_k with the angle α are presented in Table 1. It can be seen from Table 1 that the order of singularity in the stress and electric displaement fields depends strongly on the inclined angle α. It is also found from the results that the values of $\mathrm{Im}(m_k)$ are very small for the material combination PZT-5H and PZT-5.

Table 1 The values of m_k versus angle α

α	0°	5°	15°	25°	35°	45°	90°
$\mathrm{Re}(m_1)$	0.5000	0.5304	0.3493	0.2447	0.1382	0.0299	0.5000
$\mathrm{Re}(m_2)$	0.5417	0.4457	0.4004	0.3553	0.3060	0.2533	0.0678
$\mathrm{Im}(m_1\times10^3)$	0.0	−0.7225	0.0903	0.1734	0.0565	−0.0446	0.0002
$\mathrm{Im}(m_2\times10^3)$	−0.3466	−2.147	−0.0037	−0.0684	−0.1400	−2.368	0.0013

Ⅳ. CONCLUSIONS

This paper treats the plane problem of a crack that terminates at the interface between two piezoelectric materials. The study shows that this problem can be transformed into a non-homogeneous Hilbert problem by employing the concept of the axial conjugate. The numerical results indicate that the order of singularity in the traction-charge field at crack tips depends strongly on the inclined angle α, but the imaginary part of m_k is very small to any values of α for the combination PZT-5H and PZT-5.

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