BEM of postbuckling analysis of thin plates

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A new fundamental solution of deflection for plate buckling problems is first derived through the separation of in-plane stresses into an essential part and a disturbed part and by use of the resolution theorem of differential operator. A set of new boundary element formulas for the analysis of postbuckling problems of elastic plates are presented. On the basis of these formulas, in-plane and out-of-plane displacements can be decoupled. The solution procedure for the boundary value problems of the in-plane displacement becomes very simple. Some examples of a circular plate are considered, and their results are in good agreement with the known ones. It appears that the proposed method is very effective for solving the postbuckling problem of plates with arbitrary geometry.

Keywords: postbuckling, thin plate, boundary element method

Introduction

Recently, several researchers have studied finite deformation behavior of plates by using the boundary element method (BEM). Kamiya et al.[1] first presented an integral equation approach to the finite deflection of plates with the aid of Berger's equation. The equation is a quasi-linear one without introduction of the coupling between bending and in-plane deformation. The equation was originally derived by the variational calculus with unconditional disregard of the second invariant of membrane strains included in the strain energy expression. For this reason the applications are limited. Later on, Tanaka[2] derived a coupled set of boundary and inner domain integral equations in terms of the stress and deflection functions based on von Karman's equation. The integral equations have generality in theory. However, his formulas could not handle such a category of problems in which the in-plane displacements rather than the in-plane stresses on the boundary are prescribed because there were no explicit expressions describing the relationship between the in-plane displacements and the stress function.

In this paper, emphasis will put on a BEM solution of the postbuckling behavior of plates in which a set of three nonlinear integral equations in three displacement functions (U_1, U_2, W) is obtained. Accordingly, the limitation of solution to the case in which the in-plane stresses on the boundary of a plate are prescribed is removed. Moreover, a new fundamental solution is also derived. With the aid of the boundary element formulas in incremental form obtained here, the BEM solution procedure can be greatly simplified. Finally, as an application of the proposed method, a number of numerical examples are illustrated. The results are in good agreement with already existing solutions. It appears that the proposed method is effective for solving the problems mentioned above.

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Basic equations and their fundamental solution

Basic equations

Consider a thin isotropic plate of uniform thickness h and of arbitrary shape subjected to an external radial uniform compressive load, p_0, per unit length at the boundary (*Figure 1*). The material constants of the plate are represented by E (Young's modulus) and ν (Poisson's ratio). We use a cartesian coordinate system in which the x- and y-axes lie in the plate middle plane. The field equations governing the postbuckling behavior can be written as follows:

$$\begin{aligned} &N_{x,x} + N_{xy,y} = 0 \\ &N_{y,y} + N_{xy,x} = 0 \\ &D\nabla^4 W + p_0 \nabla^2 W \\ &\qquad = N_x W_{,xx} + 2N_{xy} W_{,xy} + N_y W_{,yy} + q \end{aligned} \tag{1}$$

where N_x, N_y, and N_{xy} represent the disturbed membrane force components, W is the deflection of the plate, $D = (Eh^3/12(1 - \nu^2))$ is the flexural rigidity, a comma followed by a subscript indicates partial differentiation with respect to that subscript, and $\nabla^2 = (\ )_{,xx} + (\ )_{,yy}$.

Substituting the well-known physical and geometric

relations into equations (1), we may obtain the following differential equations in three displacement functions (U_1, U_2, W):

$$\begin{aligned} L_1 U_1 + L_2 U_2 &= P_1 \\ L_2 U_1 + L_3 U_2 &= P_2 \\ L_4 W &= P_3 \end{aligned} \tag{2}$$

with

$$L_1(\ ) = (\ )_{,xx} + d_1(\ )_{,yy} \qquad L_2(\ ) = d_2(\ )_{,xy}$$
$$L_3(\ ) = (\ )_{,yy} + d_1(\ )_{,xx} \qquad L_4(\ ) = D(\nabla^4 + \lambda^2\nabla^2)(\ )$$
$$\lambda^2 = p_0/D \qquad d_1 = (1-\nu)/2 \qquad d_2 = (1+\nu)/2$$

in which U_1 and U_2 represent the disturbed in-plane displacement components and P_1, P_2, and P_3 are the pseudo-distributed load components defined by

$$\begin{aligned} P_1 &= -W_{,x}(W_{,xx} + d_1 W_{,yy}) - d_2 W_{,y} W_{,xy} \\ P_2 &= -W_{,y}(W_{,yy} + d_1 W_{,xx}) - d_2 W_{,x} W_{,xy} \\ P_3 &= J\{(U_{1,x} + 0.5W_{,x}^2)(W_{,xx} + \nu W_{,yy}) + (U_{2,y} + 0.5W_{,y}^2)(W_{,yy} + \nu W_{,xx})\} \\ &\quad + J(1-\mu)(U_{1,y} + U_{1,y} + W_{,x}W_{,y})W_{,xy} + q \end{aligned}$$

where

$$J = \frac{Eh}{1-\nu^2}$$

Since an incremental formulation may have a wider applicability to higher nonlinear problem, it is necessary to express equations (2) in the incremental form. Denoting the incremental variable by the superimposed dot and omitting the infinitesimal resulted from the product of incremental variables, one obtains

$$\begin{aligned} L_1 \dot{U}_1 + L_2 \dot{U}_2 &= \dot{P}_1 \\ L_2 \dot{U}_1 + L_3 \dot{U}_2 &= \dot{P}_2 \\ L_4 \dot{W} &= \dot{P}_3 \end{aligned} \tag{3}$$

where

$$\begin{aligned} \dot{P}_1 &= -\dot{W}_{,x}(W_{,xx} + d_1 W_{,yy}) - W_{,x}(\dot{W}_{,xx} + d_1 \dot{W}_{,yy}) - d_2 \dot{W}_{,y} W_{,xy} - d_2 W_{,y} \dot{W}_{,xy} \\ \dot{P}_2 &= -\dot{W}_{,y}(W_{,yy} + d_1 W_{,xx}) - W_{,y}(\dot{W}_{,yy} + d_1 \dot{W}_{,xx}) - d_2 \dot{W}_{,x} W_{,xy} - d_2 W_{,x} \dot{W}_{,xy} \\ \dot{P}_3 &= J\{(\dot{U}_{1,x} + W_{,x}\dot{W}_{,x})(W_{,xx} + \nu W_{,yy}) + (U_{1,x} + 0.5W_{,x}^2)(\dot{W}_{,xx} + \nu \dot{W}_{,yy}) \\ &\quad + (\dot{U}_{2,y} + W_{,y}\dot{W}_{,y})(W_{,yy} + \mu W_{,xx}) + (U_{2,y} + 0.5W_{,y})(\dot{W}_{yy} + \nu \dot{W}_{,xx})\} \\ &\quad + J(1-\nu)\{(\dot{U}_{1,y} + \dot{U}_{2,x} + \dot{W}_{,x}W_{,y} + W_{,x}\dot{W}_{,y})W_{,xy} + (U_{1,y} + U_{2,x} + W_{,x}W_{,y})\dot{W}_{,xy}\} + \dot{q} \end{aligned} \tag{4}$$

It follows that equations (3) are linear with respect to the incremental variables. It is worthwhile to note that the total values at the current deformation state under consideration are assumed to be known in the incremental approach.

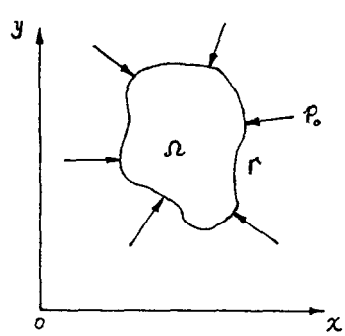

Figure 1. Geometry and loading

Fundamental solutions

According to the BEM in 2-D elasticity the fundamental solution corresponding to the first two equations of (3) is obviously Kelvin's solution (for the plane stress case) as

$$\begin{aligned} U_{ij}^*(r) &= \frac{1+\nu}{4\pi E}\left[(3-\nu)\ln\left(\frac{1}{r}\right)\delta_{ij} + (1+\nu)r_i r_j\right] \\ N_{ij}^*(r) &= -\frac{1}{4\pi r}[r_n[(1-\nu)\delta_{ij} + 2(1+\nu)r_i r_j] \\ &\quad - (1-\nu)(r_i n_j - r_j n_i)] \end{aligned} \tag{5}$$

where the senses of all the notations are given in Ref. 3.

In the following, attention will be focused on seeking the fundamental solution W^*, corresponding to the third equation of (3), which is defined by

$$D(\nabla^4 + \lambda^2\nabla^2)W^* = D\nabla^2(\nabla^2 + \lambda^2)W^* = \delta(P,Q) \tag{6}$$

where $\delta(P, Q)$ is the Dirac δ-function, and for the sake

of brevity we set

$$\nabla^2 W^* = A \tag{7}$$

It follows from equation (6) that

$$D(\nabla^2 + \lambda^2)A = \delta(P, Q) \tag{8}$$

The solution of (8) can be easily obtained as

$$A = N_0(\lambda r)/4D \tag{9}$$

in which r is the distance from the source point P to the field point Q under consideration and $N_0(\ )$ is the Bessel function of zeroth order of the second kind. In a similar manner, set

$$(\nabla^2 + \lambda^2)W^* = B \tag{10}$$

Then we have

$$D\nabla^2 B = \delta(P, Q) \tag{11}$$

By analogy with equation (9) the solution of equation (11) is

$$B = \ln r/2\pi D \tag{12}$$

Therefore subtracting equation (7) from equation (10) and according to equations (9) and (12), one yields the fundamental solution W^*:

$$W^* = [2\ln r/\pi - N_0(\lambda r)]/4D\lambda^2 \tag{13}$$

Integral equation formulation

By virtue of Betti's reciprocal theorem the integral equations corresponding to the postbuckling problems of elastic plates can be obtained in the following form:

$$C(P)\dot{U}_1(P) - \int_\Gamma [\dot{N}_j(Q)U^*_{ij}(P,Q) - \dot{U}_j(Q)N^*_{ij}(P,Q)]\,d\Gamma(Q) = \int_\Omega \dot{P}_j(Q)U^*_{ij}(P,Q)\,d\Omega(Q)$$

$$\bar{C}(P)\dot{W}(P) + \int_\Gamma [V^*_n(P,Q)\dot{W}(Q) - M^*_n(P,Q)\dot{\Theta}_n(Q) - \dot{V}_n(Q)W^*(P,Q) + \dot{M}_n(Q)\Theta^*_n(P,Q)]\,d\Gamma(Q)$$

$$+ \sum_{j=1}^{k} [\Delta M^*_{nt}(P,Q_j)\dot{W}(Q_j) - \Delta\dot{M}_{nt}(Q_j)W^*(P,Q_j)] = \int_\Omega \dot{P}_3(Q)W^*(P,Q)\,d\Omega(Q) \tag{14}$$

$$\bar{\bar{C}}(P)\dot{W}_{,n_0}(P) + \int_\Gamma [V^*_{n,n_0}(P,Q)\dot{W}(Q) - M^*_{n,n_0}(P,Q)\dot{\Theta}_n(Q) - \dot{V}_n(Q)W^*_{,n_0}(P,Q)$$

$$+ \dot{M}_n(Q)\Theta^*_{n,n_0}(P,Q)]\,d\Gamma(Q) + \sum_{j=1}^{k} [\Delta M^*_{nt,n_0}(P,Q_j)\dot{W}(Q_j) - \Delta\dot{M}_{nt}(Q_j)W^*_{,n_0}(P,Q_j)]$$

$$= \int_\Omega \dot{P}_3(Q)W^*_{,n_0}(P,Q)\,d\Omega(Q)$$

where the repeated index j implies the Einstein summation convention with $j \in \{1, 2\}$, n is the outward normal at the boundary Γ, n_0 is the outward normal at the source point on Γ, $\Delta(\ )_{Q_j} = (\ )_{Q_j^+} - (\ )_{Q_j^-}$ is the discontinuity jump at the corner point Q_j, $(\ )_{Q_j^-}$ and $(\ )_{Q_j^+}$ stand for the quantities before and after the corner Q_j, respectively, and k is the number of all the corners. According to the thin plate theory, the bending moment M^*_n and the equivalent shear force V^*_n corresponding to the fundamental solution W* can be evaluated through the high-order derivatives of W*. The details can be found in Ref. 4.

To obtain a weak solution of equations (14), as in the usual boundary element method the boundary Γ and the domain Ω of the plate are divided into a series of boundary elements and internal cells, respectively. After performing the discretization by use of various kinds of boundary elements (e.g., constant element, linear element, or high-order element), the boundary integral equations (14) become a set of linear algebraic equations including the incremental boundary variables $\dot{U}_1$, $\dot{U}_2$, $\dot{N}_1$, $\dot{N}_2$, $\dot{W}$, $\dot{\Theta}_n$, $\dot{M}_n$, and $\dot{V}_n$ for each loading step:

$$[S]\{\dot{U}\} - [Q]\{\dot{N}\} = \{\dot{R}_1\} \tag{15a}$$

$$[H]\{\dot{\Delta}\} + [G]\{\dot{T}\} = \{\dot{R}_2\} \tag{15b}$$

It can be seen from equations (4) that $\dot{P}_1$ and $\dot{P}_2$ depend only upon $\dot{W}$. So as long as the value of $\dot{W}$ in the domain Ω is known *a priori*, we can compute the pseudo-loading vector $\{\dot{R}_1\}$. After applying the boundary conditions and reordering equations (15a) and (15b) we obtain

$$[E]\{\dot{X}\} = \{\dot{R}_3\} \tag{16a}$$

$$[F]\{\dot{Y}\} = \{\dot{R}_4\} \tag{16b}$$

where $\{\dot{X}\}$ contains the unknown in-plane displacements and tractions and $\{\dot{Y}\}$ contains the unknown bending deflections and internal forces. For this case, all the coefficient matrices in equations (15a) and (15b) can be determined in terms of the boundary integrations of the fundamental solutions (U^*_{ij} and W^*) and their higher-order derivatives with respect to the unit tangential vector and outward normal vector to the plate boundary.

Iterative solution

By virtue of the property of equations (15a) and (15b) mentioned above, a reasonable iterative solution scheme is developed. In the process of iteration, only an initial value of lateral deflection of the plate is required because $\{\dot{R}_1\}$ in equation (15a) and $\{\dot{R}_3\}$ in equation (16a) are functions independent of $\dot{U}_1$ and $\dot{U}_2$. Suppose that $U_1^{(k)}$, $U_2^{(k)}$, and $W^{(k)}$ express the kth approximations that can be obtained from the preceding cycle of iteration. For the purpose of the $(k + 1)$th solution the iterative procedure is illustrated as follows:

1. Assume the initial value $\dot{W}_0$ in Ω.
2. Calculate $\{\dot{R}_1\}$ on the right-hand side of equation (15a) by means of equations (4) and (5).
3. Calculate $\{\dot{R}_3\}$ on the right-hand side of equation (16a) in view of the given boundary conditions.
4. Solve equation (16a) for the boundary unknown vector $\{\dot{X}\}$ and then determine the values of $\dot{U}_1$ and $\dot{U}_2$ in Ω.
5. Calculate $\{\dot{R}_4\}$ on the right-hand side of equation (16b) by means of equations (4), (14), and (15b) and the given boundary conditions.
6. Solve equation (16b) for the boundary unknown vector $\{\dot{Y}\}$ and then determine the values of $\dot{W}$ in Ω.
7. If $|[\dot{W}(i) - \dot{W}_0(i)]/\dot{W}(i)|_{\max} \leq \epsilon$ (ϵ is a convergence tolerance), proceed to the next loading step and $W^{(k+1)} = W^{(k)} + \dot{W}$. Otherwise, modify the initial value $\dot{W}_0$ and then continue the iteration.

It is important to note that once the matrices $[E]$ and $[F]$ in equations (16a) and (16b) have been formed, they can be stored in the core and used in each cycle of iteration without any change. That is because these matrices depend only upon the geometric and material parameters of plates. Obviously, it can save a large amount of computing time.

Numerical examples

To illustrate the efficiency and feasibility of the proposed method, some examples of circular plates with various boundary conditions subjected to an external radial, uniform, compressive load N_r at the boundary are considered. In all the computations, one quarter of the problems are analyzed. To study the convergence properties of the present approach, eight constant elements on the boundary and three meshes, as shown in *Figure 2*, are used. The geometric and material parameters are

$$\nu = 0.3 \qquad R/h = 48$$

where R is radius of a plate.

The load step p_0 is taken as $0.2N_{rcr}$ (N_{rcr} is the linear buckling load of the plate under consideration). The

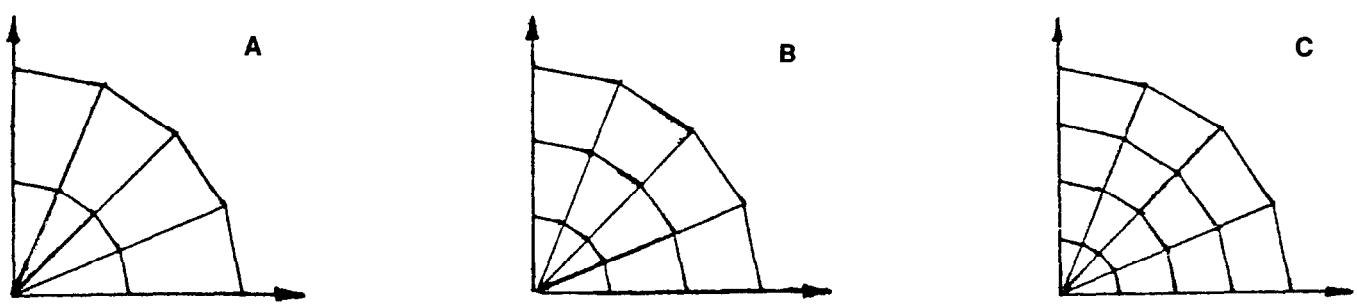

Figure 2. Meshes for the analysis of the circular plate

Table 1. Postbuckling path for circular plate with a simply supported boundary

	β		1.0	1.2	1.4	1.6	1.8	2.0
$\frac{W_m}{h}$	B	8 cells	0	0.952	1.324	1.608	2.037	2.326
	E	12	0	0.891	1.243	1.542	1.803	2.095
	M	16	0	0.882	1.235	1.533	1.792	2.086
	Ref. 5†		0	0.860	1.220	1.490	1.720	1.920

† Values obtained from Figure 68 on p. 169.

Table 2. Postbuckling path for circular plate with a clamped boundary

	β		1.0	1.2	1.4	1.6	1.8	2.0
$\frac{W_m}{h}$	B	8 cells	0	0.805	1.622	2.035	2.735	3.326
	E	12	0	1.103	1.478	1.822	2.103	2.495
	M	16	0	1.015	1.431	1.765	2.052	2.327
	Ref. 5†		0	0.990	1.400	1.710	1.980	2.210

† Values obtained from Figure 68 on p. 169.

convergence tolerance is $\epsilon = 0.0005$. Compressive load coefficient $\gamma = N_r R^2/D$ and $\gamma_{cr} = N_{rcr} R^2/D$.

Example 1: Simply supported circular plate

In this case, $\gamma_{cr} = 4.198$, and the numerical results describing the relationship between the maximum deflection W_m/h occurring at the center and the compressive load coefficient $\beta = \gamma/\gamma_{cr}$ are shown in *Table 1*. Comparison is made with the perturbation solution.[5]

Example 2: Clamped circular plate

In this case, $\gamma_{cr} = 14.68$ and the numerical results obtained and the perturbation solution are shown in *Table 2* for comparison.

It can be seen from *Tables 1* and *2* that the results obtained by employing the present method agree excellently with those of Ref. 5. In the course of computations, convergence was achieved with between five and nine iterations for each load increment. It is worthwhile to note that the iterative method gives well results up to $\beta = 2.0$ for both examples 1 and 2, and for higher values of β the iterative solution has been gradually shown to be unstable, a feature that points to a need for further explorations about the mathematical convergence aspects of the interative method.

Concluding remarks

The achievement of the fundamental solution W^* presented here allows the time of computing the pseudo-distributed normal load to be greatly reduced, and in our iterative calculations, only one initial value of lateral deflection is required. Two sample computations have shown that the postbuckling behavior of plates can be successfully investigated by the proposed method. Therefore this method may appear to be promising.

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