Efficient hypersingular line and surface integrals direct evaluation by complex variable differentiation method

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- Hadamard finite part integral
- Cauchy principal value integral
- Barycentric rational interpolation
- Complex variable differentiation method

A B S T R A C T

We present an efficient numerical scheme to evaluate hypersingular integrals appeared in boundary element methods. The hypersingular integrals are first separated into regular and singular parts, in which the singular integrals are defined as limits around the singularity and their values are determined analytically by taking the finite part values. The remaining regular integrals can be evaluated using rational interpolatory quadrature or complex variable differentiation (CVDM) for the regular function when machine precision like accuracy is required. The proposed method is then generalised for evaluating hypersingular surface integrals, in which the inner integral is treated as the hypersingular line integral via coordinate transformations. The procedure is implemented into 8-node rectangular boundary element and 6-node triangular element for numerical evaluation. Finally, several numerical examples are presented to demonstrate the efficiency of the present method. To the best of our knowledge, the proposed method is more accurate, faster and more generalised than other methods available in the literature to evaluate hypersingular integrals.

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1. Introduction

Hypersingular integrals are important for many problems in engineering, physics and mathematics. Such integrals arise when the finite domain of integration contains singularity points. This is often due to the employment of fundamental solution based numerical methods such as method of fundamental solutions (MFS) [1–3], Boundary element method (BEM) [4–6] and hybrid Trefftz-fundamental solution based finite element method [7–9] for solving boundary value problems. Popular applications of the hypersingular integrals include problems in electromagnetic scattering [10], acoustics [11,12], heat transfer [13], piezoelectric materials [14,15], fracture mechanics [16], boundary stress problem in elasticity [17], and it is vital in developing the symmetric Galerkin boundary integral equations [18,19]. Like the Cauchy principal value integral, an accurate evaluation of the hypersingular integrals often requires the knowledge of their conditions for existence [20] along with their geometrical definitions and properties [21,22], or is defined as pseudo-differential operators [23,24]. Consequently, the so called Hadamard finite part of the hypersingular integral can be taken for evaluation.

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Unlike the Cauchy principal value integral, which is well known and is implemented in popular and high level numerical software such as Mathematica and Matlab, the numerical evaluation of hypersingular integrals is still not mainstreamed and remains of research interest. In the literature, the first numerical attempt to evaluate the hypersingular integrals dates back to 1966, with Ninham [25] using the asymptotic expansion of the Euler Maclaurin formula in conjunction with the midpoint trapezoidal rule. Other approaches by means of quadrature rules can be found in the works by Paget [26,27]. If one chooses to split a hypersingular integral into regular and singular parts, the regular part of the integrand can be approximated by interpolation such as the generalised Lagrange polynomial [28,29]. The merit of this interpolatory quadrature is that the regular function’s derivative, which arises naturally for singular integrals of higher order, needs not be found explicitly. Instead, the derivative is simply approximated by differentiating the corresponding interpolants. Alternatively, Hui and Shia [30] derived a formula using the explicit derivative of the regular function with Gaussian quadrature to evaluate the hypersingular integrals. Apart from the usual interpolatory quadrature, Kolm and Rokhlin [31] employed the Fourier–Legendre series, which is a type of spectral methods, to approximate the regular function and its derivative for various orders of principal value integrals simultaneously. This approach was further simplified and generalised for endpoint singularities by Carley [32] using the semi-analytical formulation for improper integrals proposed by Brandao [33]. It is worth noting that in Brandao’s formulation, the simpler finite part of the singular integral is extracted and determined analytically. The numerical error is mainly due to the integration of the regular integral in addition to the error from the regular integrand and its derivatives’ approximations.

Although the method proposed by Kolm and Rokhlin [31] demonstrates convergence result with Legendre polynomials of higher degree, it is not without drawbacks for practical applications. As with the interpolatory quadrature, the modified quadrature weights in Kolm and Rokhlin’s method need to be re-evaluated when there is a change of bounds in the hypersingular integral. As a result, such a method is not suitable for higher dimensional applications when the bounds of the inner integral are a function of the outer integral. Without a simple change of interval bypassing the weight re-evaluation, the computational cost will be so high that it would defeat the purpose of constructing a generalised numerical quadrature scheme. In a seemingly less generalised approach, the hypersingular integrals can also be analytically examined so as to have the singularity terms regularised per application. This direct approach often leads to the more accurate result as demonstrated by Guiggiani et al. [34] for evaluating the hypersingular surface integral for three-dimensional potential problems, in which the Laurent series are employed to identify and regularise the singularity terms. However, Guiggiani’s approach may not be easily implemented when the underlying hypersingular kernels are of complicated forms. More recently, Gao [35] and her co-workers [36] proposed the use of power series for regular function approximation and for the regularisation of the singularity terms. Apart from the error due to the numerical integration, even though the approximation using the power series may converge, Gao’s method suffers some truncation error due to the power series, limiting its application in high precision application.

Herein, our method of evaluating hypersingular integrals should serve three purposes. First, the solution’s accuracy should be comparable to machine precision. Secondly, the method should be straight forward and generalised enough for fast numerical implementation with minimal computational effort and is comparable to the interpolatory quadrature method. Lastly, the method should be robust and flexible enough to handle hypersingular surface integrals which have applications in many fundamental solutions based numerical methods.

In this paper, hypersingular integral is expanded into regular and singular integrals by employing Brandao’s formulation. The regular function is first interpolated by barycentric rational polynomial [37] for comparison to the results reproduced by the Kolm and Rokhlin’s method. It was suggested in literature that such an interpolation scheme could give better approximation than polynomial interpolation due to its rational form with adjustable degree of order. Since the numerical quadrature is not reusable for hypersingular integrals with different bounds, further improvement can be made by dropping the interpolation entirely and instead, resorting to finding the derivative of the regular function numerically. In order to achieve machine precision like accuracy for the derivative approximation, complex variable differentiation method (CVDM) is employed [38,39]. Unlike the derivative approximation by the finite difference method, the CVDM approach makes use of a finite step on the complex axis while avoiding difference operation and the subsequent catastrophic cancellation on the real axis for small step size. As a result of this, the CVDM is capable of producing machine precision like accuracy [40]. The proposed method is then generalised for evaluating hypersingular surface integrals, in which coordinate transformations are performed such that the singularity and the finite part can be methodically identified and regularised. The resulting inner integral containing the singularity will have a form similar to the hypersingular line integral which can now be evaluated efficiently. The remaining integration task on the outer integral can be simply solved. Numerical results of the proposed method using 8-node rectangular boundary elements and 6-node triangular elements are then compared to the reference results produced by Guiggiani et al. [34] and Gao [35].

The remainder of this paper is organised as follows. Section 2 describes the definitions and semi-analytical values of the hypersingular line integrals. Section 3 presents the implementation of the barycentric rational interpolation and its results for the hypersingular integral evaluation. Section 4 presents the formulations and results by means of CVDM approximation. Section 5 provides a review on the definitions and properties of the hypersingular surface integrals and their generalised numerical implementation with the approach described in Section 4. Several numerical examples are considered in Section 6. Finally, some concluding remarks on the present method are presented in Section 7.
2. Definitions and properties of hypersingular integrals

In this paper, we are interested in the efficient evaluation of hypersingular line integral (1) and hypersingular surface integral [2],

\[
\int_a^b \frac{\phi(x)}{(x-t)^2} \, dx
\]

\[
\int \frac{\psi(x)}{||x_p - x_q||^3} \, dS(x_q)
\]

where \(\phi\) and \(\psi\) are the density functions for hypersingular line integral and hypersingular surface integral respectively, \(t\) the singularity point, \(a\) and \(b\) the lower and upper bounds of the singular integrals, \(x_p\) is a source point which can be any point within the domain or on the boundary and \(x_q\) is an arbitrary integration point.

As is well known in the literature [41,42], there holds a close relationship between the Cauchy principal value integral and the hypersingular integral. To ensure the existence of the singular integrals, the corresponding density function \(\phi(x)\) must satisfy some smoothness and continuity conditions [20], which will be briefly introduced in the following sections.

2.1. Cauchy principal value integral

For a Holder continuous function \(\phi(x) \in C^{0,\alpha}\) at \(x = t\), the Cauchy principal value integral in a symmetric neighbourhood of \(t \in (a, b)\) is defined as

\[
C \int_a^b \frac{\phi(x) - \phi(t)}{x-t} \, dx = \int_a^b \frac{\phi(x) - \phi(t)}{x-t} \, dx + \phi(t) \lim_{\epsilon \to 0} \left( \int_a^t \frac{1}{x-t} \, dx + \int_t^b \frac{1}{x-t} \, dx \right)
\]

for \(a < t < b\) (3)

where \(C\) indicates that the singular integral is defined as Cauchy principal value and \(\epsilon\) the symmetric neighbourhood of \(t\).

After taking the limit, we have

\[
C \int_a^b \frac{\phi(x) - \phi(t)}{x-t} \, dx = \int_a^b \frac{\phi(x) - \phi(t)}{x-t} \, dx + \phi(t) \ln \left( \frac{b-t}{t-a} \right)
\]

for \(a < t < b\) (4)

2.2. Hypersingular integral

The Hadamard finite-part integral in a symmetric neighbourhood of \(t \in (a, b)\) is defined as

\[
H \int_a^b \frac{\phi(x) - \phi(t)}{(x-t)^2} \, dx = \int_a^b \frac{\phi(x) - \phi(t)}{(x-t)^2} \, dx + \phi(t) \lim_{\epsilon \to 0} \left( \int_a^t \frac{1}{(x-t)^2} \, dx + \int_t^b \frac{1}{(x-t)^2} \, dx \right)
\]

for \(a < t < b\) (5)

where \(H\) indicates the integral is defined as Hadamard finite-part integral.

After taking the limit, we have

\[
H \int_a^b \frac{\phi(x) - \phi(t)}{(x-t)^2} \, dx = \int_a^b \frac{\phi(x) - \phi(t)}{(x-t)^2} \, dx + \phi(t) \left( \frac{1}{a-t} - \frac{1}{b-t} + \frac{2}{\epsilon} \right)
\]

for \(a < t < b\) (6)

Suppose the above limit exists and the term \(2/\epsilon\) is ignored, then (6) represents the finite-part of the otherwise divergent integral.

By repeating the above procedures, we have, for a Holder continuous function \(\phi(x) \in C^{p,\alpha}\) at \(x = t\), the finite-part integral in a neighbourhood of \(t \in (a, b)\),

\[
H \int_a^b \frac{\phi(x)}{(x-t)^{p+1}} \, dx = \frac{d}{dt} \left( \phi(t) \right) - \frac{p}{(x-t)^{p+1}} \int_a^b \frac{\phi(x)}{(x-t)^p} \, dx
\]

for \(a < t < b\) (7)

where \(p \geq 1\) is the order of singularity.

To simplify the above procedures of taking numerical limits, Brandao proposed a formulation which expands the Taylor series of \(\phi(x)\) about \(t\) and subsequently transforming the finite-part integrals into regular integrals and simpler finite-part integrals for semi-analytical evaluation [33],

\[
H \int_a^b \frac{\phi(x)}{(x-t)^{p+1}} \, dx = \int_a^b \frac{1}{(x-t)^{p+1}} \left( \phi(x) - \sum_{k=0}^{p} \frac{\phi^{(k)}(t)(x-t)^k}{k!} \right) \, dx + \sum_{k=0}^{p} \frac{\phi^{(k)}(t)}{k!} \int_a^b \frac{1}{(x-t)^{p+1-k}} \, dx
\]

for \(a \leq t \leq b\) (8)

Using (8), the finite-part value of the hypersingular integral (1) can be expressed as

\[
H \int_a^b \frac{\phi(x)}{(x-t)^2} \, dx = \phi(t) \int_a^b \frac{1}{(x-t)^2} \, dx + \phi^{(1)}(t) \int_a^b \frac{1}{x-t} \, dx + \int_a^b \frac{\phi(x) - \phi^{(1)}(t)(x-t)}{(x-t)^2} \, dx
\]

for \(a \leq t \leq b\) (9)
2.3. Simpler hypersingular integrals values

Supposing (9) exists, its simpler finite-part integrals can be analytically determined by once again ignoring the divergent terms [16].

For one sided Cauchy principal value integrals

\[
C \int_{t}^{b} \frac{1}{x-t} \, dx = \lim_{\epsilon \to 0} \int_{t+\epsilon}^{b} \frac{1}{x-t} \, dx + \ln(\epsilon) = \ln(b-t) \text{ for } t = a
\]

\[
C \int_{a}^{t} \frac{1}{x-t} \, dx = -\ln(t-a) \text{ for } t = b
\]

For two sided Cauchy principal value integral, we have from (4).

\[
C \int_{a}^{b} \frac{1}{x-t} \, dx = \ln \left( \frac{b-t}{t-a} \right) \text{ for } a < t < b
\]

For one sided finite-part integrals

\[
\mathcal{H} \int_{t}^{b} \frac{1}{(x-t)^2} \, dx = \lim_{\epsilon \to 0} \int_{t+\epsilon}^{b} \frac{1}{(x-t)^2} \, dx - \frac{1}{b-t} \text{ for } t = a
\]

\[
\mathcal{H} \int_{a}^{t} \frac{1}{(x-t)^2} \, dx = \frac{1}{a-t} - \frac{1}{b-t} \text{ for } t = b
\]

For two sided finite-part integral, we have from (6).

\[
\mathcal{H} \int_{a}^{b} \frac{1}{(x-t)^2} \, dx = \frac{1}{a-t} - \frac{1}{b-t} \text{ for } a < t < b
\]

Similarly, for higher order of finite-part integral, we have

\[
\mathcal{H} \int_{t}^{b} \frac{1}{(x-t)^p+1} \, dx = -\frac{1}{p(b-t)^p} \text{ for } t = a
\]

\[
\mathcal{H} \int_{a}^{t} \frac{1}{(x-t)^p+1} \, dx = \frac{1}{p(a-t)^p} \text{ for } t = b
\]

\[
\mathcal{H} \int_{a}^{b} \frac{1}{(x-t)^p+1} \, dx = \frac{1}{p} \left[ \frac{1}{(a-t)^p} - \frac{1}{(b-t)^p} \right] \text{ for } a < t < b
\]

3. Barycentric rational interpolation

Provided that the density function \( \phi(x) \) is known, (9) can be numerically evaluated with the analytical results for the simpler finite-part integrals from (10)–(15). For example, one may choose to analytically derive the derivative of \( \phi(x) \) and employ quadrature rule to evaluate the regular integral in (9). However, such approach is not convenient for implementation when there is a change of function \( \phi(x) \) due to different applications or when it is of very complicated forms. Instead, \( \phi(x) \) and its derivative are often approximated. In the work by Kolm and Rokhlin [31], they employed Fourier–Legendre expansion to approximate \( \phi(x) \) and achieve convergence result. In this section, we investigate the performance of interpolatory scheme using barycentric rational polynomial proposed by Floater and Hormann [37]. As suggested in the literature, the rational interpolation scheme could give better approximations than polynomial interpolation such as the generalised Lagrange interpolation due to the adjustable degree of order on the interpolant’s denominator. Furthermore, the barycentric form’s property also ensures that the computation time for the interpolants will be much less and stable [43]. In the followings, we first derive an explicit formulation for the interpolatory quadrature scheme, followed by making use of the barycentric rational polynomial for the interpolation.

3.1. Interpolatory quadrature formulation

Let \( \xi_i(x) \) be the interpolants of function \( \phi(x) \), then

\[
\phi(x) = \sum_{i=0}^{n-1} \xi_i(x) \phi(x_i)
\]

where \( \phi_i \) is the nodal value at interpolation point \( x_i \) and \( n \) is the number of interpolatory points.
Substitution of (19) into (9) yields
\[ \mathcal{H} \int_a^b \frac{\varphi(x)}{(x-t)^2} dx = \sum_{i=0}^{n-1} \varphi(x_i) W_i^* \]  

where \( W_i^* \) is the modified quadrature weight expressed as
\[ W_i^* = \ell_i(t) \mathcal{H} \int_a^b \frac{1}{(x-t)^2} dx + \ell_i^{(1)}(t) C \int_a^b \frac{1}{x-t} dx + \int_a^b \frac{\ell_i(x) - \ell_i(t) - \ell_i^{(1)}(t)(x-t)}{(x-t)^2} dx \]

Since \( \ell_i^{(1)} \) can be found directly by differentiating the interpolant \( \ell_i(x) \), the numerical task remains to integrate the regular integral in (21). Using the popular Gaussian quadrature scheme with weights \( W_j \) and points \( P_j \) and making use of the analytical solutions from (10)–(15), we obtain

For case \( a < t < b \),
\[ W_i^* = \ell_i(t) \left( \frac{1}{a-t} - \frac{1}{b-t} \right) + \ell_i^{(1)}(t) \ln \left( \frac{b-t}{t-a} \right) + \frac{b-a}{2} \sum_{j=1}^{m} W_j \frac{\ell_i(P_j) - \ell_i(t) - \ell_i^{(1)}(t)(P_j - t)}{(P_j - t)^2} \]  

For case \( t = a \),
\[ W_i^* = \frac{\ell_i(t)}{b-t} + \ell_i^{(1)}(t) \ln(b-t) + \frac{b-a}{2} \sum_{j=1}^{m} W_j \frac{\ell_i(P_j) - \ell_i(t) - \ell_i^{(1)}(t)(P_j - t)}{(P_j - t)^2} \]  

For case \( t = b \),
\[ W_i^* = \frac{\ell_i(t)}{a-t} - \ell_i^{(1)}(t) \ln(t-a) + \frac{b-a}{2} \sum_{j=1}^{m} W_j \frac{\ell_i(P_j) - \ell_i(t) - \ell_i^{(1)}(t)(P_j - t)}{(P_j - t)^2} \]

where \( m \) is the number of quadrature points for the regular part of the integration and \( P_j^* \) is the normalisation over the interval \([-1, 1]\) for arbitrary limits \([a, b]\).
\[ P_j^* = \frac{b-a}{2} P_j + \frac{a+b}{2} \]

As was seen from (22)–(24), the modified quadrature weight for the generalised numerical quadrature scheme depends on the bounds \([a, b]\) of the regular integral. The spectral method by Kolm and Rokhlin [31] also suffers from this setback.

### 3.2. Barycentric rational polynomial

So far, we have yet specified in what form the interpolant \( \ell_i(x) \) should possess. The most popular interpolant is the generalised Lagrange polynomial [28,29]. In recent research [43–45], it turns out that it is often more advantageous to have a barycentric representation for better computation speed and stability. In fact, there exists a generalised barycentric formula for every interpolation scheme [43,46]. The interpolant \( \ell_i(x) \) in barycentric form can be expressed as
\[ \ell_i(x) = \frac{\sum_{k=0}^{n-1} W_k}{s_{x,x}} \]

where \( W_k \) is called the barycentric weights.

For instance, the weights for the barycentric Lagrange interpolation is
\[ W_k = \prod_{l=0,l \neq k}^{n-1} (x_k - x_l) \]

As was mentioned earlier, the density function could be in complicated forms depending on the different practical applications. In the case of \( \varphi(x) \) possessing a rational form such as having polynomials on both of its numerator and denominator, one may expect a rational interpolation to perform better than the more traditional Lagrange interpolation.

The barycentric rational polynomial as proposed by Floater and Hormann [37] aims to generalise the family of rational interpolants. Essentially, their method avoids the formation of poles by blending the local interpolants to form a global one. The barycentric weights for the rational interpolation can be expressed as
\[ W_k = \sum_{s=k-d,0 \leq s \leq n-1-d} (-1)^s \prod_{l=s,l \neq k}^{s+d} \frac{1}{x_k - x_l} \]
The parameter $d \in [0, n-1]$ controls the degree of blending and has implication for the rational function’s $\varphi(x)$ degree of order.

The derivative of order $\beta$ of the barycentric rational interpolants share some simple differentiation formulas [47] by iterative process,

$$\varphi^{(\beta)}(x_j) = \sum_{i=0}^{n-1} \epsilon_i^{(\beta)}(x_j) \varphi(x_i)$$  \hspace{1cm} (29)

For $\beta = 1,$

$$\epsilon_i^{(1)}(x_j) = \frac{w_i}{w_j(x_j - x_i)} \text{ for } x_i \neq x_j$$  \hspace{1cm} (30)

$$\epsilon_j^{(1)}(x_j) = -\sum_{k=0, k \neq j}^{n-1} \epsilon_k^{(1)}(x_j) \text{ for } x_i = x_j$$  \hspace{1cm} (31)

For $\beta > 1,$

$$\epsilon_i^{(\beta)}(x_j) = \beta \left( \epsilon_i^{(1)}(x_j) \epsilon_j^{(\beta-1)}(x_j) - \frac{\epsilon_j^{(\beta-1)}(x_j)}{(x_j - x_i)} \right) \text{ for } x_i \neq x_j$$  \hspace{1cm} (32)

$$\epsilon_j^{(\beta)}(x_j) = -\sum_{k=0, k \neq j}^{n-1} \epsilon_k^{(\beta)}(x_j) \text{ for } x_i = x_j$$  \hspace{1cm} (33)

By substituting (26), (28), (30) and (31) into Eqs. (22)–(24) for the modified weights $W_i^\ast$, the finite-part of the hypersingular integral with interpolatory quadrature scheme, i.e. (20) can then be evaluated.

3.3. Numerical example

To demonstrate the efficiency of the interpolatory quadrature scheme using the barycentric rational polynomial, a benchmark example for a specified density function $\varphi(x)$ over the interval $[-1, 1]$ is considered.

$$\varphi(x) = \sin(2x) + \cos(3x)$$  \hspace{1cm} (34)

The analytical solution $l(t)$ of the hypersingular integral for the above density functions is

$$l(t) = \int_{-1}^{1} \frac{\varphi(x) dx}{(x-t)^2}$$  \hspace{1cm} (35)

For varying $t,$ Eq. (20) can be modified to represent the numerical quadrature $\hat{l}(t),$

$$\hat{l}(t) = \sum_{i=0}^{n-1} \varphi(x_i, t) W_i^\ast$$  \hspace{1cm} (36)

To illustrate the competitiveness of the scheme using (36), results computed using Matlab are compared to those reproduced by the Kolm and Rokhlin’s method [31] and analytical solutions by Mathematica. The approximated integrals are evaluated at 20 different singularity point $t$ coinciding with the normalised Legendre points $P_k^*$. The number of quadrature points $m$ for the regular part integration is fixed at 40 and the parameter $d$ for the barycentric rational interpolation is set as $n - 1$ to maximise the interpolation’s effectiveness. To measure the precision of this method, the mean absolute percentage error for incremental number of interpolation points $n$ from 6 to 30 is computed as

$$E = \frac{1}{n} \sum_{k=1}^{n} \left| \frac{\hat{l}(P_k^*) - l(P_k^*)}{l(P_k^*)} \right| \times 100\%$$  \hspace{1cm} (37)

where $n_t$ is the number of test points.

The results for the mean absolute percentage error of hypersingular integral evaluation using numerical quadrature of barycentric rational interpolation scheme and the Kolm and Rokhlin’s method are plotted in Fig. 1. The numerical quadrature weights $W_i^\ast$ for 14 Legendre nodes are tabulated in Table 1.

As can be seen from Fig. 1, both the barycentric rational interpolation scheme and the method by Kolm and Rokhlin have similar numerical accuracy when the number of quadrature points used is concerned. This is evidenced by the fact that the modified quadrature weights $W_i^\ast$ do not vary much between these two methods. For instance, the $W_i^\ast$ of the 14-node quadrature of both methods only differ by less than 11 decimal places as shown in Table 1. At $n = 20$, the precision of both methods starts to reach the maximum and is unable to improve further beyond $10^{-15}$. This is in agreement with the Matlab’s default setting of double precision floating point format. Results show that numerical quadrature using
Fig. 1. Mean absolute percentage error of hypersingular integral evaluation using numerical quadrature of barycentric rational interpolation scheme and Kolm and Rokhlin’s method.

Table 1
14-node quadratures for t = −0.9862838086968120.

<table>
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<tr>
<th>P_i</th>
<th>Barycentric rational interpolation scheme</th>
<th>Kolm and Rokhlin’s method</th>
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rational interpolation scheme is an alternative to the spectral method. When approximating integrand consisting of different order of singularities, we anticipate that the barycentric rational interpolation scheme would have the similar accuracy as to the method by Kolm and Rokhlin. To this end, it is noted that the above numerical schemes are attractive only if the hypersingular integrals are to be evaluated within the same interval [a, b] and with the same singularity point t so that the modified quadrature weights can be recycled.

4. Complex variable differentiation method

Although the above methods yield convergence result when polynomials of higher degree are employed, the corresponding modified quadrature weights are not reusable and therefore need to be re-evaluated when there is a change of bounds in the hypersingular integral. In order to reduce the computational cost, further improvement can be made by dropping the interpolation entirely and instead, resorted to finding the derivative of the regular function numerically. In order to achieve machine precision like accuracy with minimal computational effort for the derivative approximation, complex variable differentiation method (CVDM) [38–40] is employed in this section. Unlike the conventional finite difference approach, the CVDM makes use of finite step on the complex axis while avoiding difference operation and the subsequent catastrophic cancellation on the real axis for small step size.
Let \( i \) and \( h \) represent the imaginary unit and the complex step of a real and analytic function \( \varphi(x + ih) \). The CVDM is formulated by first expanding the Taylor series of \( \varphi(x + ih) \) around \( x \),

\[
\varphi(x + ih) = \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(x)(ih)^k}{k!}
\]

The first derivative is then extracted and rearranged from the imaginary part of the above expansion,

\[
\varphi^{(1)}(x) = \frac{\text{Im}(\varphi(x + ih))}{h} + O(h^2)
\]

where \( \text{Im} \) represents the operation of taking imaginary part.

Based on the above strategy, one can also construct a recursive Richardson extrapolation to further increase the convergence rate for the CVDM. For instance, the \( n \)th order of the first derivative approximation \( \varphi^{(1)}(x) = D_{k,n,l=1}(x) \) can be formulated as

\[
D_{k,0}(x) = \frac{\text{Im}(\varphi(x + i\frac{h}{2})) - \text{Im}(\varphi(x - i\frac{h}{2}))}{\frac{h}{2^{n+1}}}
\]

\[
D_{k,l}(x) = \frac{4^l D_{k,l-1}(x) - D_{k-1,l-1}(x)}{4^{l-1} - 1}
\]

where \( k, l = 0, 1, 2, \ldots, n \).

The hypersingular integral in the finite-part form of (9) can be efficiently evaluated using Eq. (39) or (41). It is noted that the convergence rate for the 0th order Richardson extrapolation in (40) is the same as the one in (39). For simplicity, the numerical formulations of hypersingular integral using one complex-step approximation i.e. (39) are listed as follows:

For case \( a < t < b \),

\[
\mathcal{H} \int_a^b \frac{\varphi(x)(x-t)^2}{(x-t)^2} dx = \varphi(t) \left( \frac{1}{a - t} - \frac{1}{b - t} \right) + \frac{\text{Im}(\varphi(t + ih))}{h} \ln \left( \frac{b - t}{t - a} \right) + \frac{b - a}{2} \sum_{j=1}^{m} W_j \frac{\varphi(P_j) - \varphi(t) - \frac{\text{Im}(\varphi(t + ih))}{h}(P_j - t)}{(P_j - t)^2}
\]

For case \( t = a \),

\[
\mathcal{H} \int_a^b \frac{\varphi(x)(x-t)^2}{(x-t)^2} dx = -\varphi(t) \frac{b - t}{b - t} + \frac{\text{Im}(\varphi(t + ih))}{h} \ln(b - t) + \frac{b - t}{2} \sum_{j=1}^{m} W_j \frac{\varphi(P_j) - \varphi(t) - \frac{\text{Im}(\varphi(t + ih))}{h}(P_j - t)}{(P_j - t)^2}
\]

For case \( t = b \),

\[
\mathcal{H} \int_a^b \frac{\varphi(x)(x-t)^2}{(x-t)^2} dx = \varphi(t) \frac{a - t}{a - t} - \frac{\text{Im}(\varphi(t + ih))}{h} \ln(t - a) + \frac{t - a}{2} \sum_{j=1}^{m} W_j \frac{\varphi(P_j) - \varphi(t) - \frac{\text{Im}(\varphi(t + ih))}{h}(P_j - t)}{(P_j - t)^2}
\]

Again, the same benchmark example as in (34) is considered. The results for the mean absolute percentage error (37) of hypersingular integral evaluation using CVDM of various complex-step sizes and Richardson extrapolation orders are plotted in Fig. 2.

As can be seen from Fig. 2, the hypersingular integral evaluation using CVDM has a very high convergence rate. For instance, the maximum accuracy can be easily achieved by choosing the size \( h \) be smaller than \( 10^{-7} \) in a one-step approximation according to the 0th order Richardson extrapolation results. The convergence rate also improves significantly with the use of higher order Richardson extrapolations. In practice, the one-step approximation will suffice provided that the decimal places of precision gained are higher than the respected complex-step size, i.e. \( -\log_{10} h \). As this would imply there always exists a sufficiently small size of \( h \) in a one-step approximation capable of producing machine precision accuracy. Lastly, it should be emphasised that by transforming the derivative finding to complex-step operation, the hypersingular integral can be evaluated directly bypassing the interpolatory quadrature which would otherwise require around 20 interpolatory points to achieve similar accuracy as comparing to the complex variable scheme of using just one complex-step.

5. Hypersingular surface integral

Hypersingular surface integrals, usually arisen from the use of fundamental solution as trial function in boundary integral equations, is firstly formulated by subtracting a vanishing exclusion zone around the singularity point \( x_p \),

\[
\int_{\mathcal{S}} \frac{\psi(x)}{r^3(x_p, x)} ds(x) = \lim_{\epsilon \rightarrow 0^+} \int_{\mathcal{S} - \mathcal{S}_\epsilon} \frac{\psi(x_p, x)}{r^3(x_p, x)} ds(x) + \lim_{\epsilon \rightarrow 0^+} \int_{\mathcal{S}_\epsilon} \frac{\psi(x_p, x)}{r^3(x_p, x)} ds(x)
\]
Fig. 2. Mean absolute percentage error of hypersingular integral evaluation using CVDM of various complex-step sizes and Richardson extrapolation orders.

Fig. 3. Exclusion surface around a singularity point at corner.

where \( r \) is the Euclidean distance between the source point \( x_p \) and the arbitrary integration point \( x_Q \),

\[
r(x_p, x_Q) = \|x_p - x_Q\| \quad (46)
\]

In (45), \( S_\varepsilon \) is the exclusion surface around the singularity point \( x_p \) with vanishing neighbourhood \( \varepsilon \) and \( C_\varepsilon \) is the region of spherical surface replacing \( S_\varepsilon \) around the vanishing neighbourhood of \( x_p \) as shown in Fig. 3.

If the singularity point \( x_p \) is a corner point with discontinuous outward normal, the last integral of (45) will give rise to additional free terms and the coefficients of which can be evaluated analytically or numerically by integrating the vanishing region \( C_\varepsilon \) as was demonstrated in [48,49]. The task is then simplified to evaluate the finite part of the hypersingular surface integral \( I \),

\[
I = \mathcal{H} \int_S \frac{\psi(x_p, x_Q)}{r^3(x_p, x_Q)} dS(x_Q) \quad (47)
\]
In analogy to the regularisation process for hypersingular line integral (8), the hypersingular surface integral can also be regularised as follows,

\[
I = \psi(x_p, x_p) \mathcal{H} \int_S \frac{1}{r^3(x_p, x_q)} dS(x_q) + \frac{\partial \psi(x_p, x_p)}{\partial r} \int_S \frac{1}{r^2(x_p, x_q)} dS(x_q)
\]

\[
+ \int_S \frac{\psi(x_p, x_q) - \psi(x_p, x_p) - r(x_p, x_q) \partial \psi(x_p, x_p) / \partial r}{r^3(x_p, x_q)} dS(x_q)
\]

(48)

The above formulation is difficult to materialise for two reasons: first, the differentiation of the density function \( \psi \) with respect to \( r \) needs to be analytically derived or numerically approximated which may not be a straightforward process, given that \( r \) needs not be an explicit variable of \( \psi \). Secondly, the principal value integrals in (48) would require proper coordinate transformation such that it is aligned with \( r \) before taking the finite part value. We will resolve these issues by employing polar coordinate transformation onto the local orthogonal curvilinear coordinate of the discretised surface in boundary element setting. Then, the complex variable scheme of approximating hypersingular surface integral as was discussed previously in (43) is employed to evaluate the hypersingular surface integral now equipped with compatible integration variable and the respected finite part value.

5.1. Surface discretisation

If we discretise the surface of integration in (47) by boundary elements, the global Cartesian coordinate \( x_{i-1, 2, 3} \) can then be interpolated by element’s nodal points \( x_{j}^k \) using shape functions \( N_k \) of local orthogonal curvilinear coordinate \( \xi_{j-1, 2} \) in two dimensions,

\[
x_i = \sum_{k=1}^{n} N_k(\xi_1, \xi_2) x_{j}^k
\]

(49)

where \( k \) and \( n \) are the nodal index and the total number of nodal points in an element, respectively.

For surface area represented by 8-node rectangular element or 6-node triangular element, the shape functions \( N_k \) and their derivatives \( \partial N_k / \partial \xi_j \) can be found in Appendix A.

The surface integral of (47) using the above local coordinate transformation becomes

\[
\int_S dS(x_q) = \int_{-1}^{1} \int_{-1}^{1} J_k(\xi) d\xi_1 d\xi_2
\]

\[
J_k(\xi_1, \xi_2) = \left| \frac{\partial x_q(\xi_1, \xi_2)}{\partial \xi_1} \times \frac{\partial x_q(\xi_1, \xi_2)}{\partial \xi_2} \right| = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial x_1(\xi_1, \xi_2) / \partial \xi_1 & \partial x_2(\xi_1, \xi_2) / \partial \xi_1 & \partial x_3(\xi_1, \xi_2) / \partial \xi_1 \\ \partial x_1(\xi_1, \xi_2) / \partial \xi_2 & \partial x_2(\xi_1, \xi_2) / \partial \xi_2 & \partial x_3(\xi_1, \xi_2) / \partial \xi_2 \end{vmatrix}
\]

(51)

where \( J_k(\xi_1, \xi_2) \) is the Jacobian for the transformation from the global coordinate to the local coordinate and \( (e_1, e_2, e_3) \) is a set of basis vectors in the Cartesian coordinate. The term \( \partial x_q(\xi) / \partial \xi_j \) is computed by differentiating (49) and substitution from (A.3)–(A.6).

5.2. Polar coordinate transformation

To evaluate the surface integral consisting of radial singularity \( r^3(\xi^p, \xi) \), polar coordinate transformation [17–36] is often applied around the singularity point \( \xi^p \).

\[
(\xi_1(\xi^p, \rho, \theta), \xi_2(\xi^p, \rho, \theta)) = (\rho \cos \theta + \xi_1^p, \rho \sin \theta + \xi_2^p)
\]

(52)

The surface integration can then be expressed as

\[
\int_S dS(x_q) = \int_0^{2\pi} \int_0^\rho J_k(\xi(\xi^p, \rho, \theta)) l_\rho(\rho) d\rho d\theta
\]

(53)

where \( \rho \) and \( \theta \) are the radial distance and polar angle, \( l_\rho(\rho) \) is the corresponding Jacobian for the transformation from the local coordinate to the polar coordinate. \( \rho_i(\xi^p, \theta) \) is the path of radial integration depending on the singularity point and the value of the polar angle. \( J_k(\xi^p, \xi(\xi^p, \rho, \theta)) \) is numerically computed by directly substituting (52) into (A.3)–(A.6) then into (51) using the differential form of (49).

To facilitate direct implementation of surface integration, we have explicitly derived the integral bounds for the polar coordinate transformation through internal cells discretisation. For the 8-node rectangular element, \( \theta \) and \( \rho_i(\xi^p, \theta) \) are partitioned into four subregions as illustrated in Fig. 4 while for the 6-node triangular element, they are partitioned into three subregions as illustrated in Fig. 5. We refer reader to Appendix B for further algebraic details. Alternatively, radial integration method (RIM) [35,50] can be employed for the integration by transforming the surface integral into equivalent line integrals.
Fig. 4. Regions of integration for the 8-node plate element in polar coordinate.

Fig. 5. Regions of integration for the 6-node triangular element in polar coordinate.

We anticipate that the accuracy of both methods will not differ much given that their integral values converge quickly with increased number of quadrature points.

The surface integration then becomes

\[
\int_{\mathcal{S}} dS(\xi_0) = \sum_{l=1}^{m_R} \int_{\theta_l^{\alpha}(\xi)}^{\theta_l^{\beta}(\xi)} \int_{0}^{\rho_1(\xi, \rho, \theta)} J_{\rho}(\rho) d\rho d\theta
\]  

(54)

where \( m_R \) is the total number of regions with respect to the choice of the boundary element. \( \theta_l^{\alpha}(\xi) \) and \( \theta_l^{\beta}(\xi) \) are respectively the lower and upper bounds of the outer integration for each partitioned region \( l \).

5.3. Asymptotic expansions of \( r \)

Apart from the transformation for the surface integral and its regular integrand, the singular part, i.e. the Euclidean distance \( r \) of the integrand also needs to be transformed accordingly. The Taylor expansion of the Euclidean distance projected on the Cartesian axes \( x_i(\xi, \xi^p, \rho, \theta) \) about the singularity point \( x_i^p \) is [34],

\[
r_i(\xi, \rho, \theta) = r_i(\xi, \rho, \theta) = \rho A_i(\xi, \rho, \theta) + \rho B_i(\xi, \rho, \theta) + \rho^2 C_i(\xi, \rho, \theta)
\]  

(55)

where

\[
r_i(\xi, \rho, \theta) = x_i(\xi, \xi^p, \rho, \theta) - x_i^p
\]  

(56)

\[
A_i(\xi, \rho, \theta) = \frac{\partial x_i}{\partial \xi_1} \bigg|_{\xi^p} \cos \theta + \frac{\partial x_i}{\partial \xi_2} \bigg|_{\xi^p} \sin \theta
\]  

(57)

\[
B_i(\xi, \rho, \theta) = \frac{1}{2} \left( \frac{\partial^2 x_i}{\partial \xi_1^2} \bigg|_{\xi^p} \cos^2 \theta + \frac{\partial^2 x_i}{\partial \xi_1 \xi_2} \bigg|_{\xi^p} \cos \theta \sin \theta \right) + \frac{1}{2} \left( \frac{\partial^2 x_i}{\partial \xi_2^2} \bigg|_{\xi^p} \sin^2 \theta \right)
\]  

(58)

\[
C_i(\xi, \rho, \theta) = \frac{1}{2} \left( \frac{\partial^3 x_i}{\partial \xi_1^2 \partial \xi_2} \bigg|_{\xi^p} \cos^2 \theta \sin \theta + \frac{1}{2} \left( \frac{\partial^3 x_i}{\partial \xi_1 \partial \xi_2^2} \bigg|_{\xi^p} \cos \theta \sin^2 \theta \right)
\]  

(59)
After extracting the radial distance $\rho$ term which represents the singularity order, we have the following simplified form of $r(\xi^p, \rho, \theta)$ comprising the singular part $\rho$ and the regular part $\rho_\mathcal{E}(\xi^p, \rho, \theta)$,

$$r(\xi^p, \rho, \theta) = \rho \rho_\mathcal{E}(\xi^p, \rho, \theta)$$

(60)

where

$$\rho_\mathcal{E}(\xi^p, \rho, \theta) = \sqrt[3]{\sum_{i=1}^{3} \left[ A_i(\xi^p, \rho, \theta) + \rho B_i(\xi^p, \rho, \theta) + \rho^2 C_i(\xi^p, \rho, \theta) \right]^2}.$$  

(61)

5.4. Asymptotic expansions of density function $\psi$

Owing to the fact that the density function $\psi$ also depends on $\rho$ and $\theta$ after the coordinate transformations, we seek to derive the asymptotic expansion of $\psi(\xi^p, \xi(\xi^p, \rho, \theta))$ subjected to the same transformation process of $r(\xi^p, \rho, \theta)$. For fundamental solution based hypersingular integrand, it will suffice to deal with spatial variables of $\psi$. Although similar procedures can be found in [17,36], the list of the possible spatial variables composing the density function $\psi$ may not be exhaustive. The most commonly seen of such variables include the already discussed components of $r$ with respect to the Cartesian direction, that is $r_i$ from (55). The other useful variables include the unit outward normal $n$, the spatial derivative of the Euclidean distance $\frac{\partial r}{\partial n}$ and its normal derivative $\frac{\partial}{\partial n}$. They can be easily evaluated as follows:

$$n(\xi^p, \xi(\xi^p, \rho, \theta)) = \frac{\partial x_q(\xi(\xi^p, \rho, \theta))}{\partial \xi_1} \times \frac{\partial x_q(\xi(\xi^p, \rho, \theta))}{\partial \xi_2}$$

(62)

$$\frac{\partial r(\xi^p, \rho, \theta)}{\partial \xi_i} = \frac{r_i(\xi^p, \rho, \theta)}{r(\xi^p, \rho, \theta)} = \frac{A_i(\xi^p, \rho, \theta) + \rho B_i(\xi^p, \rho, \theta) + \rho^2 C_i(\xi^p, \rho, \theta)}{\sqrt[3]{\sum_{i=1}^{3} \left[ A_i(\xi^p, \rho, \theta) + \rho B_i(\xi^p, \rho, \theta) + \rho^2 C_i(\xi^p, \rho, \theta) \right]^2}}$$

(63)

$$\frac{\partial r(\xi^p, \xi(\xi^p, \rho, \theta))}{\partial n} = \frac{3}{\partial \xi_i} \left[ \frac{\partial r(\xi^p, \rho, \theta)}{\partial \xi_i} \right] = \frac{3}{\partial \xi_i} \left[ \frac{\partial r(\xi^p, \xi(\xi^p, \rho, \theta))}{\partial \xi_i} \right].$$

(64)

5.5. Proposed formulation of hypersingular surface integral

We now have acquired all necessary tools to further progress the hypersingular surface integral from (47). Applying firstly the local coordinate transformation to the hypersingular integral (47) yields.

$$l = \int_{-1}^{1} \int_{-1}^{1} \frac{\psi(\xi^p, \xi)}{r^3(\xi^p, \xi)} j_1(\xi) d\xi_1 d\xi_2$$

(65)

Then, after applying the polar coordinate transformation, we have

$$l = \int_{0}^{2\pi} \int_{0}^{\rho(\xi^p, \theta)} \frac{\varphi(\xi^p, \xi(\xi^p, \rho, \theta))}{\rho^2} d\rho d\theta$$

(66)

where

$$\varphi(\xi^p, \rho, \theta) = \frac{\psi(\xi^p, \xi(\xi^p, \rho, \theta)) j_1(\xi^p, \xi(\xi^p, \rho, \theta))}{\rho^2 j_1(\xi^p, \xi(\xi^p, \rho, \theta))}. $$

(67)

The Eq. (67) is significant in the sense that all the regular parts of the hypersingular integrand (2) have been grouped together and expressed in a form depending on $\rho$ where the finite part value is based upon [34–36]. As the result, the inner integral of (66) can be perceived and treated as the hypersingular line integral as in (1). Finally, we regularise this line integral through Brandao’s formulation, i.e. by substituting (9) into (66). The proposed formulation for evaluating the hypersingular surface integral is then obtained.

$$l = \int_{0}^{2\pi} \left[ \frac{\varphi(\xi^p, \xi(\xi^p, 0, \theta))}{\rho^2} \int_{0}^{\rho(\xi^p, \theta)} \frac{1}{\rho^2} d\rho + \frac{\partial \varphi(\xi^p, \xi(\xi^p, 0, \theta))}{\partial \rho} \int_{0}^{\rho(\xi^p, \theta)} \frac{1}{\rho^2} d\rho \right] d\theta$$

(68)

This equation is essentially the equivalent representation form of (48).

It is noted that in the work by Guiggiani et al. [34], Laurent series expansion is applied to the whole hypersingular integrand for regularisation. By contrast, in the work by Gao [35] and her co-workers [36], only the regular part of the hypersingular integrand is approximated by the use of power series for the regularisation.
Herein, we propose that the regular part of the hypersingular integrand can be cast into Brandao's formulation and complemented with the CVDM, so as to provide a more efficient mean for the regularisation of the hypersingular surface integrals. After obtaining (68), the remaining task is to obtain an explicit formulation using the previously proposed complex variable scheme (42)-(44) for efficient numerical implementation.

To obtain the numerical formula for the hypersingular surface integral, we first apply (54) to partition the area of integration in (66).

\[
I = \sum_{l=1}^{m} \int_{\partial_{l}}(\xi^{*}) \int_{0}^{\rho_{l}(\xi^{*}, \rho)} \frac{\varphi(\xi^{P}, \xi(\xi^{P}, \rho, \theta))}{\rho^{2}} d\rho d\theta = \sum_{l=1}^{m} l_{l}
\]

(69)

where \(l_{l}\) are the hypersingular surface integrals of each partitioned region.

To perform outer integration in (69) using Gaussian quadrature, a change of interval for the normalisation of \(\theta\) is required,

\[
l_{l} = \int_{\partial_{l}}(\xi^{*}) \int_{0}^{\rho_{l}(\xi^{*}, \rho)} \frac{\varphi(\xi^{P}, \xi(\xi^{P}, \rho, \theta))}{\rho^{2}} d\rho d\theta
\]

(70)

\[
= \frac{\theta_{l}^{b}(\xi^{P}) - \theta_{l}^{a}(\xi^{P})}{2} \int_{-1}^{1} \int_{0}^{\rho_{l}(\xi^{*}, \rho)} \frac{\varphi(\xi^{P}, \xi(\xi^{P}, \rho, \theta^{*}_{l}(\xi^{P}, \theta)))}{\rho^{2}} d\rho d\theta
\]

(70)

where

\[
\theta_{l}^{*}(\xi^{P}, \theta) = \frac{\theta_{l}^{b}(\xi^{P}) - \theta_{l}^{a}(\xi^{P})}{2} \theta + \frac{\theta_{l}^{b}(\xi^{P}) + \theta_{l}^{a}(\xi^{P})}{2}
\]

(71)

Further, let \(P_{l}^{ax}\) and \(W_{k}^{ax}\) be the points and weights of the Gaussian quadrature with \(m_{ax}\) be the total number of points for the outer integral, then

\[
l_{l} = \frac{\theta_{l}^{b}(\xi^{P}) - \theta_{l}^{a}(\xi^{P})}{2} \sum_{k=1}^{m_{ax}} W_{k}^{ax} \int_{0}^{\rho_{l}(\xi^{*}, \rho)} \frac{\varphi(\xi^{P}, \xi(\xi^{P}, \rho, \theta^{*}_{l}(\xi^{P}, P_{k}^{ax})))}{\rho^{2}} d\rho
\]

(72)

After employing the formerly proposed complex variable scheme from (43) for the inner hypersingular line integral evaluation, the numerical formulation for the hypersingular surface integral \(l_{l}\) is

\[
l_{l} = \frac{\theta_{l}^{b}(\xi^{P}) - \theta_{l}^{a}(\xi^{P})}{2} \sum_{k=1}^{m_{ax}} W_{k}^{ax} \left\{ -\varphi(\xi^{P}, \xi(\xi^{P}, 0, \theta^{*}_{l}(\xi^{P}, P_{k}^{ax}))) / \rho_{l}(\xi^{P}, \theta^{*}_{l}(\xi^{P}, P_{k}^{ax})) \right\}
\]

(73)

\[
+ \text{Im}(\varphi(\xi^{P}, \xi(\xi^{P}, \rho, \theta^{*}_{l}(\xi^{P}, P_{k}^{ax})))) \ln(\rho_{l}(\xi^{P}, \theta^{*}_{l}(\xi^{P}, P_{k}^{ax}))) / h 
\]

\[
+ \rho_{l}(\xi^{P}, \theta^{*}_{l}(\xi^{P}, P_{k}^{ax})) \sum_{j=1}^{m_{l}} W_{j}[\varphi(\xi^{P}, \xi(\xi^{P}, P_{j}, \theta^{*}_{l}(\xi^{P}, P_{k}^{ax}))) - \varphi(\xi^{P}, \xi(\xi^{P}, 0, \theta^{*}_{l}(\xi^{P}, P_{k}^{ax})))]
\]

(73)

\[
- \text{Im}(\varphi(\xi^{P}, \xi(\xi^{P}, \rho, \theta^{*}_{l}(\xi^{P}, P_{k}^{ax})))) / h \right\} / \rho_{l}^{2}
\]

where

\[
P_{l}^{x}(\xi^{P}, \theta^{*}_{l}(\xi^{P}, P_{k}^{ax})) = \frac{\rho_{l}(\xi^{P}, \theta^{*}_{l}(\xi^{P}, P_{k}^{ax}))}{2} P_{j} + \frac{\rho_{l}(\xi^{P}, \theta^{*}_{l}(\xi^{P}, P_{k}^{ax}))}{2}
\]

(74)

This completes the numerical implementation of hypersingular surface integral using the proposed method.

To elucidate thoroughly on how the algorithms of the proposed method work, we provide the following pseudocode for numerically evaluating the hypersingular surface integral \(I\).

---

Choose \(\xi^{P}\)
Compute \(\theta_{l}(\xi^{P}), \theta_{2}(\xi^{P}), ..., \theta_{m}(\xi^{P})\) from (B.3)
Loop for \(l = 1, 2, ..., m_{l}\)
  Loop for \(k = 1, 2, ..., m_{ax}\)
    Compute \(\theta_{l}^{*}(\xi^{P}, \theta)\) for \(\theta = P_{k}^{ax}\) from (71)
    Compute \(\rho_{l}(\xi^{P}, \theta)\) according to the range of \(\theta\) for \(\theta = \theta_{l}^{*}(\xi^{P}, P_{k}^{ax})\) from (B.4)
    Compute \(\varphi(\xi^{P}, \xi(\xi^{P}, \rho, \theta^{*}_{l}(\xi^{P}, P_{k}^{ax})))\) and \(\varphi(\xi^{P}, \xi(\xi^{P}, \rho, \theta_{l}^{*}(\xi^{P}, P_{k}^{ax})))\) by using Table 2.
    Loop for \(j = 1, 2, ..., m\)
      Compute \(P_{j}(\xi^{P}, \theta)\) for \(\theta = \theta_{l}^{*}(\xi^{P}, P_{k}^{ax})\) from (74)
      Compute \(\varphi(\xi^{P}, \xi(\xi^{P}, P_{j}, \theta^{*}_{l}(\xi^{P}, P_{k}^{ax})))\) by using Table 2 with changed substitutions: \(\rho = P_{j}, \theta = \theta_{l}^{*}(\xi^{P}, P_{k}^{ax})\)
    End loop \(j\)
  End loop \(k\)
End loop \(l\)
Compute \(I\) from (69)
6. Numerical simulation

The proposed method of hypersingular surface integral evaluation is applied to three benchmark examples so as to demonstrate its accuracy, efficiency and robustness for solving distinct complex problems. The examples include: (1) a trapezium meshed with distorted 8-node rectangular elements, (2) a rectangle meshed with the regular 8-node rectangular elements comparing to the distorted 6-node triangular elements of various degrees, and (3) a quarter cylindrical panel represented by curved 8-node rectangular element. Similar to the numerical example for hypersingular line integral, the computations are performed in Matlab with the default double precision floating point format. Since our previous study on line integral showed that further reduction on the size of the complex-step would not affect the numerical accuracy once the result has reached the machine precision, a complex-step size of $10^{-12}$ will be employed throughout the simulations.

6.1. Example 1a: trapezium meshed with a distorted rectangular element

In this first example, distortion effects on the 8-node rectangular element is studied for the hypersingular surface integral of the following form:

$$ l = \int_S \frac{-1}{r^3} \left( 3r_3 \frac{\partial r}{\partial n} - n_3 \right) dS \tag{75} $$

where the surface of integration is over a trapezium region with corner points located at \((-1, -1, 0), (1.5, -1, 0), (0.5, 1, 0), (-1, 1, 0)\) as shown in Fig. 6. Three singularity points namely, \(a(0, 0), b(0.66, 0)\) and \(c(0.66, 0.66)\) are placed with respect to the local orthogonal curvilinear coordinate \(\xi\). In order to cast the hypersingular surface integral into our proposed formulation of (66), we have the following modified density function \(\varphi\) from (67):

$$ \varphi(\xi^p, \rho, \theta) = -\left( 3r_3 \frac{\partial r}{\partial n} - n_3 \right) \frac{J_3}{\rho^2} \tag{76} $$

![Fig. 6. Schematic of trapezium meshed with an 8-node rectangular element and with various singularity points a, b and c.](image)
Table 3
Numerical evaluation of hypersingular surface integrals on trapezium meshed with a distorted 8-node rectangular element at various test locations with respect to the different orders of Gaussian quadrature.

<table>
<thead>
<tr>
<th>m</th>
<th>Point a</th>
<th>Point b</th>
<th>Point c</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Reference [34]</td>
<td>Proposed</td>
<td>Reference [34]</td>
</tr>
<tr>
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<td>-9.154587</td>
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<td>16</td>
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<td>Exact</td>
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<td>-15.32850</td>
</tr>
</tbody>
</table>

Fig. 7. Trapezium meshed with four 8-node rectangular element.

We employ a maximum of 30-point Gaussian quadrature for the hypersingular surface integral evaluation at three of the previously discussed test points. The corresponding integral values are compared to the reference results by Guiggiani et al. [34]. Our numerical results in Table 3 show the convergences of the hypersingular integrals values at the test locations with respect to the increasing number of Gaussian quadrature orders of up to 20 units. As comparing to the results by Guiggiani for up to 10-point Gaussian quadrature, both methods show almost identical results in this particular example. But beyond the Gaussian quadrature order of 10, our results exemplify further that if the location of the singularity is well away from the element’s boundary such as the test point a, the proposed method is highly efficient in obtaining convergence result of 7 significant figures with the use of only 10 Gaussian integration points on a distorted 8-node rectangular element. The convergence will be slightly degraded when the singularity point is close to the element’s boundary. For instance, the test points b and c will require 4 and 6 higher orders of Gaussian quadrature respectively to achieve results in 7 significant figures. Overall, the present method is able to provide highly accurate and stable results with exponential convergence for economically viable number of Gaussian integration points.

6.2. Example 1b: improved meshing of trapezium using rectangular elements

To further improve the convergence of the hypersingular surface integral, the trapezium is divided at the test point a(0, 0) into 4 regions consisting of 2 non-distorted elements and 2 distorted elements as shown in Fig. 7. As discussed previously in (45), there is no difficulty in evaluating the hypersingular surface integral at geometric corner with the addition of


### Table 4
Numerical evaluation of hypersingular surface integral on trapezium at test point \(a\) with improved meshing using 4 rectangular elements.

<table>
<thead>
<tr>
<th>(m)</th>
<th>Point (a)</th>
</tr>
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<tbody>
<tr>
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<td>6</td>
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<td>18</td>
<td>-5.749236751228</td>
</tr>
<tr>
<td>20</td>
<td>-5.749236751228</td>
</tr>
</tbody>
</table>

**Fig. 8.** Relative errors of hypersingular surface integral with singularity point \(a\) in Examples 1a and 1b.

free terms. It is further noted that since the corner point \(a\) has a continuous outward normal across all 4 divided elements, the free terms can subsequently be dropped when evaluating the integral value. The numerical results employing the improved meshing are presented in Table 4. Comparing to the results in Example 1a, the results with improved meshing show better convergence with reduced order of Gaussian quadrature. To achieve the same numerical accuracy of 7 significant figures, only 6 Gaussian integration points are required in contrast to the 10 Gaussian points in the previous example. To put the numerical performance into perspective, we compute the relative errors with respect to the order of Gaussian quadrature in both examples. The results in Fig. 8 show that the improved meshing scheme is able to provide significantly faster convergence rate until it reaches and settles on the machine precision. It is in line with Table 4 that the precision limit of 13 significant figures can be efficiently achieved by using a mere of 12 Gaussian integration points.

### 6.3. Example 2a: rectangle meshed with regular rectangular elements

In this example, the effect of the singularity point on the accuracy of the numerical results, when close to the boundary, is studied. The hypersingular surface integral has the following form:

\[
I = \int_S \frac{r_1}{r^3} dS
\]

where the surface of integration is over a rectangular region with corner points at \((-1, 0, 0), (1, 0, 0), (1, 4, 0), (-1, 4, 0)\) as shown in Fig. 9. Again, three test points of \(a(0.5, -0.5), b(0.75, -0.5)\) and \(c(0.99, -0.5)\) are chosen for hypersingular surface integral evaluation using the following modified density function \(\varphi\),

\[
\varphi(\xi^p, \rho, \theta) = \frac{r_1 l_i}{\rho_k^3}
\]

For this numerical computation, the integration surface is divided into an element with hypersingularity and a regular element which can be evaluated using normal integration procedure without any special treatment. Gaussian quadrature
order of up to 52 points is employed in this study. For instance, the distribution of 4-point Gaussian quadrature in the triangular integration subregions (cf. (54)) is illustrated in Fig. 9. The hypersingular surface integral values at the three test points are compared to the reference results by Gao [35]. The numerical results in Table 5 show that the proximity of the singularity point to the element’s boundary does not seem to have a noticeable impact on the convergence of the evaluated results as comparing to the distorted element in the previous example. The computational cost of attaining 12 significant figures accuracy are shown to be at 16-point Gaussian quadrature. Indeed, the present method demonstrates an exponential convergence as shown in Fig. 11, in which the current example is labelled as case I. These results also outperform the recent power series method by Gao [35] which indicates an accuracy of 5 significant figures at best.

6.4. Example 2b: rectangle meshed with distorted triangular elements of various degrees

Based on the Example 2a, the viability of employing 6-node triangular elements to evaluate hypersingular surface integral and their distortion effects are studied. It is noted that the special treatment using the proposed method is only applied to the triangular elements containing the singularity point. The distribution of Gaussian point inside these elements will have an implication on the result accuracy. To examine the progressive impacts of the degree of distortion, we first mesh the rectangle with regular 6-node triangular elements and compute its integral value at test point a for later comparison. The aspect ratio (AR) of this meshing is 1.4142 and is labelled as case II in our numerical simulation. In case III, the same integral is subjected to distorted meshing where the elements division is at the centre of the rectangle, of which the aspect ratio is 26.5% higher than the case II meshing. It is noted that the singularity point is now located at edges between the two triangular elements on the lower right where the hypersingular integral is applied to. In case IV, a higher order of distortion with an aspect ratio of 2.3851 is examined. The division point is placed at the halfway between the rectangle’s centre

![Image](image_url)
and the test point $a$. In case V, the elements division is at the test point $a$ such that the proposed hypersingular integral treatment will need to be applied to all 4 triangular elements, of which the highest aspect ratio is 3.5777. The schematics of case II to case V along with their Gaussian quadrature distributions corresponding to the proposed hypersingular integral treatment are illustrated in Fig. 10.

The numerical results in Table 6 and the respected relative errors in Fig. 11 show that elements with lower degree of distortion, which can be measured by the magnitude of the aspect ratio, are consistently able to provide better convergences. For instance, to obtain 7 significant figures of accuracy, a 10-point Gaussian quadrature would be required for the regular rectangular element in case I, a 16-point Gaussian quadrature for the highly regular triangular element in case II, or a 34-point Gaussian quadrature for the highly distorted triangular element in case V. Generally, the 8-node rectangular element enjoys a better convergence than the 6-node triangular element due to the higher degree of freedom its polynomial interpolation imposes on. Furthermore, the number of integration subregions resulting from the polar coordinate transformation is 4 for the rectangular element as comparing to 3 for the triangular element. The implication of this is the increased number of integration points which further contributes to the stronger result by the rectangular element. Overall, the rectangular element, the highly regular and the highly distorted triangular elements using the present method are able to produce accurate results that converge to near machine precision as illustrated in Fig. 11.
Table 6
Numerical evaluation of hypersingular surface integrals using rectangular elements and triangular elements with various aspect ratios.

<table>
<thead>
<tr>
<th>m</th>
<th>Case I AR = 1</th>
<th>Case II AR = 1.4142</th>
<th>Case III AR = 1.7889</th>
<th>Case IV AR = 2.3851</th>
<th>Case V AR = 3.5777</th>
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<td>Exact</td>
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<td></td>
<td></td>
<td>1.947746</td>
</tr>
</tbody>
</table>

Fig. 11. Relative errors of hypersingular surface integrals using rectangular elements and triangular elements with various aspect ratios.

6.5. Example 3: quarter cylindrical panel with curved rectangular element

In this final example, the performance of using the curved rectangular element is studied. The hypersingular surface integral has form similar to Example 1,

\[
I = \int_S \frac{-1}{4\pi r^3} \left( 3r_3 \frac{\partial r}{\partial n} - n_3 \right) dS
\]

(79)
where the surface of integration is over a quarter cylindrical panel of radius 1 unit by length 2 units with corner points located at \(\{1, 0, 0\}, \{1, 2, 0\}, \{0, 2, 1\}, \{0, 0, 1\}\) as shown in Fig. 12. Three test points namely, \(a(0, 0)\), \(b(0.66, 0)\) and \(c(0.66, 0.66)\) of local orthogonal curvilinear coordinate \(\xi\) are chosen for evaluations. The modified density function \(\varphi\) of the present method has the following form:

\[
\varphi(\xi^p, \rho, \theta) = -\frac{1}{4\pi} \left(3\frac{\partial r}{\partial n} - n_3\right) \frac{J_\xi}{\rho^3}
\]

(80)

Gaussian quadrature of up to 30 points is employed for evaluating the hypersingular surface integrals. The corresponding integral values of each test point are compared to the reference results by Guiggiani et al. [34] and Feng et al. [36]. As similar to the previous examples, our numerical results in Table 7 show that the present method has convergence results of up to 12 significant figures. While the convergence rate will slightly deteriorate when the singularity point approaches to the element’s boundary, e.g. at test point \(c\), it is nevertheless still very economical to obtain 8 significant figures accuracy with only 14-point Gaussian quadrature, or the near machine precision of 12 significant figures for 22-point Gaussian quadrature in our worst-case scenario. To give a more substantial picture on the competitiveness of our proposed method, we list our convergence results of each test point in 12 significant figures and directly compare them to the reference results [34,36] in Table 8. If we take our convergence results of 12 significant figures as reference value, then the results of Guiggian’s method are deviated from that by \(10^{-6}\) unit while the results of Feng’s method are deviated from \(10^{-4}\) to \(10^{-6}\) units. Moreover, Fig. 13 shows that the proposed method features exponential decay behaviour for the relative error and a faster convergence.

Table 7
Convergence rate of the proposed method and the literature [34] at various singularity points for increasing number of Gaussian quadrature points.

<table>
<thead>
<tr>
<th>(m)</th>
<th>Point a</th>
<th></th>
<th>Point b</th>
<th></th>
<th>Point c</th>
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<td>Reference</td>
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</tbody>
</table>

Fig. 12. Schematic of a curved boundary element representing the quarter cylindrical panel.
Table 8
Hypersingular integral results for the quarter cylindrical panel at 3 different singularity points (Note that 0’s are added in the reference results as only 6 significant figures are found in their publications).

<table>
<thead>
<tr>
<th>Method</th>
<th>Point a</th>
<th>Point b</th>
<th>Point c</th>
</tr>
</thead>
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</tr>
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<td>Feng et al. [36]</td>
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<td>Proposed</td>
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</table>

Fig. 13. Convergence rate of the proposed method and the literature [34] at singularity point c for increasing number of Gaussian quadrature points.

Fig. 14. Contour plot of the hypersingular integral values with respect to various singularity points.

as comparing to the method by Guiggiani. To illustrate the continuity of the hypersingular surface integral, we provide a contour plot of the integral’s value with its singularities located inside an enclosed region by the test points, as indicated by the grey area in Fig. 12. As shown in Fig. 14, the contour plot displays an expected smooth transition of the integral value subjecting to the curved geometric boundaries.
7. Conclusion

To conclude, hypersingular line integral is first expanded into regular and singular integrals by employing Brandao’s formulation. The singular integrals with simpler finite parts are determined analytically while the regular integral consisting of a derivative term is approximated by the complex variable differentiation method (CVDM) to produce machine precision like accuracy. For the evaluation of hypersingular surface integral, coordinate transformations are carried out such that the integration variables are oriented to the singularity. The resulting inner integral can be readily cast into the previously proposed numerical form for hypersingular line integrals so as to have the singularity and the finite part automatically regularised.

It should be emphasised that the core novelty of the line integrals treatment is the employment of the CVDM to approximate the derivative of the density function so that a near machine precision accuracy is obtained. For the hypersingular surface integrals, despite more mathematical manipulations being involved in the numerical procedures, the underlying principle is that the corresponding density functions must be transformed into modified one compatible with the regularisation process. This transformation enables the previously developed hypersingular line integral formulation to be directly invoked to solve the surface integral problems.

There are some other notable features associated with the present work:

- The singularity regularisation process for the hypersingular surface integral is generalised by using Brandao’s formulation which bypasses the need for the limiting processes that are often required in other reference methods.
- When it comes to the density function approximation, the one complex-step approximation will suffice to provide near machine precision accuracy. Equipped with this powerful tool, the present method is free of stability issue while capable of providing highly accurate results. In contrast, the approximation process in other reference methods often relies on series expansion or interpolation schemes which inevitably would cost more computational resources but not necessarily improve the accuracy of results.

The outcome of the present work is a highly accurate, fast and generalised method for numerically evaluating line and surface integrals with principal values. The developed regularisation process and the numerical treatment are directly applicable to integrals with higher order singularities.

Appendix A

For surface area represented by an 8-node rectangular element as illustrated in Fig. A. 1, the shape functions \( N_k \) and their derivatives \( \frac{\partial N_k}{\partial \xi_j} \) can be expressed in the followings:

\[
N_k(\xi_1, \xi_2) = \frac{1}{4} \left( \xi_1^k \xi_1^k + 1 \right) \left( \xi_2^k \xi_2^k + 1 \right) \left( \xi_1^k \xi_2^k + \xi_2^k \xi_2^k - 1 \right) \quad \text{for } k \in \{1, 4\} \tag{A.1}
\]

\[
N_k(\xi_1, \xi_2) = \frac{1}{2} \left( \xi_1^k \xi_1^k + 1 \right) \left[ -\left( \xi_1^k \xi_2^k \right)^2 - \left( \xi_2^k \xi_2^k \right)^2 + 1 \right] \quad \text{for } k \in \{5, 8\} \tag{A.2}
\]

\[
\frac{\partial N_k(\xi_1, \xi_2)}{\partial \xi_1} = \frac{1}{4} \xi_1^k \left( 2 \xi_1^k \xi_1^k + \xi_2^k \xi_2^k \right) \left( \xi_2^k \xi_2^k + 1 \right) \quad \text{for } k \in \{1, 4\} \tag{A.3}
\]

\[
\frac{\partial N_k(\xi_1, \xi_2)}{\partial \xi_2} = \frac{1}{4} \xi_1^k \left( 2 \xi_1^k \xi_1^k + 2 \xi_2^k \xi_2^k \right) \left( \xi_2^k \xi_2^k + 1 \right) \quad \text{for } k \in \{1, 4\} \tag{A.4}
\]

\[
\frac{\partial N_k(\xi_1, \xi_2)}{\partial \xi_1} = -\frac{1}{2} \xi_1^k \left[ \left( \xi_1^k \xi_1^k \right)^2 + \left( \xi_1^k \xi_2^k \right)^2 - 1 \right] - \left( \xi_1^k \right)^2 \xi_1^k \xi_1^k \xi_1^k + \xi_2^k \xi_2^k + 1 \quad \text{for } k \in \{5, 8\} \tag{A.5}
\]

\[
\frac{\partial N_k(\xi_1, \xi_2)}{\partial \xi_2} = -\frac{1}{2} \xi_1^k \left[ \left( \xi_1^k \xi_1^k \right)^2 + \left( \xi_2^k \xi_2^k \right)^2 - 1 \right] - \left( \xi_1^k \right)^2 \xi_2^k \xi_1^k + \xi_2^k \xi_2^k + 1 \quad \text{for } k \in \{5, 8\} \tag{A.6}
\]

where \( \xi_{i=1,2}^k \) are the nodal points expressed in the local coordinate \( \xi_j \) of the element.

In the case of a 6-node triangular element as shown in Fig. A. 2, the orthogonal curvilinear coordinate system plays a central role in simplifying the Jacobian for the later polar coordinate transformation. The corresponding shape functions and their derivatives can be expressed in the followings [51]:

\[
N_1(\xi_1, \xi_2) = \frac{1}{6} (\sqrt{3} \xi_1^2 + \xi_2^2 - \sqrt{3}) (\sqrt{3} \xi_1^2 + \xi_2^2) \quad \tag{A.7}
\]

\[
N_2(\xi_1, \xi_2) = \frac{1}{6} (-\sqrt{3} \xi_1^2 + \xi_2^2 - \sqrt{3}) (-\sqrt{3} \xi_1^2 + \xi_2^2) \quad \tag{A.8}
\]

\[
N_3(\xi_1, \xi_2) = \frac{1}{3} \xi_2^2 (2 \xi_2^2 - \sqrt{3}) \quad \tag{A.9}
\]

\[
N_4(\xi_1, \xi_2) = \frac{1}{3} (\sqrt{3} \xi_1^2 + \xi_2^2 - \sqrt{3}) (-\sqrt{3} \xi_1^2 + \xi_2^2 - \sqrt{3}) \quad \tag{A.10}
\]
Fig. A.1. Local orthogonal curvilinear coordinates of 8-node plate element and its nodal points' distribution.

Fig. A.2. Local orthogonal curvilinear coordinates of 6-node triangular element and its nodal points' distribution.

\[ N_5(\xi_1, \xi_2) = \frac{2}{3} \xi_2 (\sqrt{3} \xi_1 - \xi_2 + \sqrt{3}) \]  \hspace{1cm} (A.11)

\[ N_6(\xi_1, \xi_2) = \frac{2}{3} \xi_2 (-\sqrt{3} \xi_1 - \xi_2 + \sqrt{3}) \]  \hspace{1cm} (A.12)

\[ \frac{\partial N_1(\xi_1, \xi_2)}{\partial \xi_1} = \xi_1 + \frac{\sqrt{3}}{3} \xi_2 - \frac{1}{2} \]  \hspace{1cm} (A.13)

\[ \frac{\partial N_2(\xi_1, \xi_2)}{\partial \xi_1} = \xi_1 - \frac{\sqrt{3}}{3} \xi_2 + \frac{1}{2} \]  \hspace{1cm} (A.14)

\[ \frac{\partial N_3(\xi_1, \xi_2)}{\partial \xi_1} = 0 \]  \hspace{1cm} (A.15)

\[ \frac{\partial N_4(\xi_1, \xi_2)}{\partial \xi_1} = -2\xi_1 \]  \hspace{1cm} (A.16)

\[ \frac{\partial N_5(\xi_1, \xi_2)}{\partial \xi_1} = \frac{2\sqrt{3}}{3} \xi_2 \]  \hspace{1cm} (A.17)

\[ \frac{\partial N_6(\xi_1, \xi_2)}{\partial \xi_1} = -\frac{2\sqrt{3}}{3} \xi_2 \]  \hspace{1cm} (A.18)

\[ \frac{\partial N_1(\xi_1, \xi_2)}{\partial \xi_2} = \frac{1}{3} \left( \sqrt{3} \xi_1 + \xi_2 - \frac{\sqrt{3}}{2} \right) \]  \hspace{1cm} (A.19)
\[
\frac{\partial N_2(\xi_1, \xi_2)}{\partial \xi_2} = \frac{1}{3} \left( -\sqrt{3} \xi_1 + \xi_2 - \frac{\sqrt{3}}{2} \right) 
\]
(A.20)

\[
\frac{\partial N_3(\xi_1, \xi_2)}{\partial \xi_2} = \frac{1}{3} \left( 4\xi_2 - \sqrt{3} \right) 
\]
(A.21)

\[
\frac{\partial N_4(\xi_1, \xi_2)}{\partial \xi_2} = \frac{2}{3} (\xi_2 - \sqrt{3}) 
\]
(A.22)

\[
\frac{\partial N_5(\xi_1, \xi_2)}{\partial \xi_2} = \frac{2}{3} (\sqrt{3} \xi_1 - 2\xi_2 + \sqrt{3}) 
\]
(A.23)

\[
\frac{\partial N_6(\xi_1, \xi_2)}{\partial \xi_2} = \frac{2}{3} (-\sqrt{3} \xi_1 - 2\xi_2 + \sqrt{3}) 
\]
(A.24)

### Appendix B

For the 8-node rectangular element, \( \theta \) and \( \rho_L(\xi^p, \theta) \) are partitioned into four subregions as was shown in Fig. 4. The lower and upper bounds of \( \theta \), namely \( \theta_1^p(\xi^p) \) and \( \theta_4^p(\xi^p) \) of the partitioned regions are defined as

\[
\begin{bmatrix}
\theta_1^p(\xi^p) \\
\theta_2^p(\xi^p) \\
\theta_3^p(\xi^p) \\
\theta_4^p(\xi^p) \\
\end{bmatrix} = \begin{bmatrix}
\theta_1(\xi^p) \\
\theta_2(\xi^p) \\
\theta_3(\xi^p) \\
\theta_4(\xi^p) - 2\pi \\
\end{bmatrix}
\]  
(B.1)

\[
\begin{bmatrix}
\theta_1^p(\xi^p) \\
\theta_2^p(\xi^p) \\
\theta_3^p(\xi^p) \\
\theta_4^p(\xi^p) \\
\end{bmatrix} = \begin{bmatrix}
\theta_2(\xi^p) \\
\theta_3(\xi^p) \\
\theta_4(\xi^p) \\
\theta_1(\xi^p) \\
\end{bmatrix}
\]  
(B.2)

where

\[
\theta_1(\xi^p) = \tan^{-1} \left( \frac{1 - \xi_2^p}{1 - \xi_1^p} \right), \quad \theta_2(\xi^p) = \pi - \tan^{-1} \left( \frac{1 - \xi_2^p}{1 + \xi_1^p} \right) \\
\theta_3(\xi^p) = \pi + \tan^{-1} \left( \frac{1 + \xi_2^p}{1 + \xi_1^p} \right), \quad \theta_4(\xi^p) = 2\pi - \tan^{-1} \left( \frac{1 + \xi_2^p}{1 - \xi_1^p} \right)
\]  
(B.3)

The upper bounds of the inner integration \( \rho_L(\xi^p, \theta) \) have the following trigonometric forms:

\[
\rho_L(\xi^p, \theta) = \begin{cases} 
\frac{1 - \xi_2^p}{\sin\theta} & \text{for } \theta_1(\xi^p) \leq \theta \leq \theta_2(\xi^p) \\
\frac{1 + \xi_1^p}{\cos\theta} & \text{for } \theta_2(\xi^p) \leq \theta \leq \theta_3(\xi^p) \\
\frac{1 + \xi_2^p}{\sin\theta} & \text{for } \theta_3(\xi^p) \leq \theta \leq \theta_4(\xi^p) \\
\frac{1 - \xi_1^p}{\cos\theta} & \text{for } \theta_4(\xi^p) - 2\pi \leq \theta \leq \theta_1(\xi^p) 
\end{cases}
\]  
(B.4)

For example, the radial path \( \rho_L((0,0), \theta) \) when the singularity point is located at \( \xi^p=(0,0) \) is shown in Fig. B.1.

![Fig. B.1. Radial path \( \rho_L((0,0), \theta) \) for singularity point located at the centre of the 8-node plate element.](image)
For the 6-node triangular element, $\theta$ and $\rho_\ell(\xi^P, \theta)$ are partitioned into three subregions as shown in Fig. 5. The lower and upper bounds of $\theta$ in the partitioned regions are defined as

\[
\begin{align*}
\begin{bmatrix}
\theta_1^P(\xi^P) \\
\theta_2^P(\xi^P) \\
\theta_3^P(\xi^P)
\end{bmatrix} &= \begin{bmatrix}
\theta_1(\xi^P) \\
\theta_2(\xi^P) \\
\theta_3(\xi^P) - 2\pi
\end{bmatrix},
\end{align*}
\]

(B.5)

\[
\begin{align*}
\begin{bmatrix}
\theta_1^P(\xi^P) \\
\theta_2^P(\xi^P) \\
\theta_3^P(\xi^P)
\end{bmatrix} &= \begin{bmatrix}
\theta_2(\xi^P) \\
\theta_3(\xi^P) \\
\theta_1(\xi^P)
\end{bmatrix},
\end{align*}
\]

(B.6)

where

\[
\begin{align*}
\theta_1(\xi^P) &= \cos^{-1}\left(\frac{-\xi_1^P}{\sqrt{(\xi_1^P)^2 + (\sqrt{3} - \xi_2^P)^2}}\right),
\end{align*}
\]

(B.7)

\[
\begin{align*}
\theta_2(\xi^P) &= \pi + \tan^{-1}\left(\frac{\xi_2^P}{1 + \xi_1^P}\right),
\end{align*}
\]

(B.8)

\[
\begin{align*}
\theta_3(\xi^P) &= 2\pi - \tan^{-1}\left(\frac{\xi_2^P}{1 - \xi_1^P}\right),
\end{align*}
\]

(B.9)

The upper bounds of the inner integration $\rho_\ell(\xi^P, \theta)$ are derived using the point-line perpendicular distance formula and simple trigonometric functions,

\[
\rho_\ell(\xi^P, \theta) = \begin{cases}
\frac{\sqrt{3\xi_1^P - \xi_2^P + \sqrt{3}}}{2\cos\left(\theta - \frac{5\pi}{6}\right)} & \text{for } \theta_1(\xi^P) \leq \theta \leq \theta_2(\xi^P), \\
-\frac{\xi_2^P}{\sin\theta} & \text{for } \theta_2(\xi^P) \leq \theta \leq \theta_3(\xi^P), \\
\frac{\sqrt{3\xi_1^P + \xi_2^P - \sqrt{3}}}{2\cos\left(\theta - \frac{\pi}{6}\right)} & \text{for } \theta_3(\xi^P) - 2\pi \leq \theta \leq \theta_1(\xi^P)
\end{cases}
\]

(B.10)

References


