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NONLINEAR ANALYSIS OF THICK PLATES BY HT FE APPROACH

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Abstract—The paper presents a hybrid-Trefftz finite element (HT FE) model for the numerical solution of a nonlinear analysis of Reissner–Mindlin plates. Exact solutions of the Lame–Navier equations are used for representing the in-plane intraelement displacement field and an incremental form of the basic equations is adopted. With the aid of the incremental form of these equations, all nonlinear terms may be viewed as pseudo-loads, and then the coupling between in-plane and out-of-plane displacements is separated. As a result, the solution procedure for the nonlinear plate becomes very simple. The practical efficiency of the new element model has been assessed through several examples. Copyright © 1996 Elsevier Science Ltd

NOTATION

C	$5Et/12(1+\nu)$ for a homogeneous plate, $G_c(h+t)$ for a sandwich plate
C_W	a part of boundary $\partial\Omega$ of the solution domain Ω, on which deflection W is prescribed, C_{M_n}, C_R, etc. are defined similarly
D	$Et^3/12(1+\nu^2)$ for a homogeneous plate, $E(h+t)^2t/2(1-\nu^2)$ for a sandwich plate
E	modulus of elasticity
G	$E/2(1+\nu)$
G_c	core shear modulus
h	core thickness
M_{ii}	bending moment
M_{ij}	twisting moment ($i \neq j$)
N_{ij}	membrane force
n_i	components of the outward normal to the boundary $\partial\Omega$
q	lateral distributed load
Q_i	transverse shear force
r	$(x^2+y^2)^{1/2}$
R	$Q_i n_i + N_n W_{,n} + N_{ns} W_{,s}$
s_i	components of the tangent to the boundary $\partial\Omega$
t	plate thickness (or face-sheet thickness)
U_i	in-plane displacements
W	lateral deflection
δ	variational symbol
δ_{ij}	the Kronecker delta
θ	$\mathrm{arctg}(y/x)$
λ^2	$10/t^2$ for a homogeneous plate, $4(1+\nu)G_c/E(h+t)t$ for a sandwich plate
ν	Poisson's ratio
∇^2	$\partial^2/\partial x^2 + \partial^2/\partial y^2$
ψ_i	the average rotations normal to the plate mid-surface

1. INTRODUCTION

During the past decade the HT element approach, initiated more than 15 years ago [1, 2], has been considerably improved and has now become highly efficient and well established tool. The approach has been successfully applied to plane elasticity [3, 4], plate bending [5–7], shells [8], axisymmetric solid mechanics [9], Poisson's equation [10] and to others [11–15]. So far, however, there is no result by HT FE approach for nonlinear Reissner–Mindlin plates.

The purpose of this paper is to develop a simple HT FE model for nonlinear analysis of Reissner–Mindlin plates. In the analysis, an incremental form of the basic equations is used in order to make the nonlinear equation be linearized. The nonlinear terms in these incremental equations may be viewed as pseudo-loads, and then the coupling of in-plane and out-of-plane displacements is separated. As a result, the derivation for the HT FE formulation is very simple. An iteration scheme is suggested to evaluate the nonlinear terms. The practical efficiency of the proposed formulation is assessed through numerical examples.

2. GOVERNING EQUATIONS AND THEIR TREFFTZ FUNCTIONS

2.1. *Basic equations*

Consider a Reissner–Mindlin plate of uniform thickness t, occupying a two-dimensional arbitrary shaped region Ω bounded by its boundary $\partial\Omega$. Throughout this paper repeated indices i, j and k imply the summation convention of Einstein. The indices i, j and k take values in the range $\{1, 2\}$. The nonlinear behaviour of the plate is governed by the following equations [16]:

$$L_{11}U_1 + L_{12}U_2 - P_1 = 0 \tag{1}$$

$$L_{21}U_1 + L_{22}U_2 - P_2 = 0 \tag{2}$$

$$L_{33}W + L_{34}\psi_1 + L_{35}\psi_2 - P_3 = 0 \tag{3}$$

$$L_{43}W + L_{44}\psi_1 + L_{45}\psi_2 = 0 \tag{4}$$

$$L_{53}W + L_{54}\psi_1 + L_{55}\psi_2 = 0 \tag{5}$$

with

$$L_{11}(\,) = (\,)_{,xx} + d_1(\,)_{,yy}, \quad L_{12}(\,) = L_{21}(\,) = d_2(\,)_{,xy},$$

$$L_{22}(\,) = (\,)_{,yy} + d_1(\,)_{,xx}, \quad L_{33} = C\nabla^2,$$

$$L_{34}(\,) = -L_{43}(\,) = -C(\,)_{,x},$$

$$L_{35}(\,) = -L_{53}(\,) = -C(\,)_{,y},$$

$$L_{44} = DL_{11} - C, \quad L_{45} = L_{54} = DL_{12},$$

$$L_{55} = DL_{22} - C, \quad U_1 = U_x,$$

$$U_2 = U_y, \quad \psi_1 = \psi_x,$$

$$\psi_2 = \psi_y, \quad d_1 = (1-\nu)/2, \quad d_2 = (1+\nu)/2$$

in which a comma followed by a subscript indicates partial differentiation with respect to that subscript, and P_1, P_2 and P_3 are the pseudo-distributed load components given by [17]

$$P_1 = -W_{,x}(W_{,xx} + d_1 W_{,yy}) - d_2 W_{,y} W_{,xy}$$

$$P_2 = -W_{,y}(W_{,yy} + d_1 W_{,xx}) - d_2 W_{,x} W_{,xy}$$

$$\begin{aligned}P_3 = J\{&(U_{1,x} + 0.5W_{,x}^2)(W_{,xx} + \nu W_{,yy}) \\ &+ (U_{2,y} + 0.5W_{,y}^2)(W_{,yy} + \nu W_{,xx})\} \\ &+ J(1-\nu)(U_{1,y} + U_{2,x} + W_{,x}W_{,y})W_{,xy} + q,\end{aligned}$$

where

$$J = \frac{Et}{1-\nu^2}.$$

The boundary conditions are

$$\begin{aligned}&U_n = U_i n_i = \bar{U}_n \quad (\text{on } C_{U_n}),\\ &U_s = U_i s_i = \bar{U}_s \quad (\text{on } C_{U_s}),\\ &\psi_n = \psi_i n_i = \bar{\psi}_n \quad (\text{on } C_{\psi_n}),\\ &\psi_s = \psi_i s_i = \bar{\psi}_s \quad (\text{on } C_{\psi_s}),\end{aligned} \tag{6}$$

$$W = \bar{W} \quad (\text{on } C_W), \tag{7}$$

$$\begin{aligned}&N_n = N_n^l + N_n^n = \bar{N}_n \quad (\text{on } C_{N_n}),\\ &N_{ns} = N_{ns}^l + N_{ns}^n = \bar{N}_{ns} \quad (\text{on } C_{N_{ns}}),\end{aligned} \tag{8}$$

$$\begin{aligned}&M_n = M_{ij}n_in_j = \bar{M}_n \quad (\text{on } C_{M_n}),\\ &M_{ns} = M_{ij}n_is_j = \bar{M}_{ns} \quad (\text{on } C_{M_{ns}}),\\ &R = R^l + R^n = \bar{R} \quad (\text{on } C_R),\\ &N_n^l = N_{ij}^l n_i n_j, \quad N_n^n = N_{ij}^n n_i n_j,\\ &N_{ns}^l = N_{ij}^l n_i s_j, \quad N_{ns}^n = N_{ij}^n n_i s_j,\\ &R^l = Q_i n_i,\\ &R^n = N_n W_{,n} + N_{ns} W_{,s}\\ &(\partial\Omega = C_{U_n} \cup C_{N_n} = C_{U_s} \cup C_{N_{ns}}\\ &\qquad = C_W \cup C_R = C_{\psi_n} \cup C_{M_n} = C_{\psi_s} \cup C_{M_{ns}}),\end{aligned} \tag{9}$$

where overbar means the prescribed value.

The relationship between stresses and displacements are defined by

$$N_{ij} = N_{ij}^n + N_{ij}^l$$

$$N_{ij}^l = Gt\left\{U_{i,j} + U_{j,i} + \frac{2\nu}{1-\nu}U_{k,k}\delta_{ij}\right\}, \tag{10}$$

$$N_{ij}^n = Gt\left\{W_{,i}W_{,j} + \frac{\nu}{1-\nu}W_{,k}W_{,k}\delta_{ij}\right\}, \tag{11}$$

$$M_{ij} = \frac{1-\nu}{2}D\left\{\psi_{i,j} + \psi_{j,i} + \frac{2\nu}{1-\nu}\psi_{k,k}\delta_{ij}\right\}, \tag{12}$$

$$Q_i = C(W_{,i} - \psi_i). \tag{13}$$

In order to make above nonlinear equations be linearized five incremental variables $\{\dot{U}_1\ \dot{U}_2\ \dot{W}\ \dot{\psi}_1\ \dot{\psi}_2\}$ are introduced. Omitting the infinitesimal resulting from the product of incremental variables, the field eqns (1)–(5) may be rewritten in incremental form:

$$L_{11}\dot{U}_1 + L_{12}\dot{U}_2 = \dot{P}_1 \tag{14}$$

$$L_{21}\dot{U}_1 + L_{22}\dot{U}_2 = \dot{P}_2 \tag{15}$$

$$L_{33}\dot{W} + L_{34}\dot{\psi}_1 + L_{35}\dot{\psi}_2 = \dot{P}_3 \tag{16}$$

$$L_{43}\dot{W} + L_{44}\dot{\psi}_1 + L_{45}\dot{\psi}_2 = 0 \tag{17}$$

$$L_{53}\dot{W} + L_{54}\dot{\psi}_1 + L_{55}\dot{\psi}_2 = 0, \tag{18}$$

where

$$\dot{P}_1 = -\dot{W}_{,x}(W_{,xx} + d_1 W_{,yy}) - W_{,x}(\dot{W}_{,xx} + d_1 \dot{W}_{,yy}) - d_2 \dot{W}_{,y} W_{,xy} - d_2 W_{,y} \dot{W}_{,xy} \quad (19)$$

$$\dot{P}_2 = -\dot{W}_{,y}(W_{,yy} + d_1 W_{,xx}) - W_{,y}(\dot{W}_{,yy} + d_1 \dot{W}_{,xx}) - d_2 \dot{W}_{,x} W_{,xy} - d_2 W_{,x} \dot{W}_{,xy} \quad (20)$$

$$\begin{aligned}\dot{P}_3 = J\{&(\dot{U}_{1,x} + W_{,x} \dot{W}_{,x})(W_{,xx} + \nu W_{,yy}) \\ &+ (U_{1,x} + 0.5W_{,x}^2)(\dot{W}_{,xx} + \nu \dot{W}_{,yy}) \\ &+ (\dot{U}_{2,y} + W_{,y} \dot{W}_{,y})(W_{,yy} + \nu W_{,xx}) \\ &+ (U_{2,y} + 0.5W_{,y}^2)(\dot{W}_{,yy} + \nu \dot{W}_{,xx})\} \\ &+ J(1-\nu)\{(\dot{U}_{1,y} + \dot{U}_{2,x} + \dot{W}_{,x} W_{,y} \\ &+ W_{,x} \dot{W}_{,y}) W_{,xy} \\ &+ (U_{1,y} + U_{2,x} + W_{,x} W_{,y}) \dot{W}_{,xy}\} + \dot{q}. \end{aligned} \quad (21)$$

The related boundary conditions become

$$\dot{U}_n = \dot{U}_i n_i = \Delta \bar{U}_n \quad (\text{on } C_{U_n}),$$

$$\dot{U}_s = \dot{U}_i s_i = \Delta \bar{U}_s \quad (\text{on } C_{U_s}),$$

$$\dot{\psi}_n = \dot{\psi}_i n_i = \Delta \bar{\psi}_n \quad (\text{on } C_{\psi_n}),$$

$$\dot{\psi}_s = \dot{\psi}_i s_i = \Delta \bar{\psi}_s \quad (\text{on } C_{\psi_s}), \quad (22)$$

$$\dot{W} = \Delta \bar{W} \quad (\text{on } C_W), \quad (23)$$

$$\dot{N}_n = \dot{N}_{ij}^l n_i n_j = (\Delta \bar{N}_n - \dot{N}_{ij}^n n_i n_j) = \bar{N}_n^* \quad (\text{on } C_{N_n}),$$

$$\dot{N}_{ns} = \dot{N}_{ij}^l n_i s_j = (\Delta \bar{N}_{ns} - \dot{N}_{ij}^n n_i s_j) = \bar{N}_{ns}^* (\text{on } C_{N_{ns}}), \quad (24)$$

$$\dot{R} = \dot{Q}_i n_i = (\Delta \bar{R} - \dot{R}^n) = \bar{R}^* \quad (\text{on } C_R),$$

$$\dot{M}_n = \dot{M}_{ij} n_i n_j = \Delta \bar{M}_n \quad (\text{on } C_{M_n})$$

$$\dot{M}_{ns} = \dot{M}_{ji} n_i s_j = \Delta \bar{M}_{ns} \quad (\text{on } C_{M_{ns}}). \quad (25)$$

It should be pointed out that $\dot{N}_{ij}^n \ll \dot{N}_{ij}^l$, $\dot{R}^n \ll \dot{R}^l$ in practical problems. So we may move these nonlinear terms to the right-hand side of the above boundary equations. In this way the coupling between in-plane and out-of-plane displacements appearing both in field and boundary equations is separated. As a result, the ensuing derivation becomes very simple. Of course an iterative approach is required to evaluate the nonlinear terms P_i, $\bar{N}_n^*$, $\bar{N}_{ns}^*$ and $\bar{R}^*$.

2.2. *Trefftz functions*

The Trefftz functions play an important role in the derivation of the HT finite element formulation. The complete system corresponding to eqns (14) and (15) can be generated in a systematic way from Muskhelishvili's complex variable formulation [3, 18]. For the sake of convenience, we list the results obtained by Jirousek and Venkatesh [3]

$$\mathbf{U}_j^* = \begin{Bmatrix} \operatorname{Re} Z_{1k} \\ \operatorname{Im} Z_{1k} \end{Bmatrix} \quad \text{with}$$

$$Z_{1k} = (3-\nu) i z^k + (1+\nu) k i z \bar{z}^{k-1} \quad (26)$$

$$\mathbf{U}_{j+1}^* = \begin{Bmatrix} \operatorname{Rm} Z_{2k} \\ \operatorname{Im} Z_{2k} \end{Bmatrix} \quad \text{with}$$

$$Z_{2k} = (3-\nu) z^k - (1+\nu) k z \bar{z}^{k-1} \quad (27)$$

$$\mathbf{U}_{j+2}^* = \begin{Bmatrix} \operatorname{Re} Z_{3k} \\ \operatorname{Im} Z_{3k} \end{Bmatrix} \quad \text{with } Z_{3k} = (1+\nu) i \bar{z}^k \quad (28)$$

$$\mathbf{U}_{j+3}^* = \begin{Bmatrix} \operatorname{Re} Z_{4k} \\ \operatorname{Im} Z_{4k} \end{Bmatrix} \quad \text{with } Z_{4k} = -(1+\nu) \bar{z}^k, \quad (29)$$

where $z = x + iy$, $\bar{z} = x - iy$ and $i = \sqrt{-1}$, $\operatorname{Re}(Z)$ and $\operatorname{Im}(Z)$ stand for taking the real part and the imaginary part of Z, respectively, and where $\mathbf{U}_j^*$ satisfies

$$\mathbf{L}_{in} \mathbf{U}_j^* = 0 \quad (30)$$

with

$$\mathbf{L}_{in} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}, \quad \mathbf{U}_j^* = \begin{Bmatrix} U_1^* \\ U_2^* \end{Bmatrix}_j. \quad (31)$$

The rest is to derive the Trefftz functions corresponding to eqns (16)–(18). As was done in Ref. [16], these three equations can be transformed into a convenient form

$$\nabla^2 f - \lambda^2 f = 0 \quad (32)$$

$$D \nabla^4 g + \dot{P}_3 = 0. \quad (33)$$

The corresponding displacements and rotations are obtained from the following relations:

$$\dot{W} = g - D \nabla^2 g / C \quad (34)$$

$$\dot{\psi}_1 = g_{,1} + f_{,2} \quad (35)$$

$$\dot{\psi}_2 = g_{,2} - f_{,1}. \quad (36)$$

The eqns (32) and (33) are, respectively, the modified Bessel equation and the biharmonic equation. Their Trefftz solutions can be generated from the following sequence:

$$f_j = I_k(\lambda r)\cos k\theta,\quad f_{j+1} = I_k(\lambda r)\ \sin k\theta \qquad (k = 0, 1, \dots) \quad (37)$$

and

$$g_i = r^k \cos(k-2)\theta,$$

$$g_{i+1} = r^k \sin(k-2)\theta \quad (k = 2, 3, \dots)$$

$$g_{i+2} = r^k \cos k\theta,$$

$$g_{i+3} = r^k \sin k\theta \quad (k = 1, 2, \dots), \qquad (38)$$

where $I_k(\,)$ is the modified Bessel function of first kind with order j.

2.3. *Assumed fields*

The element formulation follows the standard HT element methodology [3]. Unlike the conventional FE models, however, the HT formulation is based on a hybrid method that includes the use of an auxiliary interelement displacement to link up the internal displacement fields of the elements. Such fields are chosen so as to satisfy *a priori* the governing differential equations. With this method, the domain Ω is subdivided into elements, and over each element "e" the assumed intraelement fields are

$$\mathbf{u} = \begin{Bmatrix} \dot{U}_1 \\ \dot{U}_2 \end{Bmatrix} = \begin{Bmatrix} \Delta\mathring{U}_1 \\ \Delta\mathring{U}_2 \end{Bmatrix} + \begin{Bmatrix} \mathbf{N}_1 \\ \mathbf{N}_2 \end{Bmatrix} \mathbf{c}_{\text{in}} = \mathring{\mathbf{u}} + \mathbf{N}_{\text{in}}\mathbf{c}_{\text{in}}$$

$$\mathbf{w} = \begin{Bmatrix} \dot{W} \\ \dot{\psi}_1 \\ \dot{\psi}_2 \end{Bmatrix} = \begin{Bmatrix} \Delta\mathring{W} \\ \Delta\mathring{\psi}_1 \\ \Delta\mathring{\psi}_2 \end{Bmatrix} + \begin{Bmatrix} \mathbf{N}_3 \\ \mathbf{N}_4 \\ \mathbf{N}_5 \end{Bmatrix} \mathbf{c}_{\text{out}} = \mathring{\mathbf{w}} + \mathbf{N}_{\text{out}}\mathbf{c}_{\text{out}}, \qquad (39)$$

where $\mathbf{c}_{\text{in}}$ and $\mathbf{c}_{\text{out}}$ are two undetermined coefficient vectors and $\mathring{\mathbf{u}}$, $\mathring{\mathbf{w}}$, $\mathbf{N}_{\text{in}}$ and $\mathbf{N}_{\text{out}}$ are known functions which satisfy

$$\mathbf{L}_{\text{in}}\mathring{\mathbf{u}} = \begin{Bmatrix} \dot{P}_1 \\ \dot{P}_2 \end{Bmatrix}, \quad \mathbf{L}_{\text{in}} \begin{Bmatrix} \mathbf{N}_1 \\ \mathbf{N}_2 \end{Bmatrix} = 0 \quad (\text{on } \Omega_e)$$

$$\mathbf{L}_{\text{out}}\mathring{\mathbf{w}} = \begin{Bmatrix} \dot{P}_3 \\ 0 \\ 0 \end{Bmatrix}, \quad \mathbf{L}_{\text{out}} \begin{Bmatrix} \mathbf{N}_3 \\ \mathbf{N}_4 \\ \mathbf{N}_5 \end{Bmatrix} = 0 \quad (\text{on } \Omega_e)$$

$$\mathbf{c} = \begin{Bmatrix} \mathbf{c}_{\text{in}} \\ \mathbf{c}_{\text{out}} \end{Bmatrix},$$

$$\mathbf{U} = \{\dot{U}_1 \quad \dot{U}_2 \quad \dot{W} \quad \dot{\psi}_1 \quad \dot{\psi}_2\}^{\text{T}}$$

$$\mathbf{L}_{\text{out}} = \begin{bmatrix} L_{33} & L_{34} & L_{35} \\ L_{43} & L_{44} & L_{45} \\ L_{53} & L_{54} & L_{55} \end{bmatrix}$$

and where $\mathbf{N}_{\text{in}}$ and $\mathbf{N}_{\text{out}}$ are formed by a suitably truncated complete system of eqns (26)–(29), (37) and (38).

All that is left is to determine the parameters $\mathbf{c}$ so as to enforce on $\mathbf{U}$ the interelement conformity ($\mathbf{U}^{\text{e}} = \mathbf{U}^{\text{f}}$ on $\partial\Omega_{\text{e}} \cap \partial\Omega_{\text{f}}$) and the related boundary conditions, where "e" and "f" stand for any two neighbouring elements.

One of the possible ways [3, 7] is to link the Trefftz type solutions (39) through an interface displacement frame surrounding the element, which is approximated in terms of the same DOF, $\mathbf{d}$, as used in the conventional elements

$$\tilde{\mathbf{U}} = \tilde{\mathbf{N}}\mathbf{d} \qquad (40)$$

where

$$\tilde{\mathbf{U}} = \begin{Bmatrix} \tilde{\mathbf{u}} \\ \tilde{\mathbf{w}} \end{Bmatrix} \qquad (41)$$

$$\tilde{\mathbf{u}} = \{\tilde{U}_1 \quad \tilde{U}_2\}^{\text{T}} = \begin{bmatrix} \tilde{\mathbf{N}}_1 \\ \tilde{\mathbf{N}}_2 \end{bmatrix} \mathbf{d}_{\text{in}} \qquad (42)$$

$$\tilde{\mathbf{w}} = \{\tilde{W} \quad \tilde{\psi}_1 \quad \tilde{\psi}_2\}^{\text{T}} = \begin{bmatrix} \tilde{\mathbf{N}}_3 \\ \tilde{\mathbf{N}}_4 \\ \tilde{\mathbf{N}}_5 \end{bmatrix} \mathbf{d}_{\text{out}} \qquad (43)$$

$$\mathbf{d} = \begin{Bmatrix} \mathbf{d}_{\text{in}} \\ \mathbf{d}_{\text{out}} \end{Bmatrix} \qquad (44)$$

and where $\mathbf{d}_{\text{in}}$ and $\mathbf{d}_{\text{out}}$ stand for vectors of in-plane and out-of-plane displacement nodal parameters, and $\tilde{\mathbf{N}}_i (i = 1, 2, 3, 4, 5)$ are the conventional FE interpolation functions. For example, along the side A-B of a particular element (see Fig. 1), a simple interpolation of the frame displacements may be given in the form

$$\tilde{\mathbf{u}} = \begin{bmatrix} N_{\text{AB1}} \\ N_{\text{AB2}} \end{bmatrix} (\mathbf{d}_{\text{in}})_{\text{AB}} \qquad (45)$$

$$\tilde{\mathbf{w}} = \begin{bmatrix} N_{\text{AB3}} \\ N_{\text{AB4}} \\ N_{\text{AB5}} \end{bmatrix} (\mathbf{d}_{\text{out}})_{\text{AB}}, \qquad (46)$$

where

$$\begin{bmatrix} N_{\text{AB1}} \\ \\ N_{\text{AB2}} \end{bmatrix} = \begin{bmatrix} \frac{1-\rho}{2} & 0 & \frac{1+\rho}{2} & 0 \\ 0 & \frac{1-\rho}{2} & 0 & \frac{1+\rho}{2} \end{bmatrix} \qquad (47)$$

$$\begin{bmatrix} N_{\text{AB3}} \\ N_{\text{AB4}} \\ N_{\text{AB5}} \end{bmatrix} = \begin{bmatrix} \tilde{N}_1 & 0 & 0 & \tilde{N}_2 & 0 & 0 & \tilde{N}_3 \\ 0 & \tilde{N}_4 & 0 & 0 & \tilde{N}_5 & 0 & 0 \\ 0 & 0 & \tilde{N}_4 & 0 & 0 & \tilde{N}_5 & 0 \end{bmatrix} \qquad (48)$$

$$(\mathbf{d}_{\text{in}})_{\text{AB}} = \{\dot{U}_{1\text{A}} \quad \dot{U}_{2\text{A}} \quad \dot{U}_{1\text{B}} \quad \dot{U}_{2\text{B}}\}^{\text{T}} \qquad (49)$$

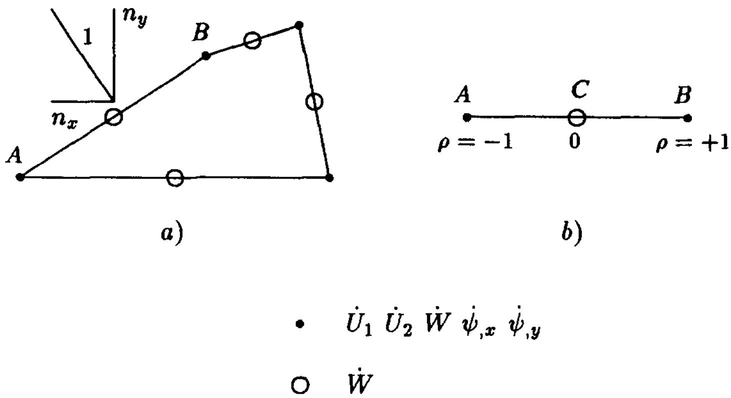

Fig. 1. A quadrilateral HT-element.

$$(\mathbf{d}_{\text{out}})_{\text{AB}} = \{\dot W_{\text{A}}\ \ \dot\psi_{x\text{A}}\ \ \dot\psi_{y\text{A}}\ \ \dot W_{\text{B}}\ \ \dot\psi_{x\text{B}}\ \ \dot\psi_{y\text{B}}\ \ \dot W_{\text{C}}\}^{\text{T}} \tag{50}$$

in which ρ is shown in Fig. 1, and

$$\tilde N_1 = \rho(\rho-1)/2, \quad \tilde N_2 = \rho(\rho+1)/2,$$

$$\tilde N_3 = 1-\rho^2, \quad \tilde N_4 = (1-\rho)/2,$$

$$\tilde N_5 = (1+\rho)/2.$$

In this paper, a quadrilateral element is chosen to be an element model with 5 DOF ($\dot U_1\ \dot U_2\ \dot W\ \dot\psi_{,x}\ \dot\psi_{,y}$) at each corner node and with 1 DOF, $\dot W$, at each mid-side node (see Fig. 1). Therefore each element has 24 DOF.

The generalized boundary forces and displacements can be easily derived from eqns (22)–(25), (39) and (40), and denote

$$\mathbf{v} = \begin{Bmatrix} \dot U_n \\ \dot U_s \end{Bmatrix} = \begin{bmatrix} n_1 & n_2 \\ s_1 & s_2 \end{bmatrix} \begin{Bmatrix} \dot U_1 \\ \dot U_2 \end{Bmatrix} = \mathring{\mathbf{v}} + \mathbf{Q}_1\mathbf{c}_{\text{in}} \tag{51}$$

$$\mathbf{F} = \begin{Bmatrix} \dot N_n \\ \dot N_{ns} \end{Bmatrix} = \begin{bmatrix} n_1^2 & n_2^2 & 2n_1n_2 \\ n_1s_1 & n_2s_2 & n_1s_2+n_2s_1 \end{bmatrix} \begin{Bmatrix} \dot N_x \\ \dot N_y \\ \dot N_{xy} \end{Bmatrix}$$

$$= \mathring{\mathbf{N}} + \mathbf{Q}_2\mathbf{c}_{\text{in}} \tag{52}$$

$$\mathbf{w}_{\text{b}} = \begin{Bmatrix} \dot W \\ \dot\psi_n \\ \dot\psi_s \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & n_1 & n_1 \\ 0 & s_1 & s_2 \end{bmatrix} \begin{Bmatrix} \dot W \\ \dot\psi_1 \\ \dot\psi_2 \end{Bmatrix} = \mathring{\mathbf{w}}_{\text{b}} + \mathbf{Q}_3\mathbf{c}_{\text{out}} \tag{53}$$

$$\mathbf{M} = \begin{Bmatrix} \dot R \\ \dot M_n \\ \dot M_{ns} \end{Bmatrix} = \mathbf{AG} = \mathring{\mathbf{M}} + \mathbf{Q}_4\mathbf{c}_{\text{out}} \tag{54}$$

$$\tilde{\mathbf{v}} = \begin{Bmatrix} \tilde U_n \\ \tilde U_s \end{Bmatrix} = \begin{bmatrix} n_1 & n_2 \\ s_1 & s_2 \end{bmatrix} \begin{Bmatrix} \tilde U_1 \\ \tilde U_2 \end{Bmatrix} = \mathbf{Q}_5\mathbf{d}_{\text{in}} \tag{55}$$

$$\tilde{\mathbf{w}}_{\text{b}} = \begin{Bmatrix} \tilde W \\ \tilde\psi_n \\ \tilde\psi_s \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & n_1 & n_2 \\ 0 & s_1 & s_2 \end{bmatrix} \begin{Bmatrix} \tilde W \\ \tilde\psi_1 \\ \tilde\psi_2 \end{Bmatrix} = \mathbf{Q}_6\mathbf{d}_{\text{out}}, \tag{56}$$

where

$$\mathbf{Q}_i = \begin{bmatrix} \mathbf{Q}_{i1} \\ \mathbf{Q}_{i2} \end{bmatrix}, \quad (i = 1, 2, 5)$$

$$\mathbf{Q}_i = \begin{bmatrix} \mathbf{Q}_{i1} \\ \mathbf{Q}_{i2} \\ \mathbf{Q}_{i3} \end{bmatrix}, \quad (i = 3, 4, 6)$$

$$\mathring{\mathbf{v}} = \{\Delta\mathring{U}_n\ \ \Delta\mathring{U}_s\}^{\text{T}},$$

$$\mathring{\mathbf{N}} = \{\Delta\mathring{N}_N\ \ \Delta\mathring{N}_{ns}\}^{\text{T}},$$

$$\mathring{\mathbf{w}}_{\text{b}} = \{\Delta\mathring{W}\ \ \Delta\mathring{\psi}_n\ \ \Delta\mathring{\psi}_s\}^{\text{T}},$$

$$\mathring{\mathbf{M}} = \{\Delta\mathring{R}\ \ \Delta\mathring{M}_n\ \ \Delta\mathring{M}_{ns}\}^{\text{T}},$$

$$\mathbf{A} = \begin{bmatrix} n_x & n_y & 0 & 0 & 0 \\ 0 & 0 & n_1^2 & n_2^2 & 2n_1n_2 \\ 0 & 0 & n_1s_1 & n_2s_2 & n_1s_2+n_2s_1 \end{bmatrix},$$

$$\mathbf{G} = \{\dot Q_x\ \ \dot Q_y\ \ \dot M_x\ \ \dot M_y\ \ \dot M_{xy}\}^{\text{T}}.$$

2.4. *Particular solution*

The particular solutions of $\mathring{\mathbf{u}}$ and $\mathring{\mathbf{w}}$ can be obtained by means of their source functions. The source functions corresponding to eqns (14)–(18) has been given in Refs [16, 19]:

$$U_{ij}^*(p,q) = \frac{1+\nu}{4\pi E}[-(3-\nu)\ln r_{pq}\delta_{ij} + (1+\nu)r_{pq,i}r_{pq,j}]$$

$$U_{33}^*(p,q) = -\frac{1}{2\pi D\lambda^2}\left[\frac{2}{1-\nu}\ln(\lambda r_{pq}) - \frac{\lambda^2 r_{pq}^2}{4}(\ln(\lambda r_{pq})-1)\right]$$

$$U_{43}^*(p,q) = -\frac{r_{pq}r_{pq,1}}{4\pi D}[\ln(\lambda r_{pq}) - 1/2]$$

$$U_{53}^*(p,q) = -\frac{r_{pq}r_{pq,2}}{4\pi D}[\ln(\lambda r_{pq}) - 1/2]$$

$$r_{pq} = \sqrt{(x_q-x_p)^2+(y_q-y_p)^2},$$

where $U^*_{mn}(p,q)$ means the in-plane displacements (for $m=1,2$) or deflection (for $m=3$) or rotations (for $m=4,5$) at field point q of an infinite plate when a unit point force ($n=1,2,3$) is applied at the source point p. Using these source functions, the particular solutions can be expressed by

$$\mathring{\mathbf{u}}=\begin{Bmatrix}\Delta\mathring{U}_1\\ \Delta\mathring{U}_2\end{Bmatrix}=\iint_\Omega \dot{P}_j\begin{Bmatrix}U^*_{1j}\\ U^*_{2j}\end{Bmatrix}\mathrm{d}\Omega \tag{57}$$

$$\mathring{\mathbf{w}}=\begin{Bmatrix}\Delta\mathring{W}\\ \Delta\mathring{\psi}_1\\ \Delta\mathring{\psi}_2\end{Bmatrix}=\iint_\Omega \dot{P}_3\begin{Bmatrix}U^*_{33}\\ U^*_{43}\\ U^*_{53}\end{Bmatrix}\mathrm{d}\Omega. \tag{58}$$

The area integration in eqns (57) and (58) will be performed by numerical quadrature using the Gauss–Legendre rule.

2.5. *Modified principle*

The HT FE formulation for nonlinear analysis of thick plates can be established by means of a modified principle [7]. The related functional used for deriving the HT FE formulation can be constructed in the form [7]

$$\Gamma^m_{\text{in}}=\sum_{\text{e}}\left\{\Gamma^{\text{e}}_{\text{in}}+\int_{\partial\Omega^2_{\text{e}}}(\bar{N}^*_n-\dot{N}_n)\dot{U}_n\,\mathrm{d}c+\int_{\partial\Omega^4_{\text{e}}}(\bar{N}^*_{ns}-\dot{N}_{ns})\dot{U}_s\,\mathrm{d}c-\int_{\partial\Omega^{11}_{\text{e}}}\mathbf{F}^{\mathrm{T}}\tilde{\mathbf{v}}\,\mathrm{d}c\right\} \tag{59}$$

$$\Gamma^m_{\text{out}}=\sum_{\text{e}}\left\{\Gamma^{\text{e}}_{\text{out}}+\int_{\partial\Omega^6_{\text{e}}}(\bar{R}^*-\dot{R})\dot{W}_n\,\mathrm{d}c+\int_{\partial\Omega^8_{\text{e}}}\times(\Delta\bar{M}_n-\dot{M}_n)\dot{\psi}_n\,\mathrm{d}c+\int_{\partial\Omega^{10}_{\text{e}}}(\Delta\bar{M}_{ns}-\dot{M}_{ns})\dot{\psi}_s\,\mathrm{d}c-\int_{\partial\Omega^{11}_{\text{e}}}\mathbf{M}^{\mathrm{T}}\tilde{\mathbf{w}}_{\text{b}}\,\mathrm{d}c\right\}, \tag{60}$$

where

$$\Delta X=\dot{X}$$

$$\Gamma^{\text{e}}_{\text{in}}=\iint_{\Omega_{\text{e}}}\dot{V}_{\text{in}}\,\mathrm{d}\Omega-\int_{\partial\Omega^1_{\text{e}}}\dot{N}_n\Delta\bar{U}_n\,\mathrm{d}c-\int_{\partial\Omega^3_{\text{e}}}\dot{N}_{ns}\Delta\bar{U}_s\,\mathrm{d}c$$

$$\Gamma^{\text{e}}_{\text{out}}=\iint_{\Omega_{\text{e}}}\dot{V}_{\text{out}}\,\mathrm{d}\Omega-\int_{\partial\Omega^5_{\text{e}}}\dot{R}\Delta\bar{W}\,\mathrm{d}c-\int_{\partial\Omega^7_{\text{e}}}\dot{M}_n\Delta\bar{\psi}_n\,\mathrm{d}c-\int_{\partial\Omega^9_{\text{e}}}\dot{M}_{ns}\Delta\bar{\psi}_s\,\mathrm{d}c$$

$$\dot{V}_{\text{in}}=\frac{1-2\nu}{6Eh}\dot{N}_{kl}\dot{N}_{kl}+\frac{1+\nu}{2Eh}\dot{N}^*_{ij}\dot{N}^*_{ij}$$

$$\dot{V}_{\text{out}}=\frac{1}{2D(1-\nu^2)}[(\dot{M}_x+\dot{M}_y)^2+2(1+\nu)\times(\dot{M}^2_{xy}-\dot{M}_x\dot{M}_y)]+\frac{1}{2C}(\dot{Q}^2_x+\dot{Q}^2_y)$$

$$\dot{N}^*_{ij}=\dot{N}_{ij}-\tfrac{1}{3}\dot{N}_{kk\delta_{ij}}$$

and where eqns (14)–(18) are assumed to be satisfied, *a priori*. The boundary $\partial\Omega_{\text{e}}$ of a particular element consists of the following parts:

$$\begin{aligned}\partial\Omega_{\text{e}}&=\partial\Omega^1_{\text{e}}+\partial\Omega^2_{\text{e}}+\partial\Omega^{11}_{\text{e}}=\partial\Omega^3_{\text{e}}+\partial\Omega^4_{\text{e}}+\partial\Omega^{11}_{\text{e}}\\&=\partial\Omega^5_{\text{e}}+\partial\Omega^6_{\text{e}}+\partial\Omega^{11}_{\text{e}}=\partial\Omega^7_{\text{e}}+\partial\Omega^8_{\text{e}}+\partial\Omega^{11}_{\text{e}}\\&=\partial\Omega^9_{\text{e}}+\partial\Omega^{10}_{\text{e}}+\partial\Omega^{11}_{\text{e}},\end{aligned}$$

where

$$\begin{aligned}&\partial\Omega^1_{\text{e}}\subset C_{U_n}\cap\partial\Omega_{\text{e}},\quad \partial\Omega^2_{\text{e}}\subset C_{N_n}\cap\partial\Omega_{\text{e}},\\&\partial\Omega^3_{\text{e}}\subset C_{U_s}\cap\partial\Omega_{\text{e}},\quad \partial\Omega^4_{\text{e}}\subset C_{N_{ns}}\cap\partial\Omega_{\text{e}},\\&\partial\Omega^5_{\text{e}}\subset C_W\cap\partial\Omega_{\text{e}},\quad \partial\Omega^6_{\text{e}}\subset C_R\cap\partial\Omega_{\text{e}},\\&\partial\Omega^7_{\text{e}}\subset C_{\psi_n}\cap\partial\Omega_{\text{e}},\quad \partial\Omega^8_{\text{e}}\subset C_{M_n}\cap\partial\Omega_{\text{e}},\\&\partial\Omega^9_{\text{e}}\subset C_{\psi_s}\cap\partial\Omega_{\text{e}},\quad \partial\Omega^{10}_{\text{e}}\subset C_{M_{ns}}\cap\partial\Omega_{\text{e}}\end{aligned}$$

and $\partial\Omega^{11}_{\text{e}}$ is the interelement boundary of the element. Consequently, we will discuss some properties and their proof on these two functionals. In doing so, we present

(i) the modified complementary principle

$$\delta\Gamma^m_{\text{in}}=0\ \Rightarrow\ (22),\ (24)\ \text{and}$$

$$\dot{U}^{\text{e}}_n=\dot{U}^{\text{f}}_n,\quad \dot{U}^{\text{e}}_s=\dot{U}^{\text{f}}_s,\quad (\text{on } \partial\Omega_{\text{e}}\cap\partial\Omega_{\text{f}}) \tag{61}$$

$$\delta\Gamma^m_{\text{out}}=0\ \Rightarrow\ (23),\ (25)\ \text{and}$$

$$\dot{W}^{\text{e}}=\dot{W}^{\text{f}},\quad \dot{\psi}^{\text{f}}_n,\quad \dot{\psi}^{\text{e}}_s=\dot{\psi}^{\text{f}}_s\quad (\text{on } \partial\Omega_{\text{e}}\cap\partial\Omega_{\text{f}}). \tag{62}$$

(ii) Theorems on the existence of extremum are (a) if the expression

$$\iint_\Omega\delta^2\dot{V}_{\text{in}}\,\mathrm{d}\Omega-\int_{C_{N_n}}\delta\dot{N}_n\delta\dot{U}_n\,\mathrm{d}c-\int_{C_{N_{ns}}}\delta\dot{N}_{ns}\delta\dot{U}_s\,\mathrm{d}c-\sum_{\text{e}}\int_{\Omega^{11}_{\text{e}}}\delta\mathbf{F}^{\mathrm{T}}\delta\tilde{v}\,\mathrm{d}c \tag{63}$$

is uniformly positive (or negative) at the neighbourhood of $\mathbf{u}_0$ ($\mathbf{u}_0=\{\dot{U}_{10},\dot{U}_{20}\}$), where $\mathbf{u}_0$ is such a value that $\Gamma^m_{\text{in}}(\mathbf{u}_0)=(\Gamma^m_{\text{in}})_0$, and where $(\Gamma^m_{\text{in}})_0$ stands for the stationary value of Γ^m_{in}, we have

$$\Gamma^m_{\text{in}} \geqslant (\Gamma^m_{\text{in}})_0 \quad (\text{or } \Gamma^m_{\text{in}} \leqslant (\Gamma^m_{\text{in}})_0). \tag{64}$$

(b) if the expression

$$\iint_\Omega \delta^2 \dot{V}_{\text{out}}\,\mathrm{d}\Omega - \int_{C_R} \delta\dot{R}\delta\dot{W}\,\mathrm{d}c - \int_{C_{M_n}} \delta\dot{M}_n\delta\dot{\psi}_n\,\mathrm{d}c - \int_{C_{M_{ns}}} \delta\dot{M}_{ns}\delta\dot{\psi}_s\,\mathrm{d}c - \sum_{\mathrm{e}}\int_{\Omega_{\mathrm{e}}^{11}} \delta\mathbf{M}^{\mathrm{T}}\delta\tilde{\mathbf{w}}_{\mathrm{b}}\,\mathrm{d}c \tag{65}$$

is uniformly positive (or negative) at the neighbourhood of $\mathbf{w}_0$, where $\mathbf{w}_0$ is such a value that $\Gamma^m_{\text{out}}(\mathbf{w}_0)=(\Gamma^m_{\text{out}})_0$, and where $(\Gamma^m_{\text{out}})_0$ stands for the stationary value of Γ^m_{out}, we have

$$\Gamma^m_{\text{out}} \geqslant (\Gamma^m_{\text{out}})_0 \quad (\text{or } \Gamma^m_{\text{out}} \leqslant (\Gamma^m_{\text{out}})_0), \tag{66}$$

where "e" and "f" stand for any two neighbouring elements and where $\tilde{\mathbf{U}}^{\mathrm{e}}=\tilde{\mathbf{U}}^{\mathrm{f}}$ is identical on $\partial\Omega_{\mathrm{e}}\cap\partial\Omega_{\mathrm{f}}$ due to the assumed frame field $\tilde{\mathbf{U}}$ [see eqn (40)].

Proof: from the first, we derive the stationary conditions for functional Γ^m_{in}. To this end, taking variation of Γ^m_{in} and noting that eqns (14) and (15) hold *a priori* by the previous assumption, one obtains

$$\begin{aligned}\delta\Gamma^m_{\text{in}} \overset{(14)(15)}{=\!=\!=} &\int_{C_{U_n}} (\dot{U}_n-\Delta\bar{U}_n)\delta\dot{N}_n\,\mathrm{d}c \\ &+\int_{C_{U_s}} (\dot{U}_s-\Delta\bar{U}_s)\delta\dot{N}_{ns}\,\mathrm{d}c \\ &-\int_{C_{N_n}} (\dot{N}_n-\bar{N}^*_n)\delta\dot{U}_n\,\mathrm{d}c \\ &-\int_{C_{N_{ns}}} (\dot{N}_{ns}-\bar{N}^*_{ns})\delta\dot{U}_s\,\mathrm{d}c \\ &+\sum_{\mathrm{e}}\int_{\partial\Omega_{\mathrm{e}}^9} [(\dot{U}_n-\tilde{U}_n)\delta\dot{N}_n\,\mathrm{d}c \\ &+(\dot{U}_s-\tilde{U}_s)\delta\dot{N}_{ns}]\,\mathrm{d}c,\end{aligned} \tag{67}$$

where the constrained equality $\overset{(14)(15)}{=\!=\!=}$ stands for the equations (14) and (15) being satisfied, *a priori*. Therefore, The Euler equations for eqn (67) are eqns (22), (24) and

$$\dot{U}^{\mathrm{e}}_n=\dot{U}^{\mathrm{f}}_n,\quad \dot{U}^{\mathrm{e}}_s=\dot{U}^{\mathrm{f}}_s,\quad (\text{on } \partial\Omega_{\mathrm{e}}\cap\partial\Omega_{\mathrm{f}}).$$

The principle eqn (61) has thus been proved.

As for the theorem on the existence of extremum, we may prove it by means of the so-called second variational approach [20, 21]. In doing so, taking variation of $\delta\Gamma^m_{\text{in}}$ and using the constrained conditions (14) and (15), we see

$$\begin{aligned}\delta^2\Gamma^m_{\text{in}} =& \iint_\Omega \delta^2\dot{V}_{\text{in}}\,\mathrm{d}\Omega - \int_{C_{N_n}} \delta\dot{N}_n\delta\dot{U}_n\,\mathrm{d}c \\ &-\int_{C_{N_s}} \delta\dot{N}_{ns}\delta\dot{U}_s\,\mathrm{d}c - \sum_{\mathrm{e}}\int_{\partial\Omega_{\mathrm{e}}^{11}} \delta\mathbf{F}^{\mathrm{T}}\delta\tilde{\mathbf{v}}\,\mathrm{d}c \\ =& \text{ expression (63).}\end{aligned} \tag{68}$$

So the theorem has been proved from the sufficient condition on the existence of a local extreme of a functional [20]. With the same way we can easily prove the above properties for Γ^m_{out}, and we omit those details here.

2.6. *Element matrix*

The element matrix may be established by setting $\delta(\Gamma^m_{\text{in}})_{\mathrm{e}}=0$ and $\delta(\Gamma^m_{\text{out}})_{\mathrm{e}}=0$. To simplify the derivation, we first transform all domain integrals in eqns (59) and (60), except loading terms, into boundary ones. In fact by reason of solution properties of the intraelement trial functions, the functions $(\Gamma^m_{\text{in}})_{\mathrm{e}}$ and $(\Gamma^m_{\text{out}})_{\mathrm{e}}$ can be simplified to

$$\begin{aligned}(\Gamma^m_{\text{in}})_{\mathrm{e}} =& \frac{1}{2}\iint_{\Omega_{\mathrm{e}}} \dot{P}_i\dot{U}_i\,\mathrm{d}\Omega - \int_{\partial\Omega_{\mathrm{e}}^1} \dot{N}_n\Delta\bar{U}_n\,\mathrm{d}c \\ &-\int_{\partial\Omega_{\mathrm{e}}^3} \dot{N}_{ns}\Delta\bar{U}_s\,\mathrm{d}c \\ &-\int_{\partial\Omega_{\mathrm{e}}^2} (\dot{N}_n-\bar{N}^*_n)\dot{U}_n\,\mathrm{d}c \\ &-\int_{\partial\Omega_{\mathrm{e}}^4} (\dot{N}_{ns}-\bar{N}^*_{ns})\dot{U}_s\,\mathrm{d}c \\ &+\frac{1}{2}\int_{\partial\Omega_{\mathrm{e}}} \mathbf{F}^{\mathrm{T}}\mathbf{v}\,\mathrm{d}c - \int_{\partial\Omega_{\mathrm{e}}^{11}} \mathbf{F}^{\mathrm{T}}\tilde{\mathbf{v}}\,\mathrm{d}c\end{aligned} \tag{69}$$

$$\begin{aligned}(\Gamma^m_{\text{out}})_{\mathrm{e}} =& \frac{1}{2}\iint_{\Omega_{\mathrm{e}}} \dot{P}_5\dot{W}\,\mathrm{d}\Omega - \int_{\partial\Omega_{\mathrm{e}}^5} \dot{R}\,\Delta\bar{W}\,\mathrm{d}c \\ &-\int_{\partial\Omega_{\mathrm{e}}^7} \dot{M}_n\Delta\bar{\psi}_n\,\mathrm{d}c - \int_{\partial\Omega_{\mathrm{e}}^9} \dot{M}_{ns}\Delta\bar{\psi}_s\,\mathrm{d}c \\ &-\int_{\partial\Omega_{\mathrm{e}}^6} (\dot{R}-\bar{R}^*)\dot{W}\,\mathrm{d}c \\ &-\int_{\partial\Omega_{\mathrm{e}}^8} (\dot{M}_n-\Delta\bar{M}_n)\dot{\psi}_n\,\mathrm{d}c \\ &-\int_{\partial\Psi_{\mathrm{e}}^{10}} (\dot{M}_{ns}-\Delta\bar{M}_{ns})\dot{\psi}_s\,\mathrm{d}c \\ &+\frac{1}{2}\int_{\partial\Omega_{\mathrm{e}}} \mathbf{M}^{\mathrm{T}}\mathbf{w}_b\,\mathrm{d}c - \int_{\partial\Omega_{\mathrm{e}}^{11}} \mathbf{M}^{\mathrm{T}}\tilde{\mathbf{w}}_b\,\mathrm{d}c.\end{aligned} \tag{70}$$

The substitution of eqns (39), (40) and (51)–(56) into (69) and (70) yield

$$(\Gamma^m_{\rm in})_e = -\mathbf{c}^{\rm T}_{\rm in}\mathbf{H}_{\rm in}\mathbf{c}_{\rm in}/2 + \mathbf{c}^{\rm T}_{\rm in}\mathbf{S}_{\rm in}\mathbf{d}_{\rm in} + \mathbf{c}^{\rm T}_{\rm in}\mathbf{r}_1 + \mathbf{d}^{\rm T}_{\rm in}\mathbf{r}_2 + \text{terms without } \mathbf{c}_{\rm in} \text{ or } \mathbf{d}_{\rm in} \quad (71)$$

$$(\Gamma^m_{\rm out})_e = -\mathbf{c}^{\rm T}_{\rm out}\mathbf{H}_{\rm out}\mathbf{c}_{\rm out}/2 + \mathbf{c}^{\rm T}_{\rm out}\mathbf{S}_{\rm out}\mathbf{d}_{\rm out} + \mathbf{c}^{\rm T}_{\rm out}\mathbf{r}_3 + \mathbf{d}^{\rm T}_{\rm out}\mathbf{r}_4 + \text{terms without } \mathbf{c}_{\rm out} \text{ or } \mathbf{d}_{\rm out}, \quad (72)$$

where

$$\mathbf{H}_{\rm in} = \mathbf{H}^*_{\rm in} + (\mathbf{H}^*_{\rm in})^{\rm T}$$

$$\mathbf{H}^*_{\rm in} = -\frac{1}{2}\int_{\partial\Omega_e}\mathbf{Q}^{\rm T}_2\mathbf{Q}_1\,{\rm d}c + \int_{\partial\Omega^2_e}\mathbf{Q}^{\rm T}_{21}\mathbf{Q}_{11}\,{\rm d}c + \int_{\partial\Omega^4_e}\mathbf{Q}^{\rm T}_{22}\mathbf{Q}_{12}\,{\rm d}c \quad (73)$$

$$\mathbf{S}_{\rm in} = -\int_{\partial\Omega^{11}_e}\mathbf{Q}^{\rm T}_2\mathbf{Q}_5\,{\rm d}c \quad (74)$$

$$\mathbf{r}_1 = \frac{1}{2}\iint_{\Omega_4}(\mathbf{N}^{\rm T}_1\dot{\mathbf{P}}_1 + \mathbf{N}^{\rm T}_2\dot{\mathbf{P}}_2)\,{\rm d}c + \frac{1}{2}\int_{\partial\Omega_e}(\mathbf{Q}^{\rm T}_1\mathring{\mathbf{N}} + \mathbf{Q}^{\rm T}_2\mathring{\mathbf{u}})\,{\rm d}c - \int_{\partial\Omega^1_e}\Delta\bar{U}_n\mathbf{Q}^{\rm T}_{21}\,{\rm d}c - \int_{\partial\Omega^3_e}\Delta\bar{U}_s\mathbf{Q}^{\rm T}_{22}\,{\rm d}c + \int_{\partial\Omega^2_e}\bar{N}^*_n\mathbf{Q}^{\rm T}_{11}\,{\rm d}c + \int_{\partial\Omega^4_e}\bar{N}^*_{ns}\mathbf{Q}^{\rm T}_{12}\,{\rm d}c - \int_{\partial\Omega^2_e}\Delta\mathring{N}_n\mathbf{Q}^{\rm T}_{51}\,{\rm d}c - \int_{\partial\Omega^4_e}\Delta\mathring{N}_{ns}\mathbf{Q}^{\rm T}_{52}\,{\rm d}c - \int_{\partial\Omega^1_e}\Delta\mathring{U}_n\mathbf{Q}^{\rm T}_{21}\,{\rm d}c - \int_{\partial\Omega^3_e}\Delta\mathring{U}_s\mathbf{Q}^{\rm T}_{22}\,{\rm d}c \quad (75)$$

$$\mathbf{r}_2 = \int_{\partial\Omega^{11}_e}\mathbf{Q}^{\rm T}_5\mathring{\mathbf{N}}\,{\rm d}c \quad (76)$$

$$\mathbf{H}_{\rm out} = \mathbf{H}^*_{\rm out} + (\mathbf{H}^*_{\rm out})^{\rm T}$$

$$\mathbf{H}^*_{\rm out} = -\frac{1}{2}\int_{\partial\Omega_e}\mathbf{Q}^{\rm T}_4\mathbf{Q}_3\,{\rm d}c + \int_{\partial\Omega^6_e}\mathbf{Q}^{\rm T}_{41}\mathbf{Q}_{31}\,{\rm d}c + \int_{\partial\Omega^8_e}\mathbf{Q}^{\rm T}_{42}\mathbf{Q}_{32}\,{\rm d}c + \int_{\partial\Omega^{10}_e}\mathbf{Q}^{\rm T}_{43}\mathbf{Q}_{33}\,{\rm d}c \quad (77)$$

$$\mathbf{S}_{\rm out} = -\int_{\partial\Omega^{11}_e}\mathbf{Q}^{\rm T}_4\mathbf{Q}_6\,{\rm d}c \quad (78)$$

$$\mathbf{r}_3 = \frac{1}{3}\iint_{\Omega_e}\mathbf{N}^{\rm T}_3\dot{P}_5\,{\rm d}c + \frac{1}{2}\int_{\partial\Omega_e}(\mathbf{Q}^{\rm T}_3\mathring{\mathbf{M}} + \mathbf{Q}^{\rm T}_4\mathring{\mathbf{w}}_b)\,{\rm d}c - \int_{\partial\Omega^5_e}\Delta\bar{W}\mathbf{Q}^{\rm T}_{41}\,{\rm d}c - \int_{\partial\Omega^7_e}\Delta\bar{\psi}_n\mathbf{Q}^{\rm T}_{42}\,{\rm d}c - \int_{\partial\Omega^9_e}\Delta\bar{\psi}_s\mathbf{Q}^{\rm T}_{43}\,{\rm d}c - \int_{\partial\Omega^6_e}[\Delta\mathring{W}\mathbf{Q}^{\rm T}_{41} + \mathbf{Q}^{\rm T}_{31}(\Delta\mathring{R} - \bar{R}^*)]\,{\rm d}c - \int_{\partial\Omega^8_e}[\Delta\mathring{\psi}_n\mathbf{Q}^{\rm T}_{42} + \mathbf{Q}^{\rm T}_{32}(\Delta\mathring{M}_n - \Delta\bar{M}_n)]\,{\rm d}c - \int_{\partial\Omega^{10}_e}[\Delta\mathring{\psi}_s\mathbf{Q}^{\rm T}_{43} + \mathbf{Q}^{\rm T}_{33}(\Delta\mathring{M}_{ns} - \Delta\bar{M}_{ns})]\,{\rm d}c \quad (79)$$

$$\mathbf{r}_4 = -\int_{\partial\Omega^{11}_e}\mathbf{Q}^{\rm T}_6\mathring{\mathbf{M}}\,{\rm d}c. \quad (80)$$

It should be pointed out that all terms not involving $\mathbf{c}$ or $\mathbf{d}$ are of no significance for an approximate solution and are therefore not listed explicitly.

To obtain the element matrices, taking vanishing variation of eqns (71) and (72) with respect to $\mathbf{c}$ at the element level, we have

$$\frac{\partial(\Gamma^m_{\rm in})_e}{\partial\mathbf{c}^{\rm T}_{\rm in}} = -\mathbf{H}_{\rm in}\mathbf{c}_{\rm in} + \mathbf{S}_{\rm in}\mathbf{d}_{\rm in} + \mathbf{r}_1 \quad (81)$$

$$\frac{\partial(\Gamma^m_{\rm out})_e}{\partial\mathbf{c}^{\rm T}_{\rm out}} = -\mathbf{H}_{\rm out}\mathbf{c}_{\rm out} + \mathbf{S}_{\rm out}\mathbf{d}_{\rm out} + \mathbf{r}_3 \quad (82)$$

which lead to

$$\mathbf{c}_{\rm in} = \mathbf{G}_{\rm in}\mathbf{d}_{\rm in} + \mathbf{g}_{\rm in} \quad (83)$$

$$\mathbf{c}_{\rm out} = \mathbf{G}_{\rm out}\mathbf{d}_{\rm out} + \mathbf{g}_{\rm out}, \quad (84)$$

where

$$\mathbf{G}_{\rm in} = \mathbf{H}^{-1}_{\rm in}\mathbf{S}_{\rm in}, \quad \mathbf{g}_{\rm in} = \mathbf{H}^{-1}_{\rm in}\mathbf{r}_1, \quad (85)$$

$$\mathbf{G}_{\rm out} = \mathbf{H}^{-1}_{\rm out}\mathbf{S}_{\rm out}, \quad \mathbf{g}_{\rm out} = \mathbf{H}^{-1}_{\rm out}\mathbf{r}_3. \quad (86)$$

As a consequence, the functionals $(\Gamma^m_{\rm in})_e$ and $(\Gamma^m_{\rm out})_e$ can be expressed only in terms of $\mathbf{d}$ and other known matrices

$$(\Gamma^m_{\rm in})_e = -\mathbf{d}^{\rm T}_{\rm in}\mathbf{G}^{\rm T}_{\rm in}\mathbf{H}_{\rm in}\mathbf{G}_{\rm in}\mathbf{d}_{\rm in}/2 + \mathbf{d}^{\rm T}_{\rm in}[\mathbf{G}^{\rm T}_{\rm in}\mathbf{H}_{\rm in}\mathbf{g}_{\rm in} + \mathbf{r}_2] + \text{terms without } \mathbf{d}_{\rm in} \quad (87)$$

$$(\Gamma^m_{\rm out})_e = -\mathbf{d}^{\rm T}_{\rm out}\mathbf{G}^{\rm T}_{\rm out}\mathbf{H}_{\rm out}\mathbf{G}_{\rm out}\mathbf{d}_{\rm out}/2 + \mathbf{d}^{\rm T}_{\rm out}[\mathbf{G}^{\rm T}_{\rm out}\mathbf{H}_{\rm out}\mathbf{g}_{\rm out} + \mathbf{r}_4] + \text{terms without } \mathbf{d}_{\rm out} \quad (88)$$

So the customary force–displacement relationships are in the form

$$\mathbf{K}_{in}\mathbf{d}_{in} = \mathbf{P}_{in} \quad (89)$$

$$\mathbf{K}_{out}\mathbf{d}_{out} = \mathbf{P}_{out}, \quad (90)$$

where

$$\mathbf{K}_{in} = \mathbf{G}_{in}^{T}\mathbf{H}_{in}\mathbf{G}_{in} \quad (91)$$

$$\mathbf{K}_{out} = \mathbf{G}_{out}^{T}\mathbf{H}_{out}\mathbf{G}_{out} \quad (92)$$

$$\mathbf{P}_{in} = \mathbf{G}_{in}^{T}\mathbf{H}_{in}\mathbf{g}_{in} + \mathbf{r}_2 \quad (93)$$

$$\mathbf{P}_{out} = \mathbf{G}_{out}^{T}\mathbf{H}_{out}\mathbf{g}_{out} + \mathbf{r}_4 \quad (94)$$

and where $\mathbf{K}_{in}$ and $\mathbf{K}_{out}$ can be calculated in the usual way, while $\mathbf{P}_{in}$ and $\mathbf{P}_{out}$ contain the unknown variables $\dot{U}_1$, $\dot{U}_2$, $\dot{W}$. An iterative procedure is, thus, required. The procedure will be given in the coming subsection.

2.7. *Iterative scheme*

Before describing the scheme, let us study some properties of $\mathbf{P}_{in}$. It is obvious from eqns (19), (20), (75), (85) and (93) that $\mathbf{P}_{in}$ depends only upon $\dot{\mathbf{W}}$. So only an initial value $\dot{W}^0$ is required. As long as the value of $\dot{W}$ in Ω is known, we can calculate the pseudo-load $\mathbf{P}_{in}$, and then all of unknown variables in eqn (89) are in-plane displacements ($\dot{U}_1$ $\dot{U}_2$). We may solve eqn (89) for them. As a consequence, $\mathbf{P}_{out}$ can be calculated from the current values of $\{\dot{U}_1, \dot{U}_2, \dot{W}\}$. An iterative scheme may be established according to the above analysis. Specifically, suppose that U_1^k, U_2^k and W^k stand for kth approximations, which can be obtained from the proceeding cycle of iteration. The $(k+1)$th solution may be evaluated as follows:

(a) Assume the initial value $\dot{W}^0$ and W^0 in Ω. If the current loading step is not the first one, but $(k+1)$th step, $\dot{W}^0$ and W^0 may be taken as $\dot{W}^k$ and W^k, where $\dot{W}^k$ and W^k stand for the incremental and the total deflection at the kth loading step, respectively.

(b) Enter the iterative cycle for $i = 1, 2, \cdots$. Calculate $\mathbf{P}_{in}$ in eqn (89) by means of eqn (93), solve eqn (89) for the nodal displacement vector $\mathbf{d}_{in}^{(i)}$, and then determine the values of $\dot{U}_1^{(i)}$ and $\dot{U}_2^{(i)}$ in Ω.

(c) Calculate $\mathbf{P}_{out}$ using the current values of $\mathbf{U}$, then solve eqn (90) for $\mathbf{d}_{out}^{(i)}$ and determine the value of $\dot{W}^{(i)}$ in Ω.

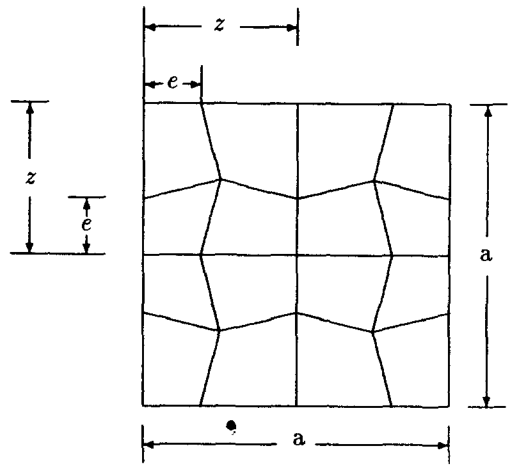

Fig. 2. Distorted mesh for example 1.

(d) If $\epsilon_i = [(\mathbf{d}^{(i)})^{T}\mathbf{d}^{(i)} - (\mathbf{d}^{(i-1)})^{T}\mathbf{d}^{(i-1)}]/(\mathbf{d}^{(i-1)})^{T}\mathbf{d}^{(i-1)} \leqslant \epsilon$ (ϵ is a convergence tolerance), proceed to the next loading step and calculate

$$\mathbf{U}^{k+1} = \mathbf{U}^{(k)} + \dot{\mathbf{U}}^{(i)}, \quad \dot{\mathbf{U}}^{k+1} = \dot{\mathbf{U}}^{(i)} \quad (95)$$

otherwise, set

$$\dot{W}^0 = \dot{W}^{(i)}, \quad W^0 = W^k + \dot{W}^{(i)} \quad (96)$$

and go back to step (b).

3. NUMERICAL APPLICATIONS

Since the main purpose of this paper is to outline the basic principles of the proposed method, the assessment has been limited to a geometrically nonlinear thin plate and two large deflection sandwich plates. In all the calculations, one quarter of the solution domain is analyzed. The convergence tolerence is $\epsilon = 0.0001$.

Example 1: *a clamped thin plate*

The square plate is subjected to a uniform distributed load q, and the geometry and material properties of the plate are

$$E = 2.1 \times 10^6\,\text{kg cm}^{-2}, \quad \nu = 0.316,$$

$$2a = 762\,\text{cm}, \quad t = 7.62\,\text{cm}, \quad Q = qa^4/Et^4$$

where $2a$ is the length of the square plate.

Table 1. Central deflection W_c/t for example 1

	Q	17.79	38.3	63.4	95	134.9
HT	2 × 2	0.2611	0.4689	0.6892	0.8987	1.1013
	4 × 4	0.2365	0.4695	0.6913	0.9035	1.1072
FE	6 × 6	0.2368	0.4702	0.6932	0.9062	1.1105
Ref. [22] (FEM)		0.2368	0.4699	0.6915	0.9029	1.1063
Exact [22]		0.237	0.471	0.695	0.912	1.121

Table 2. The coefficient β of central stress σ_x for example 1

	Q	17.79	38.3	63.4	95	134.9
HT	2×2	2.6431	5.5011	8.4243	11.314	14.109
	4×4	2.6322	5.4639	8.2972	11.205	13.945
FE	6×6	2.6113	5.3024	8.1529	11.152	13.723
Ref. [22] (FEM)		2.6319	5.4816	8.3258	11.103	13.827
Exact [22]		2.600	5.200	8.000	11.100	13.300

Table 3. Comparison of W_c/t for distorted (e and z as shown in Fig. 2) and undistorted 4×4 mesh

Q	17.79	38.3	63.4	95	134.9
Undistorted	0.2365	0.4695	0.6913	0.9035	1.1072
Distorted for $e=0.4z$	0.2347	0.4678	0.6892	0.9017	1.1041
Distorted for $e=0.3z$	0.2342	0.4659	0.6885	0.9009	1.1023
Exact •	0.237	0.471	0.695	0.912	1.121

To study the convergence properties of the present formulation, three element meshes (2×2, 4×4, 6×6) are used in the example. The maximum deflection W_c/t occurring at the center is shown in Table 1. Table 2 lists the results for central stress coefficient $\beta(\sigma_{xc}=\beta Et^2/a^2)$ vs load factor Q. Table 3 exhibits the study of sensitivity of the mesh distortion. All of these results are compared with the solutions obtained by conventional FE approach in which 16 Lagrangian nine-node elements are used [22].

Example 2: *clamped square sandwich plate*

The plate consists of two identical facings ($E=0.74\times 10^6\,\text{cm}^{-2}$, $\nu=0.3$, $2a=127\,\text{cm}$, $t=0.381\,\text{cm}$) and an aluminium honeycomb core ($G_c=0.35\times 10^4\,\text{kg cm}^{-2}$, $h=2.54\,\text{cm}$), which are subjected to a uniform load q. The boundary conditions are

$$U_1=U_2=W=\psi_x=\psi_y=0$$

on the whole boundary.

A 4×4 element mesh is used in the analysis. Table 4 shows the results for central deflection vs load parameter $Q=12a^3(1-\nu)q/(th^2E)$, and comparison is made with the results in Ref. [23] in which the same element mesh is used.

Example 3: *compressed sandwich plate*

Consider a simply supported square plate with identical isotropic facings ($2a=59.7\,\text{cm}$, $E=0.668\times 10^6\,\text{kg cm}^{-2}$, $\nu=0.3, t=0.0533\,\text{cm}$) and a 0.46 cm thick core ($G_c=0.134\times 10^4\,\text{kg cm}^{-2}$) subjected to uniform in-plane compress N_x at the boundaries $x=\pm a$ (here the origin of the coordinate frame is laid at the center of the square plate). The displacement boundary conditions used are as follows:

$$W=\psi_y=0 \quad \text{on } x=\pm a$$

$$W=\psi_x=0 \quad \text{on } y=\pm a.$$

Table 4. Central deflection W_c/h with 4×4 for example 2

Q	10	20	30	40
HT FE	0.709	1.272	1.635	1.834
Ref. [23]	0.70	1.26	1.62	1.82

Table 5. Central deflection W_c(cm) with 2×2 for example 3

$N_x(\text{kg cm}^{-1})$	54.7	57.2	61.6	66
HT FE	0	0.452	0.802	0.997
Ref. [23]	0	0.457	0.813	1.020

Assuming a symmetrical buckling pattern only a quadrant of the plate needs to be considered. Table 5 shows the load N_x vs central deflection W_C and comparison is made with the results reported in Ref. [23].

It can be seen from the above tables that the results obtained by the present method agree well with the results reported in Refs [22, 23]. As expected for example 1, it is also found from Table 1 and 2 that the numerical results converge gradually to the exact ones, along with refinement of the element meshes. Furthermore, the results in Table 3 exhibits remarkable insensitivity to the mesh distortion. In the course of computations, convergence was achieved with about 12 iterations for example 1, 24 iterations for example 2 and 18 iterations for example 3 at each load step.

4. CONCLUSIONS

An HT FE model has been presented for the nonlinear analysis of thick plates. As far as we know, most of the previous HT FE results only concern the linear problems. To some extent, this paper studies how to apply the HT FE approach to nonlinear problems. In the analysis, we use the incremental field equations and have made some modifications on nonlinear boundary equations [see eqns (24) and (25)]. The numerical results show that these modifications are practicable. We also see that these modifications greatly simplify the related derivation.

Besides, further extension is possible and straightforward. For example, the extension of a thick plate on an elastic foundation and the use of plate elements with p-approach capabilities. The extension is under way.

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