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COUPLED TORSIONAL–FLEXURAL VIBRATION OF SHAFT SYSTEMS IN MECHANICAL ENGINEERING—II. FE–TM IMPEDANCE COUPLING METHOD

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Abstract—A new approach is developed for calculating the dynamic response of multi-spool rotor systems with several intershaft bearings using the combined methodologies of the finite element method (FEM), transfer matrices (TM) and impedance methods. The approach makes use of FEM to model shaft elements and then transforms the system properties to transfer matrix mode. The intershaft bearings are treated by the impedance method. It combines the advantages of these three methods. Some numerical examples are considered to check the viability of the method.

1. INTRODUCTION

In the first part of this paper [1], a new finite element model was developed for analysing dynamic response of damped shaft systems. For large shaft systems, however, the use of FEM leads to prohibitively higher computation time and costs. In order to minimize this, a continuing effort is made by various researches to limit the dynamic degrees of freedom without reducing the accuracy of the results. Of these, the transfer matrix technique [2, 3] is of importance since it is exceptionally suitable for chain-like or branching systems that are made up of several subsystems, with each subsystem made of shaft, disks and bearings. An equation of motion for each subsystem is formulated in terms of the state vectors consisting of displacements and internal forces. With this method the size of matrix is determined by the maximum number of degrees of the subsystem and is usually much smaller than in FEM [4, 5]. However, the transfer matrix method cannot be directly used for complex systems with two or more parallel chains with several interconnections. Under this circumstance, the component mode synthesis or the transfer matrix impedance coupled method will be effective [6, 7]. Unfortunately, the applications of these two methods are restricted to natural frequency eigensolutions and steady state response analysis under harmonic excitation only.

In this study, the formulation of the coupled method of FE–TM with the impedance approach is first presented. It incorporates the advantages of the finite element, transfer matrix for the chain systems and the mechanical impedance technique for the connecting points. Some numerical examples are considered to establish the validity of the method.

2. ANALYSIS

2.1. *Time–space FE formulation*

As was undertaken by Huang [7], consider a typical spool-rotor system with several intershaft bearings. It consists of N rotors and M intershaft bearings (Fig. 1).

For a particular shaft element with discs and bearings, which is shown in Fig. 2, the finite element formulation can be written as [1]:

$$[M]\{\ddot{\mathbf{d}}(\mathbf{t})\} + [C]\{\dot{\mathbf{d}}(\mathbf{t})\} + [K]\{\mathbf{d}(\mathbf{t})\} = \mathbf{Q}(\mathbf{t}), \quad (1)$$

where $[M]$, $[C]$ and $[K]$ denote mass, damping and stiffness matrices of the element, respectively, each being of order ten; $\mathbf{d}(\mathbf{t})$ and $\mathbf{Q}(\mathbf{t})$ represent the displacement and equivalent nodal force vectors (10×1), and are defined as

$$\{\mathbf{d}(\mathbf{t})\} = \{\mathbf{d}_1 | \mathbf{d}_2\}^T$$
$$= \{v_1 \;\; w_1 \;\; \theta_{y1} \;\; \theta_{z1} \;\; \theta_{x1} | v_2 \;\; w_2 \;\; \theta_{y2} \;\; \theta_{z2} \theta_{x2}\}^T \quad (2)$$

$$\{\mathbf{Q}(\mathbf{t})\} = \{\mathbf{q}_1 | \mathbf{q}_2\}^T. \quad (3)$$

For convenience, the matrices $[M]$, $[C]$ and $[K]$ are rewritten as

$$[M] = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix},$$

$$[C] = \begin{bmatrix} c_{11} & c_{21} \\ c_{21} & c_{22} \end{bmatrix},$$

$$[K] = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}. \quad (4)$$

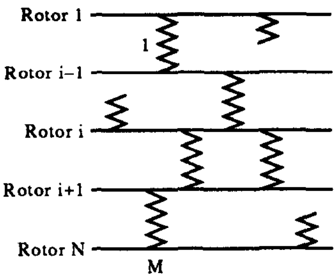


Fig. 1. A multi-spool rotor system with several intershaft components.

In the natural frequency estimation or steady state response analysis under harmonic excitation, the acceleration $\{\ddot{d}(t)\}$ and the velocity $\{\dot{d}(t)\}$ can be replaced by $-\omega^2\{d\}$ and $i\omega\{d\}$, respectively, where $i=\sqrt{(-1)}$. However, for a system subjected to arbitrary external load, these substitutions are not valid and hence an appropriate numerical integration algorithm should be used for solving eqn (1). Most of the numerical integration algorithms are based on the assumption that at any given time instant, say t_i, $\{\ddot{d}(t)\}$ and $\{\dot{d}(t)\}$ can be expressed as a linear function of $\{d(t_i)\}$ with reasonable accuracy. That is

$$\ddot{d}(t) = A(t_i)d(t_i) + B(t_i) \tag{5}$$

$$\dot{d}(t) = D(t_i)d(t_i) + E(t_i). \tag{6}$$

The substitution of (5) and (6) into (1), leads to

$$[[M]A(t_i) + [C]D(t_i) + [K]]\{d(t_i)\} + [M]\{B(t_i)\} + [C]E(t_i) = \{Q(t_i)\}. \tag{7}$$

If the Wilson θ method is used, we have

$$A(t_i) = 6/\theta^2\Delta t^2 \tag{8a}$$

$$B(t_i) = -6[d(t_{i-1}) + 6\dot{d}(t_{i-1})\theta\Delta t + \ddot{d}(t_{i-1})\theta^2\Delta t^2/3]/\theta^2\Delta t^2 \tag{8b}$$

$$C(t_i) = 3/\theta\Delta t \tag{8c}$$

$$D(t_i) = -3[d(t_{i-1}) + 2\theta\Delta t\dot{d}(t_{i-1})/3 + \ddot{d}(t_i)\theta^2\Delta t^2/3]/\theta\Delta t, \tag{8d}$$

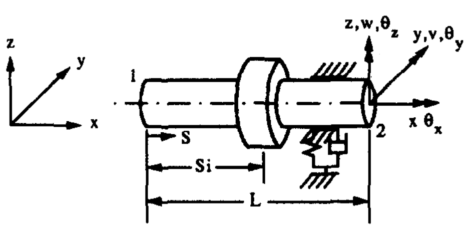


Fig. 2. Typical element and coordinate systems.

in which θ is a free parameter and the algorithms (5) and (6) are unconditionally stable when $\theta \geqslant 1.37$ (θ is taken to be 1.4 in our analysis).

2.2. *Discrete time FE–TM formulation*

The formulation of discrete time FE–TM, as given below, combines the discrete time finite element formulation and the transfer matrix formulation. For the expansion of (7) in terms of submatrices given in eqn (4), we know

$$\begin{bmatrix} J_{11} & J_{12} & R_1 \\ J_{21} & J_{22} & R_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ 1 \end{Bmatrix} = \begin{Bmatrix} q_1 \\ q_2 \\ 1 \end{Bmatrix}, \tag{9}$$

where

$$J_{11} = m_{11}A(t_i) + c_{11}D(t_i) + k_{11} \tag{10a}$$

$$J_{12} = m_{12}A(t_i) + c_{12}D(t_i) + k_{12} \tag{10b}$$

$$J_{21} = m_{21}A(t_i) + c_{21}D(t_i) + k_{21} \tag{10c}$$

$$J_{22} = m_{22}A(t_i) + c_{22}D(t_i) + k_{22} \tag{10d}$$

$$R_1 = m_{11}B_1(t_i) + m_{12}B_2(t_i) + c_{11}E_1(t_i) + c_{12}E_2(t_i) \tag{10e}$$

$$R_2 = m_{21}B_1(t_i) + m_{22}B_2(t_i) + c_{21}E_1(t_i) + c_{22}E_2(t_i) \tag{10f}$$

with notations

$$\{B(t_i)\} = \{B_1(t_i)|B_2(t_i)\}^T,$$

$$\{E(t_2)\} = \{E_1(t_2)|E_2(t_i)\}^T.$$

As was done in [5], eqn (9) can be further rewritten in terms of left and right nodal displacements and forces, such that

$$\begin{Bmatrix} d_2 \\ q_2 \\ 1 \end{Bmatrix} = \begin{bmatrix} -J_{12}^{-1}J_{11} & J_{12}^{-1} & S_1 \\ J_{21} - J_{22}J_{12}^{-1}J_{11} & J_{22}J_{12}^{-1} & S_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ q_1 \\ 1 \end{Bmatrix}, \tag{11}$$

where

$$S_1 = -J_{12}^{-1}R_1 \quad \text{and} \quad S_2 = J_{22}J_{12}^{-1}R_1 + R_2$$

or further in compact form:

$$\{\mathbf{v}\}_2^L = [P]\{\mathbf{v}\}_1^R. \tag{12}$$

Here, following the conventional TM method, the square matrix $[P]$ is called the field matrix, and the vector $\{\mathbf{v}\}$ denotes the state vector. Consequently, the point matrix $[F]$ can be derived following the method of Kumar and Sankar [4]. For the sake of conciseness we omit those details. The matrix is defined by

$$\{\mathbf{v}\}_1^R = [F]\{\mathbf{v}\}_1^L. \tag{13}$$

The transfer matrix $[T]_1$, which relates the state vectors at either end of the element, can be then formulated by combining (12) and (13). That is

$$\{\mathbf{v}\}_2^L = [P][F]\{\mathbf{v}\}_1^L = [T]_1\{\mathbf{v}\}_1^L. \tag{14}$$

Using (14) repetitively, the overall transfer matrix $[T]$, which relates the state vectors at either end of the rotor system, can be computed:

$$[T] = \prod_{i=1}^{n} [T]_i. \tag{15}$$

With the application of proper boundary conditions and the initial conditions corresponding to time $t = t_{i-1}$, the unknown generalized displacements and forces at the left (or right) end of the rotor can be solved for:

$$\{\mathbf{v}\}_n^L = [T]\{\mathbf{v}\}_1^L. \tag{16}$$

2.3. *Impedance method*

For a multi-spool rotor system with several intershaft bearings, eqn (16) can not be directly used. In order to analyse the problems, the intershaft must be separated. After separating the connections of every intershaft bearing, the constraint conditions will be a pair of generalized force vectors F_j (5 × 1) added on the connecting point j (Fig. 3). The new unknown vector F_j will appear in (16). Therefore it needs to find $5M$ supplemental equations to search for all these variables for a rotor system with M intershaft bearings. The impedance method will be adopted in establishing these supplemental equations.

For a particular intershaft bearing connecting two rotor subsystems at point j (Fig. 3), the following impedance equation must be satisfied:

$$\{F_j\} = [K]\{d_j^{(i)} - d_j^{(i+1)}\} + [C]\{\dot{d}_j^{(i)} - \dot{d}_j^{(i+1)}\}. \tag{17}$$

Using (16), (17) and the proper boundary and initial conditions, the unknown generalized displacements and forces of a multi-spool rotor system can be solved.

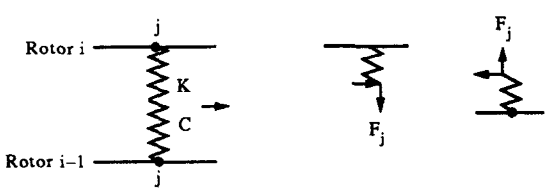


Fig. 3. Separation of intershaft parts.

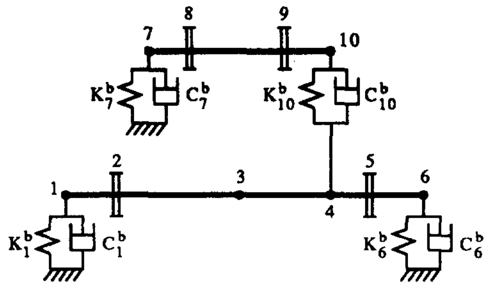


Fig. 4. Configuration of a dual rotor system.

3. NUMERICAL APPLICATION

Since the main purpose of this paper is to present the basic principles of the proposed method, its assessment has been limited to three simple examples. In all the calculations, Poisson's ratio is taken to be 0.3. As treated in part 1 of this paper [1], the complex eigenvalues, α_i, are determined in the form

$$\alpha_i = \lambda_i + i\omega_i \tag{18}$$

and the logarithmic decrement is defined as

$$\delta_i = -2\pi\lambda_i/\omega_i, \tag{19}$$

thus the stability region is $\delta_i > 0$.

Example 1

This example reproduces the results appearing in the first part of this paper of a dual rotor system shown in Fig. 4. Some initial data are listed in Table 1. The rotating speed and the area moment of inertia about the diameter axis of rotor 1 and rotor 2

Table 1. Data of a dual rotor damped system

Station	M (kg)	I_p (kg-m²)	K (N/m)	C (N-sec/m)	L (m)
1	0.0577	0	2.6269×10^7	5254	0.0762
2	10.7023	0.0859	—	0	0.1778
3	0.2499	0	—	0	0.1524
4	0.1538	0	—	0	0.0508
5	7.0869	0.0678	—	0	0.0508
6	0.0385	0	1.7513×10^7	3502	0.0508
7	0.0467	0	1.7513×10^7	3502	0.0508
8	7.202	0.0429	—	0	0.1524
9	3.692	0.0271	—	0	0.0508
10	0.0467	0	0.8756×10^7	1751	0.0508

$E = 2.068 \times 10^{11}$ N/m²

Table 2. Eigenvalues obtained by FE–TM impedance compared with CMS for example 1

Present FEM			CMS		
λ	ω	δ	λ	ω	δ
−1.991	469.12	0.0267	−2.00	469.51	0.0268
−14.07	727.84	0.1215	−14.15	728.33	0.1220
−113.65	1421.55	0.5023	−114.06	1423.34	0.5033
−107.69	2172.45	0.3115	−108.56	2174.93	0.3155
−62.87	2334.04	0.1692	−63.57	2335.42	0.1709
−70.64	3011.96	0.1474	−71.42	3014.76	0.1488
−256.11	4266.43	0.3772	−257.85	4271.58	0.3791
−63.03	5643.47	0.0702	−64.05	5650.71	0.0712

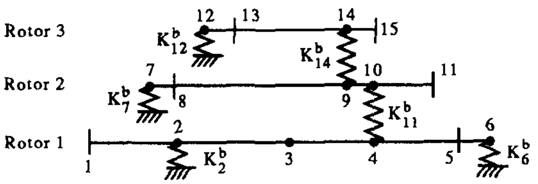


Fig. 5. Model of a tri-spool rotor system.

are 1047.2 rad/sec, 2.6467×10^{-9} m^4, 1570.8 rad/sec, and 2.1935×10^{-8} m^4, respectively. Table 2 lists the results by the proposed method and comparison is made with that by component mode synthesis (CMS) [7]. It can be seen that those results are in good agreement.

Table 3. Data of the tri-spool rotor system

Station no.	M (kg)	I_p (kg-m^2)	K (N/m)	L (m)
1	25.50	0.75	—	0.10
2	2.50	0	1×10^8	0.45
3	4.50	0	—	0.50
4	1.40	0	—	0.15
5	30.50	0.85	—	0.10
6	1.60	0	8×10^7	
7	1.10	0	7.5×10^7	0.10
8	12.50	0.35	—	0.60
9	2.10	0	—	0.10
10	1.10	0	6.5×10^7	0.05
11	15.30	0.45	—	
12	1.05	0	6×10^7	0.10
13	10.80	0.34	—	0.40
14	1.15	0	4.5×10^7	0.05
15	11.70	0.42	—	

$E = 2.07 \times 10^{11}$ N/m^2

Table 4. Eigenvalues of the tri-spool rotor system

No.	Present FE–TM f.w.	Present FE–TM b.w.	Ref. [7] f.w.	Ref. [7] b.w.
1	262.533	100.210	262.724	100.209
2	577.432	246.356	577.899	246.658
3	934.455	260.109	934.800	260.508
4	1050.167	415.892	1050.603	416.552
5	1408.785	800.657	1409.566	801.424
6	1555.089	986.698	1555.941	987.641
7	1687.411	1089.018	1688.392	1090.419
8	1782.580	1413.200	1783.707	1414.517
9	1883.298	1650.986	1885.066	1652.463
10	2460.324	1689.568	2461.945	1691.609

Table 5. Data of the rotor-bearing system shown in Fig. 6

Type of bearings	Plane cylindrical
Bearing diameter (m)	0.0254
Bearing of L/D ratio	1.0
Viscosity of oil at 25.5°C (N.S/m^2)	0.024
Disc mass (kg)	11.82
Disc diameter (m)	0.2032
Disc eccentricity (m)	0.001
Shaft diameter (m)	0.022
Total length of shaft (m)	0.5105
Young's modulus for the shaft material (N/m^2)	2.145×10^{11}

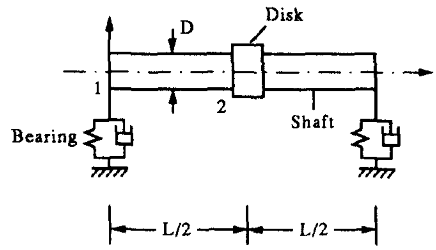


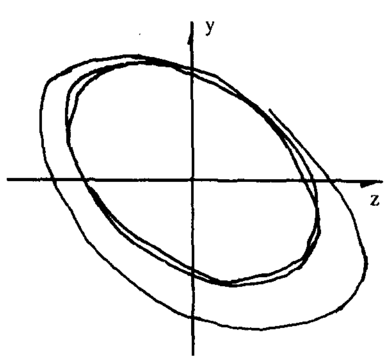
Fig. 6. The rotor system for example 3.

Example 2

Consider an undamped tri-spool isotropic system with intershaft bearings [7] (Fig. 5). Table 3 lists the data of the tri-spool rotor system. In order to allow for comparisons with other solutions given in [7], the rotating speed and the area moment of inertia about the bending axis of rotor 1, rotor 2 and rotor 3 are taken to be 314.16 rad/sec, 1.40×10^{-7} m^4, 628.32 rad/sec, 1.80×10^{-7} m^4, 942.48 rad/sec, 2.325×10^{-7} m^4, respectively. Table 4 lists the eigenvalues and compares with those in [7]. They are in good agreement.

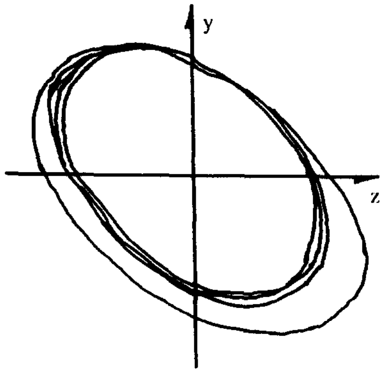
Example 3

This example is to investigate the dynamic response of a single rotor system supported on fluid film bearing [5] which is shown in Fig. 6. Table 5 lists the data of the rotor system. The transient orbital response has been studied by both the present FE–TM and the finite element method [1]. In the calculations, the number of elements and the time step used are the same for both methods. The orbital plots obtained for the rotor are shown in Figs 7 and 8, for a rotor speed of 1000 rpm. The steady state of the rotor has been obtained within six cycles in both approaches. The discrepancy between these two sets of results is almost negligible. These results are also in good agreement with those in [5].



RPM = 1000
Max. amplitude = 0.301E–04
At times = 0.152 sec
Time range = 0.0–0.321 sec

Fig. 7. Orbital response of rotor at bearing location (present FE–TM).



RPM = 1000
Max. amplitude = 0.301E–04
At times = 0.5415 sec
Time range = 0.0–0.5985 sec

Fig. 8. Orbital response of rotor at bearing location (FEM given in [1]).

4. CONCLUSION

A transient FE–TM impedance approach has been developed to study the dynamic response of multi-spool rotor systems with several intershaft components. The approach makes use of FEM to model shafts and then transforms the system properties to transfer matrix mode. The intershaft components are treated by the impedance method. It combines the advantages of these methods. It can be seen from the three examples that the results are in good agreement with those given in [5] and [7]. Although the proposed formulation and the numerical examples are confined to the multi-spool rotor system with intershaft bearings, further extensions are possible and straightforward, such as the use of non-uniform time steps, a multi-spool rotor system with general connecting components, electrical coupling in parallel synchronous generator system, etc. and we hope to present the theoretical and the numerical results of these in the near future.

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