Benchmark solution for multilayer magneto-electro-elastic plates adhesively bonded by viscoelastic interlayer

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Abstract
An analytical solution is developed for a simply supported multilayer magneto-electro-elastic plate, which is adhesively bonded by viscoelastic interlayer and subjected to transverse loading. The three-dimensional equations of magneto-electro-elasticity are used to describe the mechanical behavior in each layer of the plate, while the mechanical property of the viscoelastic interlayer is simulated by the standard linear solid model with strain memory effect. The imperfect electric conditions between adjacent layers are considered. Making use of the Fourier series expansion as well as the state-space method, a linear equation system to the solution of the problem considered is derived. By means of the Laplace transformation, the undetermined coefficients contained in the equation system are obtained analytically. The present solution can be used as the benchmark to assess the other numerical solutions. The comparison study shows a trend that the finite element solution converges to the proposed theoretical results as the mesh density increases; in contrast to the theoretical solution, the finite element solution is, however, time-consuming, in terms of mesh division and calculation. In this study, the effects of the time; interlayer thickness; and interlayer electric coefficient on the mechanical, electric, and magnetic fields are also investigated.

Keywords
Magneto-electro-elastic plate, viscoelastic interlayer, standard linear solid model, Laplace transformation

Introduction
Owing to the ability of converting energy from one form to another, smart structures consisting of piezoelectric and/or piezomagnetic materials are increasingly used in various branches of practical engineering, such as sensors, actuators, vibration control, health monitoring, and robotics. In particular, the magneto-electro-elastic (MEE) composite systems involving piezoelectric and piezomagnetic phases have received great attention because of the new characteristics known as MEE coupling effect, which does not exist in single-phase piezoelectric or piezomagnetic materials (Priya et al., 2007; Qin, 2007, 2013; Zhai et al., 2008). The individual layers in MEE-layered structures are either sintered together (Agarwal et al., 2012) or bonded by adhesives, such as epoxy, hysol, and araldite (Mustapha and Ye, 2015). The sintered MEE structures can be considered as perfectly bonded (PB) interface, that is, the out-of-plane variables are all continuous along the thickness. In the adhesively bonded case, the adhesive is much softer than its adherend, which causes inevitably interfacial slip. Since the adhesive possesses viscoelastic property in nature, the electric, magnetic, and stress fields in MEE structures are usually time dependent. Such problems exist widely in practical engineering (Jin and Wang, 2011; Wang and Pan, 2008; Yan et al., 2011) and are worthy of being investigated thoroughly.

The analytical solutions based on the rigorous theory are of great importance, as they can be used to assess the corresponding numerical solutions (Guinovart-Díaz et al., 2013; Qin, 2000; Qin and Mai, 1997; Qin and Yu, 1997). A review of the literature shows that many analytical studies have been presented for piezoelectric and MEE-layered structures. To cite a few, by the use of the
propagator matrix method, Pan (2001) developed an exact solution for simply supported multilayer MEE plates with PB interface under transverse loadings. Qin and Ye (2004) presented a thermoelctroelastic solution for describing internal bone remodeling process. The free vibrations of functionally graded MEE plates were investigated by Chen et al. (2005) using the state-space method. Wang et al. (2007) studied the scattering of an antiplane shear wave by a piezoelectric circular cylinder with an imperfect interface by using analytical method. Wu and Lu (2009) proposed a modified Pagano method to study the three-dimensional (3D) dynamic responses of functionally graded MEE materials. In their report, the free electric/magnetic potential and flux conditions were considered. The buckling and vibration analyses of layered MEE structures under uniform thermal loading were studied by Kumaravel et al. (2010). By employing the state-vector approach, Chen et al. (2007) presented analytical solutions for the propagation of harmonic waves in MEE-layered plates. The MEE responses of multiferroic fibrous composites with imperfectly bonded interface under longitudinal shear were investigated by Wang and Pan (2007). Based on the nonlocal theory and Kirchhoff plate theory, Ke et al. (2014) analyzed free vibrations of MEE nanoplates subjected to biaxial loadings, external electric potential, external magnetic potential, and temperature rise. Based on the Timoshenko beam theory, Milazzo et al. (2009) presented analytical solution for the transient analysis of an MEE bimorph beam, in which the electric and magnetic fields were assumed to be steady. Wang et al. (2009) analyzed the semi-infinite crack penetrating a piezoelectric circular inhomogeneity bonded to an infinite piezoelectric matrix through a linear viscous interface. Zhou et al. (2005) investigated the dynamic behavior of collinear symmetric interface cracks between two MEE half planes under the loading of shear waves by means of the Schmidt method. The effect of adhesive layer in the layered smart structures was analyzed in some reports (Chen and Lee, 2004; Madhav and Soh, 2007; Wu et al., 2018), while the adhesives in these studies were modeled as full elastic material. The experiment investigation indicated that the adhesive generally possesses viscoelastic property (Meshgin et al., 2009). Most of the existing models for piezoelectric and MEE-layered structures were restricted to perfect and/or elastic imperfect interface. Recently, Wu et al. (2018) developed a two-dimensional (2D) analytical model for infinite layered MEE cylindrical shell with viscoelastic interface.

For layered structures made of elastic materials bonded by viscoelastic interlayer, several analytical models have been reported in the literature. Based on the Bernoulli–Euler beam theory, Galuppi and Royer-Carfagni (2012, 2013) presented analytical solutions for the bending of two-layer beams with viscoelastic interlayer subjected to both static and time-dependent loadings. By means of Laplace transformation, analytical solutions for layered functionally graded beams bonded by viscoelastic interlayer were developed by Li et al. (2014). By means of Laplace transformation with Zakian’s numerical method, Zhang and Wang (2011) investigated the long-term behavior of fiber-strengthened beams with viscoelastic interlayer. Based on the elasticity theory, Wu et al. (2016, 2017) recently proposed an analytical solution for layered beams and plates with viscoelastic interlayer. In their work, effects of geometrical and material parameters of the interlayer on the stress and displacement fields were investigated in detail. The buckling behavior of two-layer beams with viscoelastic interlayer was studied by Galuppi and Royer-Carfagni (2014). In their report, two models, including the full viscoelastic model and the quasi-elastic model, were proposed.

In this work, an exact analytical solution for 3D-layered MEE plate subjected to transverse loading is proposed. The plate consists of a number of MEE layers which are adhesively bonded by viscoelastic interlayers. They are modeled using the 3D equations of magneto-electro-elasticity. The adhesive interlayer is described by the standard linear solid model considering the strain memory effect. The imperfect electric conditions between adjacent layers are examined. By means of Fourier series expansion and the state-space method, a linear equation system for the solution of the problem is obtained. The corresponding undetermined coefficients are determined analytically via the Laplace transformation. The present solution is compared with the other available solutions. And the effects of the time; interlayer thickness; and interlayer electric coefficient on the mechanical, electric, and magnetic fields are studied in detail.

Mathematical model

As shown in Figure 1, we consider a layered plate consisting of \( p \) MEE layers adhesively bonded by \( p-1 \) viscoelastic interlayers. The material property in the MEE layers is orthotropic and that in the adhesive interlayers...
is isotropic. The plate is simply supported at four edges and loaded by distributed mechanical loading \( q(x, y) \) acting on the top surface of the plate. A Cartesian coordinate \( O-xyz \) is established with the origin \( O \) located at the corner of the bottom surface of the plate.

In such a layered system, only the interlayer possesses time-dependent material property, which results in the time-varying continuity conditions between adjacent MEE layers. Therefore, the mechanical, electric, and magnetic fields in the MEE layers are also time dependent, although the material property of MEE layers is time invariant.

**Assumptions**

This study is based on the following four assumptions: (1) the plate deformation is small and within the range of linearity; (2) the adhesive interlayer is far thinner than the MEE adherends, that is, \( \Delta h \ll h \); (3) based on the second assumption, we assume that the interlayer deformation is linear along the thickness direction, that is, the interlayer strain is constant through the thickness; and (4) the interlayer, made of adhesives, is softer in comparison with the MEE layer. Thus, the horizontal normal stress in the interlayer is negligible.

**Basic equations for a MEE layer**

Based on the 3D equations of magneto-electro-elasticity (Chen et al., 2007), the coupled constitutive equations for the \( i \)th MEE layer can be given in the tensor form, as follows

\[
\sigma^i_{ij} = \epsilon^i_{jk} \gamma^i_k - \epsilon^i_{kj} E^i_j - d^i_{jk} H^i_k, \quad D^i_j = \epsilon^i_{jk} \gamma^i_k + \epsilon^i_{kj} E^i_j + \mu^i_{jk} H^i_k, \quad i = 1, 2, \ldots, p
\]

where \([\epsilon'] = [u_x' \ u_y' \ u_z']^T\). The equation of mechanical equilibrium together with Gauss’s laws for electricity and magnetism in the absence of free charge are given by

\[
\sigma^i_{ki,j} = 0, \quad D^i_{j,k} = 0, \quad B^i_{j} = 0, \quad i = 1, 2, \ldots, p
\]

By using the state approach (Wang et al., 2003), the partial differential equations for the out-of-plane variables can be obtained from equations (1) to (3), as follows

\[
\frac{\partial}{\partial z} X^i(x, y, z, t) = M^i(X^i(x, y, z, t), i = 1, 2, \ldots, p
\]

where \( M \) is given in equation (A2) in Appendix 2; \( X^i \) is the state vector including 10 out-of-plane variables, that is, \( X^i = [u^i_x \ u^i_y \ D^i_x \ B^i_z \ \sigma^i_{xz} \ \tau^i_{xz} \ \phi^i \ \psi^i \ u^i_z]^T \). The boundary conditions at the four edges of the plate are considered as fully simple supports and suitably grounded where the following quantities are satisfied (Wu and Lu, 2009)

\[
\begin{align*}
\sigma^x_x &= u^x_x = u^x_y = \phi^x = \psi^x = 0, \quad x = 0, a \\
\sigma^y_y &= u^y_x = u^y_y = \phi^y = \psi^y = 0, \quad y = 0, b
\end{align*}
\]

For this boundary conditions, the 10 out-of-plane variables in \( X^i \) can be expanded in the double Fourier series form

\[
\begin{array}{l}
\left[ u^i_x(x, y, z, t) \\
u^i_y(x, y, z, t) \\
D^i_x(x, y, z, t) \\
B^i_z(x, y, z, t) \\
\sigma^i_{xz}(x, y, z, t) \\
\tau^i_{xz}(x, y, z, t) \\
\phi^i(x, y, z, t) \\
\psi^i(x, y, z, t) \\
u^i_z(x, y, z, t)
\end{array}
\]

where \( \alpha_m = m\pi/a \) and \( \beta_n = n\pi/b \). By substituting equation (6) into equation (4), one obtains

\[
\frac{d}{dz} X^i_{mn}(z, t) = K^i_{mn} X^i_{mn}(z, t), \quad m, n = 1, 2, 3, \ldots, i = 1, 2, \ldots, p
\]

where \( X^i_{mn} = [u^i_{x,mn} \ u^i_{y,mn} \ D^i_{x,mn} \ B^i_{z,mn} \ \sigma^i_{xz,mn} \ \tau^i_{xz,mn} \ \phi^i_{mn} \ \psi^i_{mn} \ u^i_{z,mn}]^T \); \( K^i_{mn} \) is defined in equation (A3) in Appendix 2. The solution of equation (7) is
\[ X_{m,n}^i(z, t) = e^{K_{m,n}^i(t)} e^{K_{m,n}^i(t)} C_{m,n}^i(t), \]

\[ m, n = 1, 2, 3, \ldots, i = 1, 2, \ldots, p \]

where \( C_{m,n}^i(t) = [c_{1,m,n}^i(t) c_{2,m,n}^i(t) \cdots c_{10,m,n}^i(t)]^T \) is a vector involving undetermined time-varying coefficients. Let us define

\[
\Psi_{m,n}^i(z) = e^{K_{m,n}^i(z)} = \begin{bmatrix}
T_{1,m,n}^i(z) \\
T_{2,m,n}^i(z) \\
\vdots \\
T_{10,m,n}^i(z)
\end{bmatrix}
\]

\[ m, n = 1, 2, 3, \ldots, i = 1, 2, \ldots, p \]

By employing equations (1) and (2), other in-plane variables can be expressed in terms of the out-of-plane variables as

\[ \alpha_x^i = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ -\alpha_m c_{11} \epsilon_{m,n}^i - \beta_n c_{12} \epsilon_{m,n}^i + c_{13} \epsilon_{m,n}^i \left( \mu_{m,n}^i \right)^{-1} \right\} \sin(\alpha_{m,x}) \sin(\beta_{n,y}) \]

\[ \alpha_y^i = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ -\alpha_m c_{12} \epsilon_{m,n}^i - \beta_n c_{22} \epsilon_{m,n}^i + c_{23} \epsilon_{m,n}^i \left( \mu_{m,n}^i \right)^{-1} \right\} \cos(\alpha_{m,x}) \cos(\beta_{n,y}) \]

\[ D_x^i = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( \epsilon_{x,m,n}^i - \alpha_m \left[ c_{11} + \frac{c_{15} \epsilon_{x,m,n}^i}{c_{55}^i} \right] \phi_{m,n}^i - \alpha_m \left[ d_{11} + \frac{d_{15} \epsilon_{x,m,n}^i}{c_{55}^i} \right] \psi_{m,n}^i \right) \cos(\alpha_{m,x}) \sin(\beta_{n,y}) \]

\[ D_y^i = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( \epsilon_{y,m,n}^i - \beta_n \left[ c_{22} + \frac{c_{25} \epsilon_{y,m,n}^i}{c_{55}^i} \right] \phi_{m,n}^i - \beta_n \left[ d_{22} + \frac{d_{25} \epsilon_{y,m,n}^i}{c_{55}^i} \right] \psi_{m,n}^i \right) \sin(\alpha_{m,x}) \cos(\beta_{n,y}) \]

\[ B_x^i = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( \epsilon_{x,m,n}^i + \frac{c_{15} \epsilon_{x,m,n}^i}{c_{55}^i} \phi_{m,n}^i - \frac{d_{15} \epsilon_{x,m,n}^i}{c_{55}^i} \psi_{m,n}^i \right) \cos(\alpha_{m,x}) \cos(\beta_{n,y}) \]

\[ B_y^i = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( \epsilon_{y,m,n}^i + \frac{c_{25} \epsilon_{y,m,n}^i}{c_{55}^i} \phi_{m,n}^i - \frac{d_{25} \epsilon_{y,m,n}^i}{c_{55}^i} \psi_{m,n}^i \right) \sin(\alpha_{m,x}) \sin(\beta_{n,y}) \]

\[ \tau_{xy}^i = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[ \beta_n \epsilon_{60} \epsilon_{x,m,n}^i(r) + \alpha_m \epsilon_{60} \epsilon_{y,m,n}^i(r) \right] \cos(\alpha_{m,x}) \cos(\beta_{n,y}), i = 1, 2, \ldots, p \]

where

\[ \kappa^i = \begin{bmatrix}
\epsilon_{33} & \epsilon_{33} & \epsilon_{33} \\
\epsilon_{33} & -\epsilon_{33} & -\epsilon_{33} \\
\epsilon_{33} & -\epsilon_{33} & -\epsilon_{33}
\end{bmatrix} \]

\[ \mu^i = \begin{bmatrix}
\epsilon_{33} & \epsilon_{33} \\
\epsilon_{33} & -\epsilon_{33} \\
\epsilon_{33} & -\epsilon_{33}
\end{bmatrix} \]

\[ \omega^i_{mn} = \begin{bmatrix}
\sigma_{z,m,n}^i + \alpha_{m,n} \epsilon_{31} \epsilon_{m,n}^i + \beta_{m,n} \epsilon_{33} \epsilon_{m,n}^i \\
\sigma_{z,m,n}^i + \alpha_{m,n} \epsilon_{31} \epsilon_{m,n}^i + \beta_{m,n} \epsilon_{33} \epsilon_{m,n}^i \\
\sigma_{z,m,n}^i + \alpha_{m,n} \epsilon_{31} \epsilon_{m,n}^i + \beta_{m,n} \epsilon_{33} \epsilon_{m,n}^i
\end{bmatrix} \]

\[ G^i(t) = G_1^i e^{\mu^i_0 t} + G_2^i \]

\[ E^i(t) = 2(1 + \mu^i) G^i(t) \]

**Figure 2.** Standard linear solid model.
According to the theory for linear viscoelasticity (Kolarˇi´k and Pegoretti, 2008), the constitutive equations in the interlayer are governed by

\[
\sigma_{z, mn}^i(t) = E^i(t)\varepsilon_{z, mn}^i(0) + \int_0^t \left[ E^i(t - \xi) \frac{\partial \varepsilon_{z, mn}^i(\xi)}{\partial \xi} \right] d\xi, \quad \tau_{z, mn}^i(t) = G^i(t)\gamma_{z, mn}^i(0)
\]

Continuity and surface conditions

where

\[
\delta_{i, mn}^{(w)} = \frac{\delta_{i, mn}^{(w)}}{\Delta h}, \quad \delta_{i, mn}^{(v)} = \frac{\delta_{i, mn}^{(v)}}{\Delta h}, \quad i = 1, 2, \ldots, p - 1
\]

Continuity and surface conditions

By combining equations (6), (8), (9), and (14) to (17), the continuity conditions between the adjacent MEE layers can be rearranged as

\[
\Psi_{mn}^{i+1}(d_{i+1}^l, t) - \Psi_{mn}^i(d_{i}^l, t) = \Delta_{mn}^l(t), \quad m, n = 1, 2, 3, \ldots, i = 1, 2, \ldots, p - 1
\]

and

\[
\tau_{z, mn}^i(d_{i}^l, t) = \frac{\delta_{i, mn}^{(w)}(t)}{\Delta h} \otimes dG^i(t),
\]

The imperfect electric conditions between adjacent layers are also considered. For a dielectrically weakly conducting condition (Wang et al., 2009), the electric potential is continuous across the interface, whereas the transverse electric displacement has a discontinuity across the interface which is proportional to a certain differential expression of the electric potential (Wang and Sudak, 2007). These two conditions can be expressed by

\[
\delta_{i, mn}^{(D)} = -\chi_1[(\alpha_m)^2 + (\beta_n)^2]\phi_{mn}(d_{i}^l, t),
\]

where

\[
\delta_{i, mn}^{(D)} = \chi_2 D_{z, mn}^l(x, y, d_{i}^l, t), \quad i = 1, 2, \ldots, p - 1
\]
The propagator matrix method is not applicable. Here, the analysis of layered structures (Pan, 2001; Pan and Han, 2005). Since the continuity conditions in this model involve convolution operation in equation (13), the propagator matrix method is not applicable. Here, Cramer’s law in conjunction with Laplace transformation method is employed to determine the coefficients in equation (8). By substituting equations (6), (8), and

\[
q(x, y) = -\sum_{m = 1}^{\infty} \sum_{n = 1}^{\infty} q_{mn} \sin(\alpha_m x) \sin(\beta_n y),
\]

\[
q_{mn} = -\frac{4}{ab} \int_{0}^{a} \int_{0}^{b} q(x, y) \sin(\alpha_m x) \sin(\beta_n y) dx dy
\]

The propagator matrix method is very popular for the analysis of layered structures (Pan, 2001; Pan and Han, 2005). Since the continuity conditions in this model involve convolution operation in equation (13), the propagator matrix method is not applicable. Here, Cramer’s law in conjunction with Laplace transformation method is employed to determine the coefficients in equation (8). By substituting equations (6), (8), and

\[
M_{mn}^l = \begin{bmatrix} T_{5, mn}(0)^T & T_{6, mn}(0)^T & T_{7, mn}(0)^T \\ 0 & 0 & 0 \end{bmatrix}
\]

\[
Q_{mn} = \begin{bmatrix} 0 & 0 & 0 & q_{mn} & 0 & 0 \end{bmatrix}
\]

\[
\text{where}
\]

\[
\Omega_{mn} = \begin{bmatrix} -\Psi_{mn}^1(d_1) & \Psi_{mn}^2(d_2) & O & \cdots & \cdots & \cdots & \cdots & \cdots & O \\
O & -\Psi_{mn}^2(d_2) & \Psi_{mn}^3(d_3) & O & \cdots & \cdots & \cdots & \cdots & O \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & O \\
O & \cdots & O & -\Psi_{mn}^i(d_i) & \Psi_{mn}^{i+1}(d_{i+1}) & O & \cdots & \cdots & O \\
O & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & O \\
M_{mn}^l & O & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & O \\
\end{bmatrix}
\]

\[
\Gamma_{mn}(t) = \begin{bmatrix} C_{mn}^1(t) \\
C_{mn}^2(t) \\
\cdots \\
C_{mn}^{p-1}(t) \\
Q_{mn} \end{bmatrix},
\]

\[
G_{mn}(t) = \begin{bmatrix} \Delta_{mn}^1(t) \\
\Delta_{mn}^2(t) \\
\cdots \\
\Delta_{mn}^{p-1}(t) \\
Q_{mn} \end{bmatrix}
\]

where \(q(x, y)\) can be expanded into double Fourier series, as follows

\[
q(x, y) = -\sum_{m = 1}^{\infty} \sum_{n = 1}^{\infty} q_{mn} \sin(\alpha_m x) \sin(\beta_n y),
\]

\[
q_{mn} = -\frac{4}{ab} \int_{0}^{a} \int_{0}^{b} q(x, y) \sin(\alpha_m x) \sin(\beta_n y) dx dy
\]

The propagator matrix method is very popular for the analysis of layered structures (Pan, 2001; Pan and Han, 2005). Since the continuity conditions in this model involve convolution operation in equation (13), the propagator matrix method is not applicable. Here, Cramer’s law in conjunction with Laplace transformation method is employed to determine the coefficients in equation (8). By substituting equations (6), (8), and

\[
M_{mn}^l C_{mn}^1(t) + M_{mn}^p C_{mn}^p(t) = Q_{mn}, m, n = 1, 2, 3, \ldots
\]

where

\[
\text{where the subscript } T \text{ means the transpose of the matrix, and } 0 \text{ denotes a } 10 \times 1 \text{ null sub-matrix. By combining equations (18) and (22), a relation between } C_{mn}(t) \text{ and } \Delta_{mn}(t) \text{ can be obtained}
\]

\[
\Omega_{mn} \Gamma_{mn}(t) = G_{mn}(t), m, n = 1, 2, 3, \ldots
\]

where

\[
\text{where } \Omega_{mn} \text{ is a } 10 \times 10 \text{ null sub-matrix. By using Cramer’s law of linear algebraic equation system, the time-varying coefficients can be expressed by } \delta_{l, mn}^{(1)}, \delta_{l, mn}^{(2)}, \delta_{l, mn}^{(3)}, \delta_{l, mn}^{(3)}, \text{ and } q_{mn}, \text{ as follows}
\]

\[
\Gamma_{mn}(t) = \sum_{k=1}^{p-1} \left[ \frac{\Omega_{mn, k}^{(1)}}{\Omega_{mn}} \delta_{k, mn}(t) \right] + \frac{\Omega_{mn, k}^{(2)}}{\Omega_{mn}} \delta_{k, mn}(t)
\]

\[
\delta_{l, mn}^{(3)}(t) + \frac{\Omega_{mn, k}^{(4)}}{\Omega_{mn}} \delta_{k, mn}(t)
\]

\[
+ \frac{\Omega_{mn, k}^{(5)}}{\Omega_{mn}} q_{mn}, m, n = 1, 2, 3, \ldots, i = 1, 2, \ldots, 10p
\]

(24)
where $\Gamma^k_{mn}(t)$ is the $k$th element of $\Gamma_{mn}(t)$; the matrix with double vertical lines, for example, $[\Omega_{mn}]$, represents the determinant of the matrix; $\Omega^{(a)}_{mn}, \Omega^{(b)}_{mn}, \Omega^{(c)}_{mn}, \Omega^{(D)}_{mn}$ are obtained by replacing the $k$th column of $\Omega_{mn}$ with the vector $B^{(a)}_k, B^{(b)}_k, B^{(c)}_k, B^{(D)}_k$, and $B^{(c)}_k$, respectively, in which

$$B^{(a)}_k = \begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 0 \end{bmatrix}^T,$$

$$B^{(b)}_k = \begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 0 \end{bmatrix}^T,$$

$$B^{(c)}_k = \begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}^T,$$

$$B^{(D)}_k = \begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}^T,$$

$$B^{(c)}_k = \begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}^T,$$

$$k = 1, 2, \ldots, p - 1$$

The coefficients for the $i$th layer can be expressed by

$$c^{(i)}_{j,mn}(t) = \Gamma_{mn}^{10i} \cdot \delta^{(j)}_{i,mn}(t),$$

$m, n = 1, 2, 3, \ldots, i = 1, 2, \ldots, p, j = 1, 2, \ldots, 10$ (25)

where $c^{(i)}_{j,mn}(t)$ is the $j$th element of $C^{(i)}_{mn}(t)$.

**Laplace transformation**

By substituting equations (8), (11), and (25) into equation (19) and then conducting Laplace transformation, a set of equations for $\delta^{(a)}_{k,mn}(s), \delta^{(b)}_{k,mn}(s), \delta^{(c)}_{k,mn}(s), \delta^{(D)}_{k,mn}(s)$, and $\delta^{(c)}_{k,mn}(s)$ can be obtained as

$$(A_{mn} + F \frac{s}{s + 1/\theta_G})P_{mn}(s) = \frac{1}{s}D_{mn}$$

where

$$A_{mn} = \begin{bmatrix} A_{mn}^{11} & A_{mn}^{12} & \cdots & A_{mn}^{15} \\ A_{mn}^{21} & A_{mn}^{22} & \cdots & A_{mn}^{25} \\ \vdots & \vdots & \ddots & \vdots \\ A_{mn}^{51} & A_{mn}^{52} & \cdots & A_{mn}^{55} \end{bmatrix},$$

$$F = \begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix},$$

$$D_{mn} = \begin{bmatrix} D_{mn}^{11} & D_{mn}^{12} & \cdots & D_{mn}^{15} \\ D_{mn}^{21} & D_{mn}^{22} & \cdots & D_{mn}^{25} \\ \vdots & \vdots & \ddots & \vdots \\ D_{mn}^{51} & D_{mn}^{52} & \cdots & D_{mn}^{55} \end{bmatrix},$$

$$P_{mn}(s) = \begin{bmatrix} \delta^{(a)}_{1,mn}(s) & \cdots & \delta^{(a)}_{(p-1),mn}(s) \\ \delta^{(b)}_{1,mn}(s) & \cdots & \delta^{(b)}_{(p-1),mn}(s) \\ \delta^{(c)}_{1,mn}(s) & \cdots & \delta^{(c)}_{(p-1),mn}(s) \\ \delta^{(D)}_{1,mn}(s) & \cdots & \delta^{(D)}_{(p-1),mn}(s) \end{bmatrix}^T,$$

in which, $s$ denotes the Laplace transformation variable; the variable with an over curve means the variable is in Laplace transformation space; $I$ is a $(3p - 3) \times (3p - 3)$ unit matrix; $O_1, O_2,$ and $O_3$ are $(3p - 3) \times (2p - 2), (2p - 2) \times (3p - 3),$ and $(2p - 2) \times (2p - 2)$ null matrices, respectively; the elements in $A_{mn}$ and $D_{mn}$ are given in equation (A4) in Appendix 2. By using Cramer’s law and solving equation (26), $\delta^{(a)}_{k,mn}(s), \delta^{(b)}_{k,mn}(s), \delta^{(c)}_{k,mn}(s), \delta^{(D)}_{k,mn}(s),$ and $\delta^{(c)}_{k,mn}(s)$ can be written into the fractional expression of $s$

$$\tilde{P}_{mn}(s) = \frac{j^{n}}{\sum_{k=0}^{3p-3} \eta_{mn}^{k} s^k},$$

$n = 1, 2, 3, \ldots, j = 1, 2, \ldots, 5p - 5$ (27)

where $\tilde{P}_{mn}(s)$ is the $j$th element of $\tilde{P}_{mn}(s),$ and

$$\eta_{mn} = \sum_{i=0}^{3p-3} \sum_{j=0}^{3p-3} C^{(i)}_{p-3-i} \left( \frac{1}{\theta_G} \right)^{3p-3-i} \cdot 0 \leq k \leq 3p - 3$$

where

$$\sum_{i=0}^{3p-3} \sum_{j=0}^{3p-3} C^{(i)}_{p-3-i} \left( \frac{1}{\theta_G} \right)^{3p-3-i} \cdot 0 \leq k \leq 3p - 3, 1 \leq j \leq 3p - 3$$

and

$$\sum_{i=0}^{k} \sum_{j=0}^{3p-3} C^{(i)}_{p-3-i} \left( \frac{1}{\theta_G} \right)^{3p-3-i} \cdot 0 \leq k \leq 3p - 3, 3p - 2 \leq j \leq 5p - 5$$

in which, $C^{(i)}_a = a^{(i)} b^{(i)} (a - b);$ according to the permutation and combination theory, if any $l$ $(0 \leq l \leq 3p - 3)$ columns in the first $3p - 3$ columns of the determinant $|A_{mn}|$ are replaced by the same columns of $F,$ there will be $C^{(i)}_{3p-3}$ kinds of results, and $J^{(i)}_{mn}$ is the sum of all the results. An example for $J^{(i)}_{mn}$ is given in Appendix 3. Let us define that $|A^{(i)}_{mn}|$ is the result that the $j$th $(1 \leq j \leq 5p - 5)$ column of $|A_{mn}|$ is replaced by the column vector $D_{mn}.$ If any $l$ $(0 \leq l \leq 3p - 4)$ columns in the first $3p - 3$, except for $jth$ $(1 \leq j \leq 3p - 3),$ columns of $|A^{(i)}_{mn}|$ are replaced by the same columns of $F,$ there will be $C^{(i)}_{3p-4}$ kinds of results, and $J^{(i)}_{mn}$ is the sum of the all
results. $N^j_{mn}$ is the result that the $j$th $(3p-2 \leq j \leq 5p-5)$ column of each determinant in $R^j_{mn}$ is replaced by $D_{mn}$. Examples for $L^j_{mn}$ and $N^j_{mn}$ are given in Appendix 3. Equation (27) can be further decomposed as

$$P_{mn}(s) = \sum_{l=1}^{3p-2} \frac{r_{mn}^l}{s-s_{mn}^l}, \ m, n = 1, 2, 3, \ldots, j = 1, 2, \ldots, 5p - 5$$

where $s_{mn}^l$ $(l = 1, 2, \ldots, 3p-2)$ is the root of

$$\sum_{k=0}^{3p-3} \eta_{mn}^k s_{mn}^{l-k+1} = 0 \quad \text{and} \quad r_{mn}^l = \sum_{k=0}^{3p-3} \omega_{mn}^k(s_{mn}^l)^k / \sum_{k=0}^{3p-3} (k+1)\eta_{mn}^k(s_{mn}^l)^k.$$  

The inversed Laplace transformation of equation (28) is

$$P_{mn}(s) = \sum_{l=1}^{3p-2} r_{mn}^l e^{-s_{mn}^l t}, \ m, n = 1, 2, 3, \ldots, j = 1, 2, \ldots, 5p - 5$$

Finally, by substituting equation (29) into equation (24), and then substituting the results into equations (8) and (10), the solution of the time-dependent stress, electric displacement, magnetic induction, elastic displacement, and electric and magnetic potential fields for the plate can all be obtained.

### Numerical examples

The material parameters of piezoelectric BaTiO$_3$, magnetostrictive CoFe$_2$O$_4$ and viscoelastic epoxy for the

### Table 1. Material parameters of piezoelectric BaTiO$_3$ (Pan, 2001).

<table>
<thead>
<tr>
<th>Property</th>
<th>BaTiO$_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elastic</td>
<td></td>
</tr>
<tr>
<td>$c_{11}$</td>
<td>$228$</td>
</tr>
<tr>
<td>$c_{12}$</td>
<td>$173$</td>
</tr>
<tr>
<td>$c_{33}$</td>
<td>$170.5$</td>
</tr>
<tr>
<td>Piezoelectric</td>
<td></td>
</tr>
<tr>
<td>$d_{31}$</td>
<td>$580.3$</td>
</tr>
<tr>
<td>$d_{33}$</td>
<td>$699.7$</td>
</tr>
<tr>
<td>$d_{15}$</td>
<td>$550$</td>
</tr>
<tr>
<td>Dielectric</td>
<td></td>
</tr>
<tr>
<td>$\varepsilon_{11}$</td>
<td>$0.08$</td>
</tr>
<tr>
<td>$\varepsilon_{33}$</td>
<td>$0.093$</td>
</tr>
<tr>
<td>Magnetic-permeability</td>
<td></td>
</tr>
<tr>
<td>$\mu_{11}$</td>
<td>$-590$</td>
</tr>
<tr>
<td>$\mu_{33}$</td>
<td>$157$</td>
</tr>
</tbody>
</table>

The units of material parameters are as follows: $c_{ij}$ in $10^9$ N/m$^2$, $d_{ij}$ in C/m$^2$, $\varepsilon_{ij}$ in $10^{-9}$ C$^2$/N m$^2$, and $\mu_{ij}$ in $10^{-6}$ N s$^2$/C$^2$.

### Table 2. Material parameters of magnetostrictive CoFe$_2$O$_4$ (Pan, 2001).

<table>
<thead>
<tr>
<th>Property</th>
<th>CoFe$_2$O$_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elastic</td>
<td></td>
</tr>
<tr>
<td>$c_{11}$</td>
<td>$286$</td>
</tr>
<tr>
<td>$c_{12}$</td>
<td>$173$</td>
</tr>
<tr>
<td>$c_{33}$</td>
<td>$170.5$</td>
</tr>
<tr>
<td>Piezomagnetic</td>
<td></td>
</tr>
<tr>
<td>$q_{31}$</td>
<td>$580.3$</td>
</tr>
<tr>
<td>$q_{33}$</td>
<td>$699.7$</td>
</tr>
<tr>
<td>$d_{15}$</td>
<td>$550$</td>
</tr>
<tr>
<td>Dielectric</td>
<td></td>
</tr>
<tr>
<td>$\varepsilon_{11}$</td>
<td>$0.08$</td>
</tr>
<tr>
<td>$\varepsilon_{33}$</td>
<td>$0.093$</td>
</tr>
<tr>
<td>Magnetic-permeability</td>
<td></td>
</tr>
<tr>
<td>$\mu_{11}$</td>
<td>$-590$</td>
</tr>
<tr>
<td>$\mu_{33}$</td>
<td>$157$</td>
</tr>
</tbody>
</table>

The units of material parameters are as follows: $c_{ij}$ in $10^9$ N/m$^2$, $q_{ij}$ in N/Am, $\varepsilon_{ij}$ in $10^{-9}$ C$^2$/N m$^2$, and $\mu_{ij}$ in $10^{-6}$ N s$^2$/C$^2$.

### Table 3. Material parameters of viscoelastic epoxy (Yang et al., 2014).

<table>
<thead>
<tr>
<th>Relaxation moduli</th>
<th>Long-term moduli</th>
<th>Relaxation time</th>
<th>Poisson’s ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_1$</td>
<td>$562.2 \times 10^6$ N/m$^2$</td>
<td>$G_2$</td>
<td>$31.03 \times 10^6$ N/m$^2$</td>
</tr>
</tbody>
</table>

### Table 4. Details of the defined variables.

<table>
<thead>
<tr>
<th>Variables</th>
<th>Descriptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_{mn}$</td>
<td>$\sigma^1_{ij}$ at $x = 0.5a$, $y = 0.5b$, $z = 0$</td>
</tr>
<tr>
<td>$\tau_{mn}$</td>
<td>$\tau^1_{ij}$ at $x = 0$, $y = 0.5b$, $z = h_1$</td>
</tr>
<tr>
<td>$u_{mn}$</td>
<td>$u^1_{ij}$ at $x = 0.5a$, $y = 0.5b$, $z = 0$</td>
</tr>
<tr>
<td>$\phi_{mn}$</td>
<td>$\phi^1_{ij}$ at $x = 0.5a$, $y = 0.5b$, $z = 0$</td>
</tr>
<tr>
<td>$D_{mn}$</td>
<td>$D^1_{ij}$ at $x = 0$, $y = 0.5b$, $z = h_1$</td>
</tr>
<tr>
<td>$B_{mn}$</td>
<td>$B^1_{ij}$ at $x = 0$, $y = 0.5b$, $z = h_1$</td>
</tr>
</tbody>
</table>

following examples are given in Tables 1 to 3 (Pan, 2001; Yang et al., 2014). We beforehand define six variables as listed in Table 4.

### Convergence analysis

The present solution is in the form of double Fourier series; therefore, the convergence property should be assessed first. The series in equations (6) and (10) are truncated into finite ones, that is, $m, n = 1, 2, \ldots, N$. A simply supported three-layer rectangular plate under a uniform loading with $q(x, y) = 1$ N/m$^2$ is considered. The facial layers ($i = 1, 3$) and the core layer ($i = 2$) are made of BaTiO$_3$ and CoFe$_2$O$_4$, respectively. The dimensions of the plate are $a = 0.8$ m, $b = 1.2$ m, $h_1 = h_3 = 0.1$ m, $h_2 = 0.05$ m, $\Delta h = 2 \times 10^{-4}$ m.
Table 5. The present solution at the location of $x = 0.2 \text{ m}$, $y = 0.3 \text{ m}$, and $z = 0.1 \text{ m}$ when $t = 1000 \text{ s}$ with different series items $N = 1, 3, 5, \ldots, 13$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$u_x$</th>
<th>$u_y$</th>
<th>$u_z$</th>
<th>$\sigma_x$</th>
<th>$\tau_{xz}$</th>
<th>$\phi$</th>
<th>$D_u$</th>
<th>$\psi$</th>
<th>$B_x$</th>
</tr>
</thead>
</table>

The units of the above variables are as follows: $u_i$ in $10^{-11} \text{ m}$, $u_i'$ in $10^{-13} \text{ m}$, $\sigma_i$ and $\tau_{xz}$ in $10^{-11} \text{ N/m}^2$, $\phi$ in $10^{-3} \text{ V}$, $D_u'$ in $10^{-11} \text{ C/m}^2$, $\psi$ in $10^{-6} \text{ C/s}$, and $B_x$ in $10^{-11} \text{ Wb/m}^2$.

Table 6. Comparison of $\sigma_{xm}$, $\tau_{xz,m}$, $u_{zm}$, and $\phi_m$ when $t = 10^4 \text{ s}$ between the present solution and the FE solution.

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\sigma_{xm}$ (N/m$^2$)</th>
<th>$\tau_{xz,m}$ (10$^{-1}$ N/m$^2$)</th>
<th>$u_{zm}$ (10$^{-11}$ m)</th>
<th>$\phi_m$ (10$^{-3}$ V)</th>
<th>Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>6.500</td>
<td>-8.136</td>
<td>-2.701</td>
<td>-1.592</td>
<td>4.06</td>
</tr>
<tr>
<td>4</td>
<td>6.709</td>
<td>0.280</td>
<td>0.399</td>
<td>1.53</td>
<td>0.974</td>
</tr>
<tr>
<td>6</td>
<td>6.751</td>
<td>0.122</td>
<td>0.109</td>
<td>0.730</td>
<td>0.354</td>
</tr>
<tr>
<td>10</td>
<td>6.759</td>
<td>0.0486</td>
<td>0.145</td>
<td>0.133</td>
<td>0.236</td>
</tr>
<tr>
<td>20</td>
<td>6.781</td>
<td>0.0122</td>
<td>0.109</td>
<td>0.133</td>
<td>0.0886</td>
</tr>
</tbody>
</table>

The error is calculated using the formula $(\text{FE} - \text{present}) / \text{present}$.

The viscoelastic adhesive interlayer is made of epoxy. The electric condition of the interlayer is assumed to be unelectroded, that is, $\chi_1 = \chi_2 = 0$. The present solutions with different series items when $t = 10^4 \text{ s}$ are tabulated in Table 5. It can be observed from Table 5 that the present solution tends to be convergent as the number of series terms increases. Moreover, the convergence accuracy of the present solution reaches three significant digits when $N = 13$.

**Comparison analysis**

In this section, results from the finite element (FE) software ANSYS are compared with those from the present solution. Since ANSYS is unable to directly simulate MEE materials, we here consider a two-layer piezoelectric plate bonded by a viscoelastic interlayer. The two piezoelectric layers are made of BaTiO$_3$ and the viscoelastic adhesive interlayer is made of epoxy. The parameters are assumed to be: $q(x,y) = \sin(\pi x/a)\sin(\pi y/b) \text{ N/m}^2$, $a = b = 0.5 \text{ m}$, $h_1 = h_2 = 0.05 \text{ m}$, $\Delta h = 2 \times 10^{-4} \text{ m}$. The electric condition in the interlayer is assumed to be insulating with $\chi_1 = 0$ and $\chi_2 \to +\infty$ (Wang and Sudak, 2007). In the FE modeling, the piezoelectric layers are simulated by SOLID-5 element, and the viscoelastic interlayer is simulated by SOLID-185 element. The thicknesses of the interlayer and piezoelectric layers are divided into 1 and $\lambda$ elements, respectively, while the length and width are divided into 5 $\lambda$ elements. The FE results of $\sigma_{xm}$, $\tau_{xz,m}$, $u_{zm}$, and $\phi_m$ when $t = 10^4 \text{ s}$ for different $\lambda$ are compared with present ones (Table 6). It can be observed from Table 6 that the FE results approach the present ones along with an increase in $\lambda$. The errors of $\sigma_{xm}$, $\tau_{xz,m}$, $u_{zm}$, and $\phi_m$ are all less than 0.15% when $\lambda = 20$. However, the FE method becomes computationally expensive and time-consuming when $\lambda$ increases.

The error caused from the fourth assumption is investigated by comparing the present solution with the FE one which considers the horizontal normal stress in the interlayer. The reference structure is a two-layer BaTiO$_3$ plate with a full elastic interlayer which can be easily obtained by letting $G' = 0$ in equation (11). The parameters are fixed at $q(x,y) = \sin(\pi x/a)\sin(\pi y/b) \text{ N/m}^2$, $a = b = 1 \text{ m}$, $h_1 = h_2 = 0.1$, $\chi_3 = 0$. The comparison of $u_{zm}$ between the present solution and the FE solution with different interlayer shear modulus $G^*$ and thickness $\Delta h$ are plotted in Figure 3. It can be found that the error gently increases with the increase of $G^*$ when $\Delta h$ is small, while it becomes rapid as $\Delta h$ increases. The error is less than 2% when $\Delta h < 1 \times 10^{-3} \text{ m}$.

Since no analytical solution for layered MEE structures bonded by viscoelastic interlayer has been found in literature, the present solution is degraded into the PB case and compared with the exact solution from Pan (2001) and the FE solution based on the Reissner mixed variational theorem by Phoenix et al. (2009).
CoFe$_2$O$_4$ (F/B/F). The geometric parameters are taken as positive values, that is, equations (1) to (3).

Moreover, the present solution is compared with the analytical solution for infinite layered MEE cylindrical shell (Wu et al., 2018) with different curvature radius to show the linkage between the two works. Since the model for cylindrical shell is a 2D one, this model is degraded into 2D case by eliminating the quantities in $y$-direction, that is, $u_x$, $\tau_{xy}$, $\tau_{y}$, $\sigma_{y}^{p}$, $D_{y}^{1}$, $B_{y}^{1}$. Consider a 2D two-layer BaTiO$_3$ structure bonded by viscoelastic epoxy under sinusoidal load $q(x) = -0.5\sin(\pi x/a)N/m^2$. The parameters are fixed at $a = 4$ m, $h_1 = h_2 = 0.1$ m, $\Delta h = 5 \times 10^{-4}$ m, $\chi_1 = \chi_2 = 0$. The average length of the cylindrical shell is also taken as 4 m, while its curvature radius is variable. Table 8 shows the comparison between the present solution and the solution for layered cylindrical shell with different curvature radius. It can be observed from Table 8 that the solution for cylindrical shell tends to approach the present one as the shell curvature radius increases.

**Parameter analysis**

Consider a three-layer simply supported B/F/B plate adhesively bonded by epoxy subjected to sinusoidal loading of $q(x, y) = \sin(\pi x/a)\sin(\pi y/b)N/m^2$. In this section, the magnetic-permeability constant of CoFe$_2$O$_4$ are taken as positive values, that is, $\mu_{11}^{i} =$

### Table 7. Comparisons of $\phi'$ and $\psi'$ at the point of $x = 0.75$ m, $y = 0.25$ m, and $z = 0.3$ m, obtained by Phoenix et al. (2009), Pan (2001), and from the present solution in PB condition, respectively.

<table>
<thead>
<tr>
<th>Solution</th>
<th>$\phi'$ ($10^{-3}$ V)</th>
<th>$\psi'$ ($10^{-6}$ C/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Phoenix et al. (2009)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mesh size: $4 \times 4$</td>
<td>1.043</td>
<td>-2.180</td>
</tr>
<tr>
<td>Mesh size: $8 \times 8$</td>
<td>1.049</td>
<td>-2.181</td>
</tr>
<tr>
<td>Mesh size: $12 \times 12$</td>
<td>1.050</td>
<td>-2.183</td>
</tr>
<tr>
<td>Pan (2001)</td>
<td>1.054</td>
<td>-2.194</td>
</tr>
<tr>
<td>Present-PB</td>
<td>1.054</td>
<td>-2.194</td>
</tr>
<tr>
<td>F/B/F</td>
<td>2.144</td>
<td>-8.648</td>
</tr>
</tbody>
</table>

### Table 8. Comparison between the present solution and the solution for layered cylindrical shell with different curvature radius.

<table>
<thead>
<tr>
<th>Solutions</th>
<th>$\sigma_{zm}$ (N/m$^2$)</th>
<th>$\tau_{xzm}$ (N/m$^2$)</th>
<th>$u_{zm}$ (m)</th>
<th>$\psi_m$ (V)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solution for cylindrical shell</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R_c = 2$ m</td>
<td>230.2</td>
<td>-8.261</td>
<td>-4.410 $\times 10^{-8}$</td>
<td>-0.08965</td>
</tr>
<tr>
<td>$R_c = 3$ m</td>
<td>161.4</td>
<td>-5.911</td>
<td>-2.275 $\times 10^{-8}$</td>
<td>-0.06366</td>
</tr>
<tr>
<td>$R_c = 4$ m</td>
<td>144.7</td>
<td>-5.353</td>
<td>-1.876 $\times 10^{-8}$</td>
<td>-0.05746</td>
</tr>
<tr>
<td>$R_c = 5$ m</td>
<td>137.7</td>
<td>-5.121</td>
<td>-1.723 $\times 10^{-8}$</td>
<td>-0.05487</td>
</tr>
<tr>
<td>$R_c = 6$ m</td>
<td>133.9</td>
<td>-4.999</td>
<td>-1.646 $\times 10^{-8}$</td>
<td>-0.05350</td>
</tr>
<tr>
<td>$R_c = 7$ m</td>
<td>131.6</td>
<td>-4.927</td>
<td>-1.602 $\times 10^{-8}$</td>
<td>-0.05267</td>
</tr>
<tr>
<td>Present solution</td>
<td>129.5</td>
<td>-4.861</td>
<td>-1.563 $\times 10^{-8}$</td>
<td>-0.05193</td>
</tr>
</tbody>
</table>

$R_c$ means the curvature radius for the cylindrical shell.
Figure 4. The distributions of the present solution along the thickness at $x = 0.75$ m, $b = 0.25$ m, with $x_1 = 0, \tilde{a} = 2 \times 10^{-4}$ m at different times $t = 50, 500, \text{and} 5000$ s, respectively.
\[ \mu_{22} = 590 \times 10^{-6} \text{Ns}^2/\text{C}^2 \]. The parameters are assumed to be \( a = b = 1 \text{ m} \), \( h_1 = h_2 = h_3 = 0.1 \text{ m} \), \( x_1 = 0 \), while \( t, \Delta h \) and \( \chi_1 \) are variable.

The distributions of the present solution along the thickness direction at \( x = 0.75 \text{ m} \), \( b = 0.25 \text{ m} \), with \( x_1 = 0 \), \( D_{h1} = 2 \times 10^{-4} \text{ m} \) at different time \( t = 50, 500, \) and 5000 s and in PB condition are shown in Figure 4. It can be observed from Figure 4 that the jumps of \( u_i^1 \) and \( u_i^2 \) between adjacent layers increase with time, since the shear and elastic modulus of the interlayer in equations (11) and (12) decrease with time. Furthermore, the maximum of mechanical variables, in absolute values, increase with time. Compared with the PB case, the maximum absolute values of \( u_i^1 \), \( u_i^2 \), \( \sigma_i^1 \), and \( \tau_i^1 \) at \( t = 5000 \text{ s} \) increase by 45.2\%, 98.28\%, 32.3\%, and 2.09\%, respectively. Due to the MEE coupling effect, the distributions of electric and magnetic variables also change with the time. The maximum absolute value of \( D_j^1 \) decrease with the time, those of \( B_i^1 \) and \( B_j^1 \) increase with time and that of \( D_j^1 \) is almost invariable with time. Compared with the PB case, the maximum absolute value of \( D_j^1 \) at \( t = 5000 \text{ s} \) decreases by 9.93\%, while those of \( B_i^1 \) and \( B_j^1 \) increase by 76.2\% and 36.2\%, respectively.

The variations of \( \sigma_{zm} \), \( \tau_{zm} \), \( D_{zm} \), and \( B_{zm} \) with the time for different interlayer thicknesses are given in Figure 5. It can be observed from Figure 5 that \( \sigma_{zm} \) increases with time, while \( \tau_{zm} \), \( D_{zm} \) and \( B_{zm} \), in absolute values, decrease with time. They all tend to be constant, since the material property in the interlayer tends to be invariable in the long term, that is, \( G^1(t) \rightarrow G_z^1 \) when \( t \rightarrow \infty \). \( \sigma_{zm} \) increases as \( \Delta h \) increases, while the absolute values of \( \tau_{zm}, D_{zm}, \) and \( B_{zm} \) decrease as \( \Delta h \) increases. The results in PB case are close to the initial values (\( t \rightarrow 0 \)) of the model considering viscoelastic interlayer. In comparison with the PB results, the long-term absolute value (\( t \rightarrow \infty \)) of \( \sigma_{zm} \) when \( \Delta h = 3 \times 10^{-4} \text{ m} \) increases by 32.8\%, while those of \( \tau_{zm}, D_{zm}, \) and \( B_{zm} \) decrease by 26.3\%, 69.9\%, and 1.13\%, respectively.

The variations of \( D_{zm} \) and \( \phi_m \) with \( \chi_1 \) when \( t = 10^4 \text{ s} \) and \( \Delta h = 2 \times 10^{-4} \text{ m} \) are given in Figure 6. It can be observed from Figure 6 that \( D_{zm} \) decreases with \( \chi_1 \), while \( \phi_m \) increases with \( \chi_1 \). They all tend to be invariable as \( \chi_1 \) increases. Compared with the unelectroded condition, that is, \( \chi_1 = 0 \), the absolute value of \( D_{zm} \) when \( \chi_1 \rightarrow \infty \) increases by 556\%, while that of \( \phi_m \) decreases by 33.3\%.

Conclusion

An exact analytical solution for multilayer MEE plates adhesively bonded by viscoelastic interlayer under transverse loading is presented to study the time-dependent behavior of the plate. The results obtained support the following conclusions:
1. The present solution expressed in the series form has a good convergence property.

2. The FE solution with small mesh size is close to the present one; however, the FE solution becomes time-consuming as the mesh size decreases.

3. The time-dependent distributions of stress, electric, and magnetic fields in the plate are affected by the viscoelastic property of the interlayer. They all tend to be constant after about $t = 5000$ s. Compared with the results in perfect bond case, the maximum values of the horizontal displacement, transverse displacement, horizontal normal stress, transverse shear stress, and horizontal and transverse magnetic inductions increase by 45.2%, 98.28%, 32.3%, 2.09%, 76.2%, and 36.2%, respectively, while that of transverse electric displacement decrease by 9.93%.

4. The longitudinal normal stress on the plate surface increases with the increase of the interlayer thickness. The shear stress, longitudinal electric displacement, and magnetic induction near the interlayer decrease with the increase of the interlayer thickness. In comparison with the results from the perfect bond case, the long-term absolute value of horizontal normal stress when $\Delta h = 3 \times 10^{-4}$ m increases by 32.8%, while those of transverse shear stress, horizontal electric displacement, and horizontal magnetic induction decrease by 26.3%, 69.9%, and 1.13%, respectively.

5. The electric condition between adjacent layers has considerable effects on the electric displacement and potential. Compared with the unelectroded condition, the absolute value of horizontal electric displacement when interlayer electric coefficient tends to infinity increases by 556%, while that of electric potential decreases by 33.3%.

**Declaration of conflicting interests**

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**References**


\section*{Appendix 1}

\textbf{Notation}

- \(a; b; H\): length, width, and thickness of the plate, respectively.
- \(c^i_{jk}; e^i_{hj}; q^i_{ij}\): tensors of elastic, piezoelectric, piezomagnetic, dielectric, magnetic-permeability, and magneto-electric constants, respectively.
- \(d^i_{ik}; d^i_{jk}; \mu^i_{jk}\): interlayer electric coefficients.
- \(\theta^s_G; G^1; G^2\): relaxation time, relaxation moduli, and long-term moduli, respectively.
- \(\alpha^i_{jk}; \beta^i_{jk}; \gamma^i_{jk}\): tensors of stress, strain, electric displacement, electric field, magnetic induction, and magnetic field, respectively.
- \(\chi_1; \chi_2\): number of layers in the plate.
- \(\rho; p\): density, elastic displacement, electric, and magnetic potentials, respectively.
- \(\Delta h\): thickness of the \(i\)th layer and the interlayer, respectively.
- \(E^i; G^i; \mu^i\): elastic modulus, shear modulus, and Poisson’s ratio in the interlayer, respectively.
- \(d^i_{ik}; d^i_{jk}\): \(z\) coordinate values at the lower and upper surfaces of the \(i\)th layer, respectively.

\section*{Appendix 2}

The variables and material constants of MEE in equation (1) are defined as follows

\begin{equation}
\begin{align*}
|\sigma'| &= [\sigma'_x \quad \sigma'_y \quad \sigma'_z \quad \tau'_{yz} \quad \tau'_{xz} \quad \tau'_{xy}]^T, \quad |\gamma'| = [\epsilon'_x \quad \epsilon'_y \quad \epsilon'_z \quad \gamma'_y \quad \gamma'_z \quad \gamma'_x]^T \\
|D'| &= [D'_x \quad D'_y \quad D'_z], \quad |E'| = [E'_x \quad E'_y \quad E'_z]^T, \quad |B'| = [B'_x \quad B'_y \quad B'_z]^T, \quad |H'| = [H'_x \quad H'_y \quad H'_z]^T \\
|\epsilon'| &= \begin{bmatrix}
\epsilon'_{11} & 0 & 0 \\
0 & \epsilon'_{22} & 0 \\
0 & 0 & \epsilon'_{33}
\end{bmatrix}, \quad |\mu'| &= \begin{bmatrix}
\mu'_{11} & 0 & 0 \\
0 & \mu'_{22} & 0 \\
0 & 0 & \mu'_{33}
\end{bmatrix}
\end{align*}
\end{equation}

The matrix \(M\) in equation (4) is defined as

\begin{equation}
M = \begin{bmatrix}
0 & M_1 \\
M_2 & 0
\end{bmatrix}, \quad M_1 = \begin{bmatrix}
1 & 0 & -\epsilon'_{24} \frac{\partial}{\partial x} & -\epsilon'_{24} \frac{\partial}{\partial y} & -\epsilon'_{15} \frac{\partial}{\partial x} & -\epsilon'_{15} \frac{\partial}{\partial y} & -\frac{\partial}{\partial x} & -\frac{\partial}{\partial y} \\
0 & 1 & \epsilon'_{24} \frac{\partial}{\partial y} & \epsilon'_{24} \frac{\partial}{\partial y} & \epsilon'_{15} \frac{\partial}{\partial x} & \epsilon'_{15} \frac{\partial}{\partial x} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\
0 & 0 & f'_1 \frac{\partial^2}{\partial x^2} + f'_2 \frac{\partial^2}{\partial y^2} & f'_3 \frac{\partial^2}{\partial x^2} + f'_4 \frac{\partial^2}{\partial y^2} & 0 & 0 & 0 & 0 \\
f'_1 \frac{\partial^2}{\partial x^2} + f'_2 \frac{\partial^2}{\partial y^2} & f'_3 \frac{\partial^2}{\partial x^2} + f'_4 \frac{\partial^2}{\partial y^2} & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{equation}

\begin{equation}
M_2 = \begin{bmatrix}
-\sigma'_{11} \frac{\partial^2}{\partial x^2} - \epsilon'_{66} \frac{\partial^2}{\partial y^2} & -(\sigma'_{12} + \epsilon'_{66}) \frac{\partial^2}{\partial x \partial y} & -\nu'_{21} \frac{\partial}{\partial x} & -\nu'_{21} \frac{\partial}{\partial y} & -\nu'_{11} \frac{\partial}{\partial x} & -\nu'_{11} \frac{\partial}{\partial y} \\
(\sigma'_{12} + \epsilon'_{66}) \frac{\partial^2}{\partial x \partial y} & \sigma'_{22} \frac{\partial^2}{\partial y^2} - \epsilon'_{66} \frac{\partial^2}{\partial x^2} & -\nu'_{32} \frac{\partial}{\partial x} & -\nu'_{32} \frac{\partial}{\partial y} & -\nu'_{12} \frac{\partial}{\partial x} & -\nu'_{12} \frac{\partial}{\partial y} \\
0 & \xi'_{22} & \xi'_{23} & \xi'_{12} & \xi'_{11} & \xi'_{11} \\
0 & \xi'_{32} & \xi'_{33} & \xi'_{13} & \xi'_{11} & \xi'_{11} \\
\end{bmatrix}
\end{equation}
where “sym.” means the matrix is symmetric, and

\[ f_1^i = \frac{(e_{11})^2}{c_{55}}, \quad f_2^i = \frac{(e_{44})^2}{c_{44}}, \quad f_3^i = d_{11}^i + \frac{(e_{15})^2}{c_{55}} \]

\[ f_4^i = \frac{d_{22}^i + (e_{24})^2}{c_{44}}, \quad f_5^i = \mu_{11}^i, \quad f_6^i = \mu_{22}^i + \frac{(e_{24})^2}{c_{44}} \]

\[ \alpha_{ij}^k = c_{ij}^k - e_{ijk}^k, \quad k = 1, 2, j = 1, 2 \]

\[ \nu_{ij}^k = \frac{(\xi_{ij}^k e_{11} + \xi_{ij}^k e_{24} + \xi_{ij}^k e_{33})}{|\kappa|}, \quad k = 1, 2, j = 1, 2 \]

\[ \kappa^i = \begin{bmatrix} c_{33}^i & e_{33}^i & q_{33}^i \\ e_{33}^i & -e_{33}^i & -d_{33}^i \\ q_{33}^i & -d_{33}^i & -\mu_{33}^i \end{bmatrix} \]

in which, \( \xi_{ij}^k \) are the algebraic cofactors of \( \kappa^i \).

The matrix \( K_{mm}^i \) in equation (7) is defined as

\[
K_{mm}^i = \begin{bmatrix} 0 & K_{mm,1}^i \\ K_{mm,2}^i & 0 \end{bmatrix}, \quad K_{mm,1}^i = \begin{bmatrix} \frac{1}{c_{55}} & 0 & -\alpha_m e_{15}^i & -\alpha_m e_{24}^i & -\alpha_m \\
0 & \frac{1}{c_{44}} & -\beta_n e_{15}^i & -\beta_n e_{24}^i & -\beta_n \\
\alpha_m e_{15}^i & \beta_n e_{15}^i & (\alpha_m)^2 f_{11}^i & (\alpha_m)^2 f_{22}^i & 0 \\
\alpha_m e_{24}^i & \beta_n e_{24}^i & (\alpha_m)^2 f_{22}^i & (\alpha_m)^2 f_{22}^i & 0 \\
\alpha_m & \beta_n & 0 & 0 & 0 \end{bmatrix}
\]

\[
K_{mm,2}^i = \begin{bmatrix} \alpha_m \beta_n (\sigma_{12}^i + c_{66}^i) & -\alpha_m \nu_{21}^i & -\alpha_m \nu_{31}^i & -\alpha_m \nu_{11}^i \\
\alpha_m \beta_n (\sigma_{12}^i + c_{66}^i) & (\beta_n)^2 \nu_{22}^i & (\alpha_m)^2 c_{66}^i & -\beta_n \nu_{22}^i & -\beta_n \nu_{32}^i & -\beta_n \nu_{12}^i \\
-\alpha_m \nu_{21}^i & \beta_n \nu_{22}^i & \xi_{12}^i & \xi_{13}^i & \xi_{11}^i \\
-\alpha_m \nu_{31}^i & \beta_n \nu_{32}^i & \xi_{12}^i & \xi_{13}^i & \xi_{11}^i \\
-\alpha_m \nu_{11}^i & \beta_n \nu_{12}^i & \xi_{12}^i & \xi_{13}^i & \xi_{11}^i \end{bmatrix}
\]

The elements in \( A_m \) and \( D_m \) in equation (26) are given as follows

\[
A_{ma}^{10}(i,k) = -\frac{\Delta t}{G_j} \sum_{j=1}^{10} T_{ij}^m (d_j^i) \frac{\Omega_{mn,10}^{(i)} - \Omega_{mn}^{(i)}}{|\Omega_{mn}|}, \quad A_{ma}^{14}(i,k) = -\frac{\Delta t}{G_j} \sum_{j=1}^{10} T_{ij}^m (d_j^i) \frac{\Omega_{mn,10}^{(i)} - \Omega_{mn}^{(i)}}{|\Omega_{mn}|},
\]

\[
A_{ma}^{12}(i,k) = -\frac{\Delta t}{G_j} \sum_{j=1}^{10} T_{ij}^m (d_j^i) \frac{\Omega_{mn,10}^{(i)} - \Omega_{mn}^{(i)}}{|\Omega_{mn}|}, \quad A_{ma}^{14}(i,k) = -\frac{\Delta t}{G_j} \sum_{j=1}^{10} T_{ij}^m (d_j^i) \frac{\Omega_{mn,10}^{(i)} - \Omega_{mn}^{(i)}}{|\Omega_{mn}|},
\]

\[
A_{ma}^{20}(i,k) = -\frac{\Delta t}{G_j} \sum_{j=1}^{10} T_{ij}^m (d_j^i) \frac{\Omega_{mn,10}^{(i)} - \Omega_{mn}^{(i)}}{|\Omega_{mn}|}, \quad A_{ma}^{24}(i,k) = -\frac{\Delta t}{G_j} \sum_{j=1}^{10} T_{ij}^m (d_j^i) \frac{\Omega_{mn,10}^{(i)} - \Omega_{mn}^{(i)}}{|\Omega_{mn}|},
\]

\[
A_{ma}^{24}(i,k) = -\frac{\Delta t}{G_j} \sum_{j=1}^{10} T_{ij}^m (d_j^i) \frac{\Omega_{mn,10}^{(i)} - \Omega_{mn}^{(i)}}{|\Omega_{mn}|}, \quad A_{ma}^{24}(i,k) = -\frac{\Delta t}{G_j} \sum_{j=1}^{10} T_{ij}^m (d_j^i) \frac{\Omega_{mn,10}^{(i)} - \Omega_{mn}^{(i)}}{|\Omega_{mn}|},
\]
\[ 
\begin{align*}
\Lambda_{mn}^{11}(i, k) &= -\frac{\Delta h}{2(1 + \mu^*)G_1^2} \sum_{j=1}^{10} T_{ij, mn}(d_j^i) \left[ \frac{\Omega_{mn}^{(n)}}{\Omega_{mn}} \right] \left[ \Omega_{mn, 10i-10+j, k} \right] \\
\Lambda_{mn}^{12}(i, k) &= -\frac{\Delta h}{2(1 + \mu^*)G_1^2} \sum_{j=1}^{10} T_{ij, mn}(d_j^i) \left[ \frac{\Omega_{mn}^{(v)}}{\Omega_{mn}} \right] \left[ \Omega_{mn, 10i-10+j, k} \right] \\
\Lambda_{mn}^{35}(i, k) &= -\frac{\Delta h}{2(1 + \mu^*)G_1^2} \sum_{j=1}^{10} T_{ij, mn}(d_j^i) \left[ \frac{\Omega_{mn}^{(w)}}{\Omega_{mn}} \right] \left[ \Omega_{mn, 10i-10+j, k} \right] + \delta_{ik} G_1^2 \\
\Lambda_{mn}^{34}(i, k) &= -\frac{\Delta h}{2(1 + \mu^*)G_1^2} \sum_{j=1}^{10} T_{ij, mn}(d_j^i) \left[ \frac{\Omega_{mn}^{(6)}}{\Omega_{mn}} \right] \left[ \Omega_{mn, 10i-10+j, k} \right] \\
\Lambda_{mn}^{12}(i, k) &= -\frac{\Delta h}{2(1 + \mu^*)G_1^2} \sum_{j=1}^{10} T_{ij, mn}(d_j^i) \left[ \frac{\Omega_{mn}^{(D)}}{\Omega_{mn}} \right] \left[ \Omega_{mn, 10i-10+j, k} \right] \\
\Lambda_{mn}^{44}(i, k) &= \chi_2 \sum_{j=1}^{10} T_{ij, mn}(d_j^i) \left[ \frac{\Omega_{mn, 10i-10+j, k}}{\Omega_{mn}} \right] \\
\Lambda_{mn}^{45}(i, k) &= \chi_2 \sum_{j=1}^{10} T_{ij, mn}(d_j^i) \left[ \frac{\Omega_{mn, 10i-10+j, k}}{\Omega_{mn}} \right] \left[ \Omega_{mn}^{(w)} \right] \\
\Lambda_{mn}^{51}(i, k) &= \left[ (\alpha_m)^2 + (\beta_m)^2 \right] \chi_1 \sum_{j=1}^{10} T_{ij, mn}(d_j^i) \left[ \frac{\Omega_{mn}^{(v)}}{\Omega_{mn}} \right] \left[ \Omega_{mn, 10i-10+j, k} \right] \\
\Lambda_{mn}^{52}(i, k) &= \left[ (\alpha_m)^2 + (\beta_m)^2 \right] \chi_1 \sum_{j=1}^{10} T_{ij, mn}(d_j^i) \left[ \frac{\Omega_{mn}^{(w)}}{\Omega_{mn}} \right] \left[ \Omega_{mn, 10i-10+j, k} \right] \\
\Lambda_{mn}^{53}(i, k) &= \left[ (\alpha_m)^2 + (\beta_m)^2 \right] \chi_1 \sum_{j=1}^{10} T_{ij, mn}(d_j^i) \left[ \frac{\Omega_{mn}^{(w)}}{\Omega_{mn}} \right] \left[ \Omega_{mn, 10i-10+j, k} \right] \\
\Lambda_{mn}^{54}(i, k) &= \left[ (\alpha_m)^2 + (\beta_m)^2 \right] \chi_1 \sum_{j=1}^{10} T_{ij, mn}(d_j^i) \left[ \frac{\Omega_{mn}^{(6)}}{\Omega_{mn}} \right] \left[ \Omega_{mn, 10i-10+j, k} \right] \\
\Lambda_{mn}^{55}(i, k) &= \left[ (\alpha_m)^2 + (\beta_m)^2 \right] \chi_1 \sum_{j=1}^{10} T_{ij, mn}(d_j^i) \left[ \frac{\Omega_{mn}^{(D)}}{\Omega_{mn}} \right] \left[ \Omega_{mn, 10i-10+j, k} \right] \left[ \Omega_{mn}^{(w)} \right] \\
D_{mn}^{11}(i) &= \frac{\Delta h}{G_1^2} \sum_{j=1}^{10} T_{ij, mn}(d_j^i) \left[ \frac{\Omega_{mn}^{(q)}}{\Omega_{mn}} \right] \left[ \Omega_{mn, 10i-10+j} \right] q_{mn} \\
D_{mn}^{22}(i) &= \frac{\Delta h}{G_1^2} \sum_{j=1}^{10} T_{ij, mn}(d_j^i) \left[ \frac{\Omega_{mn}^{(q)}}{\Omega_{mn}} \right] \left[ \Omega_{mn, 10i-10+j} \right] q_{mn} \\
D_{mn}^{33}(i) &= -\chi_2 \sum_{j=1}^{10} T_{ij, mn}(d_j^i) \left[ \frac{\Omega_{mn}^{(q)}}{\Omega_{mn}} \right] \left[ \Omega_{mn, 10i-10+j} \right] q_{mn} \\
D_{mn}^{44}(i) &= -\chi_2 \sum_{j=1}^{10} T_{ij, mn}(d_j^i) \left[ \frac{\Omega_{mn}^{(q)}}{\Omega_{mn}} \right] \left[ \Omega_{mn, 10i-10+j} \right] q_{mn} \\
D_{mn}^{55}(i) &= -\chi_2 \sum_{j=1}^{10} T_{ij, mn}(d_j^i) \left[ \frac{\Omega_{mn}^{(q)}}{\Omega_{mn}} \right] \left[ \Omega_{mn, 10i-10+j} \right] q_{mn} \\
\delta_{ik} &= \begin{cases} 
1, & i = k \\
0, & i \neq k 
\end{cases}
\end{align*} \] (A3)

**Appendix 3**

The calculations of \( F_{mn}^{2, 2}, F_{mn}^{2, 1}, \) and \( N_{mn}^{4, 2} \) when \( p = 2 \) are given as an example, as follows
$$J^2_{mn} = \begin{bmatrix} 1 & A_{13}^{mn} & A_{14}^{mn} & A_{15}^{mn} \\ 0 & A_{23}^{mn} & A_{24}^{mn} & A_{25}^{mn} \\ 0 & A_{33}^{mn} & A_{34}^{mn} & A_{35}^{mn} \\ 0 & A_{43}^{mn} & A_{44}^{mn} & A_{45}^{mn} \\ 0 & A_{53}^{mn} & A_{54}^{mn} & A_{55}^{mn} \end{bmatrix} = 1 + \frac{1}{C_{12}} \begin{bmatrix} 0 & A_{12}^{mn} & 0 & A_{14}^{mn} & A_{15}^{mn} \\ 0 & A_{22}^{mn} & 0 & A_{24}^{mn} & A_{25}^{mn} \\ 0 & A_{32}^{mn} & 1 & A_{34}^{mn} & A_{35}^{mn} \\ 0 & A_{42}^{mn} & 0 & A_{44}^{mn} & A_{45}^{mn} \\ 0 & A_{52}^{mn} & 0 & A_{54}^{mn} & A_{55}^{mn} \end{bmatrix}$$

$$L^2_{mn} = \begin{bmatrix} 1 & D_{13}^{mn} & D_{14}^{mn} & A_{15}^{mn} \\ 0 & D_{23}^{mn} & D_{24}^{mn} & A_{25}^{mn} \\ 0 & D_{33}^{mn} & D_{34}^{mn} & A_{35}^{mn} \\ 0 & D_{43}^{mn} & D_{44}^{mn} & A_{45}^{mn} \\ 0 & D_{53}^{mn} & D_{54}^{mn} & A_{55}^{mn} \end{bmatrix} = 1 + \frac{1}{C_{12}} \begin{bmatrix} A_{11}^{mn} & D_{11}^{mn} & 0 & A_{14}^{mn} & A_{15}^{mn} \\ A_{21}^{mn} & D_{21}^{mn} & 0 & A_{24}^{mn} & A_{25}^{mn} \\ A_{31}^{mn} & D_{31}^{mn} & 1 & A_{34}^{mn} & A_{35}^{mn} \\ A_{41}^{mn} & D_{41}^{mn} & 0 & A_{44}^{mn} & A_{45}^{mn} \\ A_{51}^{mn} & D_{51}^{mn} & 0 & A_{54}^{mn} & A_{55}^{mn} \end{bmatrix}$$

$$N^4_{mn} = \begin{bmatrix} 1 & A_{13}^{mn} & D_{14}^{mn} & A_{15}^{mn} \\ 0 & A_{23}^{mn} & D_{24}^{mn} & A_{25}^{mn} \\ 0 & A_{33}^{mn} & D_{34}^{mn} & A_{35}^{mn} \\ 0 & A_{43}^{mn} & D_{44}^{mn} & A_{45}^{mn} \\ 0 & A_{53}^{mn} & D_{54}^{mn} & A_{55}^{mn} \end{bmatrix} = 1 + \frac{1}{C_{12}} \begin{bmatrix} 1 & A_{12}^{mn} & 0 & D_{14}^{mn} & A_{15}^{mn} \\ 0 & A_{22}^{mn} & 0 & D_{24}^{mn} & A_{25}^{mn} \\ 0 & A_{32}^{mn} & 1 & D_{34}^{mn} & A_{35}^{mn} \\ 0 & A_{42}^{mn} & 0 & D_{44}^{mn} & A_{45}^{mn} \\ 0 & A_{52}^{mn} & 0 & D_{54}^{mn} & A_{55}^{mn} \end{bmatrix}$$