Damage analysis of thermopiezoelectric properties: Part II. Effective crack model

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Abstract

Having completed the general formulation for temperature, heat flow, displacement, electric potential and displacements and mechanical stresses of a piezoelectric material as presented in part I of this work, part II is concerned with a generalized self-consistent approximate method for determining the thermoelastic properties of piezoelectric materials weakened by microcracks. A representative area element is adopted; it contains a microcrack surrounded by an elliptic matrix in a solid with effective properties. Numerical results are given for a piezoelectric BaTiO3 ceramic. The effective conductivity and effective modulus are found to decrease with increasing crack density.

1. Introduction

The manufacturing of 'smart structures' made of piezoelectric materials would invariably involve the introduction of internal stresses that would lead to microcracks. Such defects tend to alter the thermal and mechanical properties and hence would affect the failure behavior. It is, therefore, essential to know the structural behavior in the presence of defects due to the combined effects of mechanical, electric and thermal loads.

Micromechanical modelling of piezocomposites has been made for determining their effective physical properties. The connectivity theory [1] assumes that the continuous solid could be modelled by a lump mass system. The analysis was extended [2] to consider discontinuous reinforcement. Interactions between continuous fibers with finite concentrations was analyzed [3] by using a concentric cylinder model to predict the effective electroelastic moduli of a continuous fiber reinforced composite. Estimated were the effective properties [4] using dilute, self-consistent, and differential micromechanical models. Formulas for estimating the overall thermo-electroelastic moduli of multiphase fibrous can be found in [5]. Two-phase fibrous piezocomposites uniform strain and electric fields can be generated [6] by certain loading conditions. The concept of uniform fields was further elaborated [7,8] in two, three and four phase composites with cylindrical microstructures. Many of the earlier studies on fracture behavior of piezoelectric structures were focused on specific physical properties (e.g. damage-free transversely isotropic materials). The concept of domain microcrack growth for analyzing microcrack-weakened brittle solids were proposed [9].

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Effective behavior of piezoelectric materials with cracks are analyzed with numerical results found for a piezoelectric $\text{BaTiO}_3$ ceramic. Crack interactions are accounted for in an approximate manner by a model that assumes a micro crack is surrounded by an elliptical matrix with effective properties. Estimated are the decay of the effective conductivity and modulus with increasing crack density.

2. Effective crack model

Assume that a piezoelectric material with microcracks can be modeled by application of a representative area element as shown in Fig. 1. The model contain a single microcrack of length $2a$ embedded in an elliptical matrix with modulus $E^M$ while the surrounding material outside the ellipse has an effective modulus $E^*$ yet unknown Fig. 1. The major and minor axes of the elliptical inclusion are chosen as

$$l_1 = a + \delta, \quad l_2 = \delta,$$

where $a$ is the half-length of the crack. For simplicity, we assume that all cracks have the same length and orientation. $\delta$ is related to the crack density by [10]

$$\varepsilon = \frac{a^2}{\pi (a + \delta) \delta} = N a^2 / \Lambda.$$

The derivation below is similar to the uncoupled theory of thermo-elasticity, while the electric and elastic fields are coupled. The temperature enters into the problem only through the constitutive equations. The effective conductivity and the effective electroelastic constants can thus be determined independently. Evaluation of the effective thermal stress and pyroelectric coefficients requires a knowledge of the effective conductivity.

3. Effective conductivity

Consider a representative area element containing a set of microcracks. Define the intensity $H_i$ by

$$H_i = -\frac{\partial \theta}{\partial x_i}.$$  

The ‘energy’ is given by [11]

$$E = q_i( x_1, x_2) H_i( x_1, x_2).$$

To produce a statistical homogeneous field in a cracked medium, it is expedient to apply boundary conditions

![Fig. 1. Schematic of representative area element with a crack.](image)
that produce homogeneous fields in a homogeneous body. For heat conduction problems, the boundary conditions are

\[ \theta(s) = H_i^0 x_i, \]

or

\[ q_n (s) = q_i^0 n_i, \]

where \( s \) denotes the external boundary. The superscript 0 stands for constant, and \( n_i \) is the outward normal to \( s \). The effective conductivity \( k_{ij}^* \) and the resistivity \( \rho_{ij}^* \) are defined as

\[ q_j = k_{ij}^* \tilde{H}_j, \]

\[ \tilde{H}_j = \rho_{ij}^* \tilde{q}_j^*, \]

where the asterisks denote the effective value, and the symbol \( \cup \) denotes area average. The conditions in Eqs. (5) and (6) can be used to determine \( k_{ij}^* \) and \( \rho_{ij}^* \). Only the procedure leading to \( \rho_{ij}^* \) will be presented. Consider the average flux

\[ \tilde{q} = \tilde{q}^0. \]

Hence, the effective resistivity \( \rho_{ij}^* \) may be given by extending the results in [12] to the cracked problem

\[ \rho_{ij}^* \tilde{q}_j^0 = \rho_{ij}^M \tilde{q}_j^0 + \frac{1}{A} \sum_{k=1}^{N} \int_{l_k} \Delta \tilde{\theta} \tilde{n}_i \, d\sigma, \]

where \( A \) and \( N \) are respectively the area and crack number of the representative area element. The length of the \( k \)th crack is \( l_k \) and the superscript \( M \) stands for matrix. Hence

\[ \Delta \tilde{\theta}(\tilde{x}_i) = \tilde{\theta}_{(1)}^M(\tilde{x}_i) - \tilde{\theta}_{(2)}^M(\tilde{x}_i), \quad \tilde{x}_2 = 0. \]

It is seen from Eq. (10) that the determination of the effective resistivity \( \rho_{ij}^* \) requires a knowledge of the jump \( \Delta \theta \) in the cracked medium when Eq. (6) is prescribed. For a microcrack-weakened medium, an exact solution for \( \Delta \theta \) is not feasible and approximate methods are usually devised to determine the effective resistivity \( \rho_{ij}^* \). A generalized self-consistent method will be adopted. The approach is based on a crack-matrix-effective medium model [10] as shown in Fig. 1. Since it is difficult to find temperature field analytically for the model, an approximate method [10] will be adopted to estimate the jump \( \Delta \theta \). In this case, the potential 'energy' is defined as

\[ J(\theta) = \int_{S_M} k_{ij}^M \theta_i \theta_j \, dS + \int_{S_E} k_{ij}^M \theta_i \theta_j \, dS - \int_{-a}^{a} \Delta \theta \tilde{q}_j^0 \tilde{n}_i \, d\sigma. \]

Among all possible solutions, the exact temperature field leads to \( J_{\text{min}} \). For convenience, let \( \theta^M \) be the temperature field for a crack embedded in a matrix material. The crack is subjected to a unit flux on the crack surface while \( \theta^{BM} \) stands for the temperature field of a crack embedded in an effective medium. These solutions can be obtained by using the technique presented in part I [13] of this work. The results are

\[ \Delta \theta^M = \frac{4}{k_M^M} \left( a^2 - x_i^2 \right)^{1/2}, \]

\[ \theta_{(1)}^M(z_i) = \frac{2}{k_M^M} \text{Re} \left[ \left( a^2 - z_i^2 \right)^{1/2} + i z_i \right], \]

\[ \theta_{(2)}^M(z_i) = -\frac{2}{k_M^M} \text{Re} \left[ \left( a^2 - z_i^2 \right)^{1/2} + i \bar{z}_i \right]. \]
The solutions for a crack embedded in an effective medium can be obtained from Eqs. (13)–(15) if \( k^M \) is replaced by \( k^* \) in Eqs. (13)–(15). The approximate temperature \( \theta \) is assumed to be the linear superposition of the above fields, i.e.,

\[
\theta = q_2^0 \left( \chi^M \theta^M + \chi^{EM} \theta^{EM} \right) \quad (M \text{ and } EM \text{ not summed}),
\]

where the superscript EM stands for effective medium while \( \chi^M \) and \( \chi^{EM} \) are the constants to be determined by the principle of minimum potential energy. Substituting Eq. (13) to Eq. (16) into Eq. (12) and minimizing the result yield

\[
\begin{bmatrix}
I_{11} & I_{12} \\
I_{12} & I_{22}
\end{bmatrix}
\begin{bmatrix}
\chi^M \\
\chi^{EM}
\end{bmatrix}
= a^2 \pi \begin{bmatrix}
1/k^M \\
1/k^*
\end{bmatrix},
\]

where

\[ I_{11} = \int_{S^M} k^M_{ij} \theta^M_{i,j} ds + \int_{S^{EM}} k^*_i \theta^M_{i,j} ds, \]

\[ I_{12} = \int_{S^M} k^M_{ij} \theta^{EM}_{i,j} ds + \int_{S^{EM}} k^*_i \theta^{EM}_{i,j} ds, \]

\[ I_{22} = \int_{S^M} k^{EM}_{ij} \theta^M_{i,j} ds + \int_{S^{EM}} k^*_i \theta^{EM}_{i,j} ds. \]

Solving Eq. (17) for \( \chi^M \) and \( \chi^{EM} \) and substituting the results into Eq. (16) the jump \( \Delta \theta \) can be determined. The quantity \( \rho^*_{22} \) can thus be found from Eq. (10) as

\[
\rho^*_{22} = \rho^M_{22} + 2 \pi \varepsilon \left( \frac{\chi^M}{k^M} + \frac{\chi^{EM}}{k^*} \right).
\]

4. Effective modulus

The present approach referred to as the generalized self-consistent procedure will be developed by way of the results for the dilute and self-consistent methods.

4.1. Dilute and self-consistent approximation

Referring to the representative area element in Fig. 1, the effective material properties can be defined as

\[
\tilde{\tilde{I}} = E^* \tilde{Z} - \beta^* \tilde{\theta},
\]

or the equivalent

\[
\tilde{Z} = F^* \tilde{I} + \alpha^* \tilde{\theta},
\]

where \( F^* \) is the inverse of \( E^* \), \( \tilde{Z} \) is average strain and electric field. The other quantities in Eqs. (22) and (23) are

\[
\alpha = F \beta
\]

\[
Z_{ij} \begin{cases} 
(u_{i,j} + u_{j,i})/2 & \text{for } i; j = 1, 2, 3, \\
\phi_{i,j} & \text{for } i = 4; j = 1, 2, 3.
\end{cases}
\]
For a cracked medium, $\tilde{N}$ and $\tilde{Z}$ in Eqs. (22) and (23) are given by
\begin{align}
\tilde{N} &= \tilde{N}^M, \\
\tilde{Z} &= \tilde{Z}^M + Z^c.
\end{align}
(26) (27)

Note that
\begin{equation}
\tilde{Z}^c_{ij} = \frac{1}{2A} \sum_{k=1}^{N} \int_{i_k} \left[ (1 + H(i - 4)) \Delta U_j \tilde{n}_j + H(3 - j) \Delta U_j \tilde{n}_j \right] \, dc,
\end{equation}
(28)

where $\tilde{n} = (\tilde{n}_1, \tilde{n}_2, 0, 0)^T$. Making use of Eqs. (26)–(28), these results
\begin{align}
E_{i j k l} Z^0_{k l} &= E_{i j k l}^M Z^0_{k l} - E_{i j k l} Z^c_{k l}, \\
F_{i j k l} \Pi^0_{k l} &= F_{i j k l}^M \Pi^0_{k l} + \tilde{Z}^c_{ij}.
\end{align}
(29) (30)

Two auxiliary problems will be considered for determining the effective modulus of the crack solid by using a set of uniform boundary conditions.

**Isothermal.** The isothermal problem pertains to specifying traction-charge or displacement and electric potential that would produce uniform field in a homogeneous material. That is
\begin{equation}
t(s) = \Pi^0 n
\end{equation}
(31)
or
\begin{equation}
U(s) = Z^0 x
\end{equation}
(32)
together with
\begin{equation}
\theta(s) = 0,
\end{equation}
(33)
where $t(s)$ stands for the traction-charge vector on the boundary $s$.

**Uniform temperature.** The uniform temperature problem corresponds to specifying the homogeneous boundary condition for traction-charge or displacement and electric potential as stated for the isothermal problem. That is
\begin{equation}
\theta(s) = \theta^0
\end{equation}
(34)
together with
\begin{equation}
t(s) = 0
\end{equation}
(35)
or
\begin{equation}
U(s) = 0.
\end{equation}
(36)

When Eqs. (31) and (33) are prescribed, it follows from the energy theorem [14] that
\begin{equation}
\tilde{N} = \Pi^0, \quad \tilde{\theta} = 0.
\end{equation}
(37)

As a consequence, Eqs. (22) and (23) are reduced to
\begin{align}
\tilde{N} &= E \cdot \tilde{Z}, \\
\tilde{Z} &= F \cdot \tilde{N}.
\end{align}
(38) (39)

For boundary conditions in Eqs. (32) and (33) it is found that
\begin{equation}
\tilde{Z} = Z^0, \quad \tilde{\theta} = 0.
\end{equation}
(40)
From the boundary conditions in Eqs. (34) and (35), Eq. (23) gives
\[ \tilde{\mathbf{Z}} = \alpha^* \theta^0. \]  
(41)

Making one of Eqs. (27), (29) and (41), the dilute approximation yields
\[ \alpha^*_{ij} = \alpha^0_{ij} - \frac{\pi}{4A} \sum_{\beta = 1}^{N} \frac{(l_{\beta})^2}{2} \bar{\Omega}_{\beta} \hat{\Omega}_{ij} \left\{ [1 + H(k - 4)] \bar{b}_i \delta_{2j} + H(3 - l) \bar{b}_j \delta_{2i} \right\}. \]  
(42)

\[ F_{ij,kl}^* = F^0_{ij,kl} + \frac{\pi}{4A} \sum_{\beta = 1}^{N} \frac{(l_{\beta})^2}{2} \bar{\Omega}_{\beta,j} \hat{\Omega}_{ij} \hat{\Omega}_{mk} \hat{\Omega}_{2i} \left\{ [1 + H(p - 4)] \bar{c}_{pm} \delta_{2q} + H(3 - q) \bar{c}_{qm} \delta_{2p} \right\}. \]  
(43)

Assume that all cracks have the same length and orientation, Eqs. (29), (30) and (42) can be reduced to
\[ \alpha^*_{ij} = \alpha^0_{ij} - \frac{\pi \varepsilon}{4} \left\{ [1 + H(i - 4)] b_i \delta_{2j} + H(3 - j) b_j \delta_{2i} \right\}, \]  
(44)

\[ E_{ij,k}^* Z^0_{kl} = E^M_{ij,k} Z^0_{kl} - \frac{\pi \varepsilon}{4} \int_{S} \left\{ [1 + H(P - 4)] C_{MN} n_{q} E^M_{25k} + H(3 - q) C_{Q5} n_{p} E^M_{25k} \right\} Z^0_{kl}. \]  
(45)

\[ F_{ij,k}^* \Pi^0_{ik} = F^M_{ij,k} \Pi^0_{ik} \left\{ [1 + H(i - 4)] C_{ij} n_{j} + H(3 - j) C_{ij} n_{j} \right\} \Pi^0_{ik}. \]  
(46)

In the self-consistent approach, the effect of microcrack interaction is taken into account approximately by embedding each microcrack directly in the effective medium. Therefore, the algebraic equations for the effective moduli are formed by replacing band C in Eqs. (44)–(46) by the as yet unknown constants \( b^* \) and \( C^* \).

### 4.2. Generalized self-consistent approximation

As before the model on Fig. 1 will be used to estimate the effective elastoelectric constants. In contrast to Eq. (12) for heat conduction, the potential energy is given by
\[ J(U) = \frac{1}{2} \int_{S} E^M_{ij,k} U_{ij,k} U_{ij,m} ds + \frac{1}{2} \int_{S} E^M_{ij,k} U_{ij,k} U_{ij,m} ds - \int_{-a}^{0} (\sigma_z^2)^T \Delta U d\zeta. \]  
(47)

A possible form of displacement or electric potential field is
\[ U = \sum_{l=1}^{4} \Pi^0_{l,k} (\chi^M_{ij} U^M_{ij} + \chi^{EM}_{ij} U^EM_{ij}). \]  
(48)

where \( \chi^M \) and \( \chi^{EM} \) are two \( 4 \times 4 \) constant matrices to be determined by the minimum potential principle. For simplicity, consider a plane strain model where coupling between in-plane stresses and electric fields take place. Further, assume that the material is transversely isotropic and let the \( x_2 \)-axis to coincide with the axis of symmetry. The plane strain constitutive equations can thus be expressed as
\[
\begin{pmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{12} \\
D_1 \\
D_2
\end{pmatrix} =
\begin{pmatrix}
\varepsilon_{11} & \varepsilon_{12} & 0 & 0 & \varepsilon_{31} \\
\varepsilon_{12} & \varepsilon_{22} & 0 & 0 & \varepsilon_{32} \\
0 & 0 & \varepsilon_{44} & \varepsilon_{15} & 0 \\
0 & 0 & \varepsilon_{15} & -\varepsilon_{11} & 0 \\
\varepsilon_{31} & \varepsilon_{32} & 0 & -\varepsilon_{33} & 0
\end{pmatrix}
\begin{pmatrix}
Z_{11} \\
Z_{22} \\
Z_{12} \\
D_1 \\
D_2
\end{pmatrix} - \begin{pmatrix}
\gamma_{11} \\
\gamma_{33} \\
2Z_{12} \\
D_1 \\
D_2
\end{pmatrix} \theta.
\]  
(49)

The inverse is
\[
\begin{pmatrix}
Z_{11} \\
Z_{22} \\
2Z_{12} \\
Z_{41} \\
Z_{42}
\end{pmatrix} =
\begin{pmatrix}
f_{11} & f_{12} & 0 & 0 & p_{31} \\
f_{12} & f_{22} & 0 & 0 & p_{33} \\
0 & 0 & f_{44} & p_{15} & 0 \\
0 & 0 & p_{15} & -1/\varepsilon_{11} & 0 \\
p_{31} & p_{33} & 0 & 0 & -1/\varepsilon_{33}
\end{pmatrix}
\begin{pmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{12} \\
D_1 \\
D_2
\end{pmatrix} + \begin{pmatrix}
\alpha_{11} \\
\alpha_{33} \\
\alpha_{33} \\
D_1 \\
D_2
\end{pmatrix} \theta.
\]  
(50)
where the familiar two-index notation is adopted [15], and \( c_{ij}, e_{ij}, f_{ij}, p_{ij} \) and \( e_{ij} \) are the reduced material constants. Further, it should be pointed out that the lower case \( c = c_{ij} \) and upper case \( C = C_{ij} \) are two different matrices.

First, let the representative area element under consideration be subjected to a homogeneous stress \( \sigma_{21}^0 = \sigma_{21}^0 \) and the other stresses are zero. Hence, Eq. (48) becomes

\[
U = \left( \chi_1^M U^M + \chi_1^{EM} U^{EM} \right) \sigma_{21}^0,
\]  

(51)

where \( \chi_1^K = \text{diag}[\chi_{11}^K, \chi_{12}^K, \chi_{13}^K] \) (\( K = M, EM \)). Substituting Eq. (51) into Eq. (48) and minimizing the result, it is found that

\[
\begin{pmatrix}
\phi_{11} & \Phi_{12} \\
\Phi_{12} & \phi_{22}
\end{pmatrix}
\begin{pmatrix}
\chi_1^M \\
\chi_1^{EM}
\end{pmatrix}
= \frac{\pi a^2}{2}
\begin{pmatrix}
C_1^M \\
C_1^{*}
\end{pmatrix},
\]  

(52)

where

\[
(\Phi_{11})_{JK} = \int_{S_{ijK}} E^M_{ijKm} U^M_{ij,K} U^M_{K,m} \, ds + \int_{S_{ijEM}} E^{*M}_{ijKm} U^M_{ij,K} U^M_{K,m} \, ds \quad (J, K \text{ not summed}),
\]

(53)

\[
(\Phi_{12})_{JK} = 2 \int_{S_{ijEM}} E^M_{ijKm} U^EM_{ij,K} U^M_{K,m} \, ds + 2 \int_{S_{ijEM}} E^{*M}_{ijKm} U^EM_{ij,K} U^M_{K,m} \, ds \quad (J, K \text{ not summed}),
\]

\[
(\Phi_{22})_{JK} = \int_{S_{ijEM}} E^M_{ijKm} U^EM_{ij,K} U^EM_{K,m} \, ds + \int_{S_{ijEM}} E^{*M}_{ijKm} U^EM_{ij,K} U^EM_{K,m} \, ds \quad (J, K \text{ not summed}),
\]

(54)

\[
C_1^M = \begin{pmatrix} C_{11}^M & C_{12}^M & C_{13}^M \end{pmatrix}^T,
\]

(55)

\[
C_1^{*} = \begin{pmatrix} C_{11}^{*} & C_{12}^{*} & C_{13}^{*} \end{pmatrix}^T,
\]

(56)

in which \( C \) as defined in Eq. (117) of [13] can be rewritten as

\[
C = -2 \text{Im}[AB^{-1}],
\]

(57)

where the eigenvalues \( p_j \) are determined by

\[
a_0 p^6 + a_1 p^4 + a_2 p^2 + a_3 = 0
\]

(58)

with

\[
a_0 = -c_{44}(c_{33} e_{33} + e_{33}^2),
\]

\[
a_1 = e_{33}(c_{13}^2 + 2 c_{13} c_{44} - c_{11} c_{33}) + (e_{15} + e_{31})[2 e_{33}(c_{13} + c_{44}) - c_{33}(e_{15} + e_{31})]
\]

\[
- c_{44}(2 e_{15} e_{33} + c_{33} e_{11}) - c_{11} e_{33},
\]

\[
a_2 = e_{11}(c_{13}^2 + 2 c_{13} c_{44} - c_{11} c_{33}) + (e_{15} + e_{31})[2 e_{15}(c_{13} + c_{44}) - c_{44}(e_{15} + e_{31})]
\]

\[
- c_{11}(2 e_{15} e_{33} + c_{44} e_{11}) - c_{44} e_{15},
\]

\[
a_3 = -c_{11}(e_{11} c_{44} + e_{11}^2),
\]

Solving Eq. (52) for \( \chi_1^M \) and \( \chi_1^{EM} \), \( \Delta U \) can be determined from Eq. (133) in [13] as

\[
\begin{pmatrix}
\Delta u_1 \\
\Delta u_2 \\
\Delta \phi
\end{pmatrix}
= \begin{pmatrix}
\chi_{11}^M C_{11}^M + \chi_{11}^{EM} C_{11}^{*} \\
\chi_{12}^M C_{21}^M + \chi_{12}^{EM} C_{21}^{*} \\
\chi_{13}^M C_{31}^M + \chi_{13}^{EM} C_{31}^{*}
\end{pmatrix}
\begin{pmatrix}
a^2 - x_1^2 \end{pmatrix}^{1/2} \sigma_{21}^0.
\]  

(59)
Putting Eq. (58) into Eq. (30) and noting Eq. (50), the result gives
\[
\begin{pmatrix}
    f_{44}^* \\
    p_{15}^*
\end{pmatrix} = \begin{pmatrix}
    f_{44}^M \\
    p_{15}^M
\end{pmatrix} + \begin{pmatrix}
    \chi_{11}^M C_{11}^M + \chi_{11}^{EM} C_{11}^* \\
    0
\end{pmatrix} \frac{\pi \varepsilon}{2} \cdot \tag{59}
\]
Now, let \( \Pi_{22}^0 = \sigma_{22}^0 \) or \( \Pi_{24}^0 = D_2 \) and others equal to zero, it follows that
\[
\begin{pmatrix}
    f_{13}^* \\
    f_{33}^* \\
    p_{33}^*
\end{pmatrix} = \begin{pmatrix}
    f_{13}^M \\
    f_{33}^M \\
    p_{33}^M
\end{pmatrix} + \begin{pmatrix}
    \chi_{22}^M C_{22}^M + \chi_{22}^{EM} C_{22}^* \\
    \chi_{33}^M C_{33}^M + \chi_{33}^{EM} C_{33}^*
\end{pmatrix} \frac{\pi \varepsilon}{2} \cdot \tag{60}
\]
or
\[
\begin{pmatrix}
    p_{31}^* \\
    -1/\varepsilon_{33}^*
\end{pmatrix} = \begin{pmatrix}
    p_{31}^M \\
    -1/\varepsilon_{33}^M
\end{pmatrix} + \begin{pmatrix}
    0 \\
    \chi_{33}^M C_{33}^M + \chi_{33}^{EM} C_{33}^*
\end{pmatrix} \frac{\pi \varepsilon}{2} . \tag{61}
\]
Eqs. (51), (52), (59)–(61) constitute the governing equations for determining the effective moduli \( F^* \) of a cracked piezoelectric solid. Once \( k_{ij}^* \) and \( F^* \) is obtained, the effective \( \alpha^* \) can be determined from
\[
\alpha_{ij}^* = \alpha_{ij}^0 \frac{\pi \varepsilon}{4} \{ [1 + H(i - 4)] b_i^* \delta_{2j} + H(3 - j) b_j^* \delta_{2i} \} . \tag{62}
\]

5. Example: cracked piezoelectric ceramic

As an illustration, consider a cracked piezoelectric ceramic (BatiO₃) [16], the properties of which are given by \( c_{11} = 150 \) GPa, \( c_{12} = 66 \) GPa, \( c_{13} = 66 \) GPa, \( c_{33} = 146 \) GPa, \( c_{44} = 44 \) GPa, \( \alpha_{11} = 8.53 \times 10^{-6} / \text{K} \), \( \alpha_{33} = 1.99 \times 10^{-6} / \text{K} \), \( \lambda_3 = 0.133 \times 10^3 \) N/CK, \( e_{31} = -4.35 \) C/m², \( e_{33} = 17.5 \) C/m², \( e_{15} = 11.4 \) C/m², \( e_{11} = 1115 \kappa_0 \), \( e_{33} = 1260 \kappa_0 \), \( \kappa_0 = 8.85 \times 10^{-12} \) C²/Nm². Since the values of the coefficient of heat conduction for BatiO₃ could not be found in the literature, the value \( k_{22}/k_{11} = 1.5 \) and \( k_{12} = 0 \) are assumed. The results are obtained for a representative area element containing microcracks with the same length \( 2a \) and orientation. The line crack is assumed to be in the \( x_1 \)-\( x_3 \) plane and parallel to \( x_1 \)-axis. In Fig. 2, the normalized \( k_{22}^*/k_{22} \) are plotted against the crack density parameter \( \varepsilon \) as defined by Eq. (2). The curves correspond to results for the dilute, self-consistent and generalized self-consistent methods. The dilute model overestimates the effective conductivity, while the self-consistent technique yield a lower bound. The results predicted by the generalized

Fig. 2. Variations of effective conductivity with crack density.
self-consistent method are in between. The normalized effective modulus $c_{33}^*/c_{33}^M$ are presented in Fig. 3 for various values of crack density $\epsilon$. The dilute and self-consistent solutions are also given in Fig. 3 for comparison. It is found again that the curve for the generalized self-consistent model lies between the curves for dilute scheme and for self-consistent method.

6. Conclusion

Presented are the solutions based on the generalized self-consistent model where the representative area volume assumes the crack to be embedded in an elliptically-shaped matrix. Numerical results are obtained for a thermopiezoelectric solid containing microcracks with same length $2a$ and same orientation, and comparison is made with those of dilute and self-consistent methods. The presently obtained heat conduction constants and elastoelectric moduli are higher than those predicted by the self-consistent scheme, but lower than those provided by dilute approximation. Although the results are confined to the case of plane strain and all microcracks with the same length and same orientation, it is easy to extend the procedure to other plane problems, such as the case of $u_3 = u_3(x_1, x_2) \neq 0$ and the cracks being randomly oriented.

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References


