Pergamon

0045-7949(95)00115-8

Computers & Structures Vol. 58, No. 1, pp. 195–201, 1996

0045-7949/96 $9.50 + 0.00

APPLICATION OF HYBRID-TREFFTZ ELEMENT APPROACH TO TRANSIENT HEAT CONDUCTION ANALYSIS

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(*Received* 11 *May* 1994)

Abstract—The paper presents a hybrid-Trefftz element approach for the numerical solution of transient linear heat conduction problems. In the proposed method, the transient heat conduction equation is first discretized with respect to time and then the resulting set of elliptic equations is solved by the corresponding time independent hybrid Trefftz element approach. At the end of the paper, the proposed method is assessed through numerical examples.

1. INTRODUCTION

The application of the finite element method (FEM) to heat conduction problems has been initiated in 1965 by Zienkiewicz and Cheung [1]. Consequently Wilson and Nickell [2] formulated time dependent finite elements with Gurtin's variational principle [3]. In 1970, Zienkiewicz and Parekh [4] applied isoparametric finite elements to two- and three-dimensional transient heat conduction problems along with a recursive process for the solution in time. In 1972, Argyris *et al.* [5–7] used real time–space finite elements to analyse structural problems. Later on, further new methods have continuously been introduced. For example, Bruch and Zyvoloski [8] solved two-dimensional transient heat conduction problem with prismatic elements based on a weighted residual procedure. Padovan [9, 10] reported work on heat conduction for linear and nonlinear anistropic materials where the material properties were space and temperature dependent. Zienkiewicz [11] proposed a Galerkin-type formulation in place of the traditional finite difference formulation for the time marching scheme. Comini and his associates [12] reported a finite element solution for nonlinear heat conduction with phase change. Tham and Cheung [13] used a quasi-variational approach to develop a parabolic time–space element, which was unconditionally stable in the solution of heat conduction problems. Yu and Hsu [14] presented a generalized finite element formulation for space–time domain for heat conduction in solids. More recently, Chen and Chen [15] proposed a new Laplace transform to remove the time derivative from the governing differential equation and then solved the associated equation by the FEM. Surana and Teong [16] developed a 27-node three-dimensional solid p-version element for steady-state heat conduction, where the temperature field in all directions of the element can be of arbitrary order. Donea [17] compared the Grank–Nicholson recurrence formula with the Galerkin approach and showed that the Galerkin method yields better accuracy for fast varying boundary conditions.

A relatively complete comparison for different time-marching schemes was made by Wood and Lewis [18]. These schemes are able to handle a wide variety of complex heat conduction problems with irregular geometry and a dependent heat source generation. It is, however, often necessary to take very small time steps in order to avoid undesirable numerically induced oscillations [19, 20] in the solution, which may result in an excessive amount of computer time [21].

In contrast to conventional FE, the hybrid-Trefftz (HT) finite element approach initiated in 1978 [22], has many advantages [23]: high accuracy, fast p-convergence rate, enhanced insensitivity to mesh distortion, great liberty in element geometry, possibility of accurately representing, without troublesome mesh adjustment, various local effects due to loading and/or geometry, etc.

The purpose of this paper is to develop a simple HT finite element model for analysis of transient heat conduction. In doing so, we first convert the original governing equation into a series of modified Helmholtz equations at discretized times and then develop the corresponding HT finite element formulation.

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2. HT FINITE ELEMENT APPROACH

2.1. *Principle of the approach*

Consider a two-dimensional heat conduction equation which describes the unsteady temperature distribution in a solid (domain Ω). This problem is governed by the differential equation

$$K\nabla^2 u + Q = \rho c\, \partial u/\partial t \tag{1}$$

subject to the initial condition on $\bar{\Omega}$:

$$u(x, y, 0) = u_0(x, y) \tag{2}$$

and the boundary conditions on $\partial\Omega$

$$u(x, y, t) = \bar{u}(x, y, t) \quad \text{on } \Gamma_1 \tag{3}$$

$$p(x, y, t) = \bar{p}(x, y, t) \quad \text{on } \Gamma_2 \tag{4}$$

$$s(x, y, t) = \bar{s}(x, y, t) \quad \text{on } \Gamma_3. \tag{5}$$

In these relations

$$p = K\,\partial u/\partial n, \quad s = hu + p, \quad \bar{s} = hu_{\text{env}} \tag{6}$$

$$\bar{\Omega} = \Omega \cup \partial\Omega, \quad \partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3, \tag{7}$$

where $u(x, y, t)$ is the temperature function, K specified thermal conductivity, $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ the two-dimensional operator, ρ the density, c the specific heat and the overhead bar designates the imposed quantities. Furthermore u_0 is the initial temperature, h is the heat transfer coefficient and u_{env} stands for environmental temperature.

For simplicity, consider the single step formula [24]:

$$Lu_n = (\nabla^2 - a)u_n = b_n \tag{8}$$

with the following boundary conditions

$$u_n = \bar{u}_n \quad \text{on } \Gamma_1, \quad p_n = \bar{p}_n \quad \text{on } \Gamma_2,$$

$$s_n = \bar{s}_n \quad \text{on } \Gamma_3, \tag{9}$$

where [see eqn (6)]

$$p_n = K\,\partial u_n/\partial n, \quad s_n = hu_n + p_n, \tag{10}$$

$$a = \frac{2\rho c}{K\Delta t}, \quad b_n = -(\nabla^2 + a)u_{n-1} - (Q_{n-1} + Q_n)/K \tag{11}$$

and where $\bar{u}_n$, $\bar{p}_n$, $\bar{s}_n$ stand for imposed quantities at the time $t = t_n$. Hereafter, to further simplify the subsequent writing, we shall omit the index "n" in eqns (8) and (9).

The main idea of the HT finite element model is to establish a finite element formulation whereby the intraelement continuity is enforced on a non-conforming internal displacement field (or other physical field) chosen so as to *a priori* satisfy the governing differential equations of the problem [22, 25, 26].

We consider again the boundary value problem defined by eqns (8)–(11). The domain is subdivided into elements and over each element "e" the assumed field is written as

$$u = \mathring{u} + \sum_{j=1}^{m} N_j c_j = \mathring{u} + \mathbf{Nc} \tag{12}$$

where c_j are undetermined coefficients and $\mathring{u}$ and N_j are known functions which satisfy

$$L\mathring{u} = b, \quad LN_j = 0 \quad \text{on } \Omega^e. \tag{13}$$

For convergence, the set of homogeneous solutions $N_1, N_2, \ldots, N_m$ must be such that for $m \to \infty$ this set is T-complete (for T-completeness see for example [27]). Therefore the matrix $\mathbf{N}$ should be formed by a suitably truncated T-complete set of homogeneous solutions (see Section 2.2.1).

Let

$$\Gamma^e = \Gamma_1^e \cup \Gamma_2^e \cup \Gamma_3^e \cup \Gamma_4^e, \tag{14}$$

where

$$\Gamma_1^e = \Gamma^e \cap \Gamma_1, \quad \Gamma_2^e = \Gamma^e \cap \Gamma_2, \quad \Gamma_3^e = \Gamma^e \cap \Gamma_3 \tag{15}$$

and where Γ_4^e is the interelement portion of Γ^e. From eqn (12) the following boundary quantities can easily be derived:

$$p = \mathring{p} + \mathbf{Pc} \quad \text{on } \Gamma_2^e \tag{16}$$

$$s = \mathring{s} + \mathbf{Sc} \quad \text{on } \Gamma_3^e. \tag{17}$$

Furthermore, to enforce on u the conformity, $u^e = u^f$ on $\Gamma^e \cap \Gamma^f$ (where "e" and "f" stand for any two neighbouring elements), we will use an auxiliary interelement frame field $\tilde{u}$ approximated in terms of the same degrees of freedom (DOF), $\mathbf{d}$, as used in the conventional elements, but confined now to the interelement portion of the element boundary. As opposed to standard HT elements (where $\tilde{u}$ extends over the whole element boundary, $\Gamma^e = \partial\Omega^e$), we have used an alternative HT formulation [26], where $\tilde{u}$ is confined to the interelement portion of Γ^e. The obvious advantage of such a formulation is the decrease in the number of DOF for the element assembly. In our case, we assume

$$\tilde{u} = \tilde{\mathbf{N}}\mathbf{d} \quad \text{on } \Gamma_4^e. \tag{18}$$

As an example, Fig. 1 displays a typical HT element with an arbitrary number of sides. In the simplest case, with linear shape function, the vector of nodal parameters is defined as

$$\mathbf{d} = \{\tilde{u}_1, \tilde{u}_2, \tilde{u}_3\} \tag{19}$$

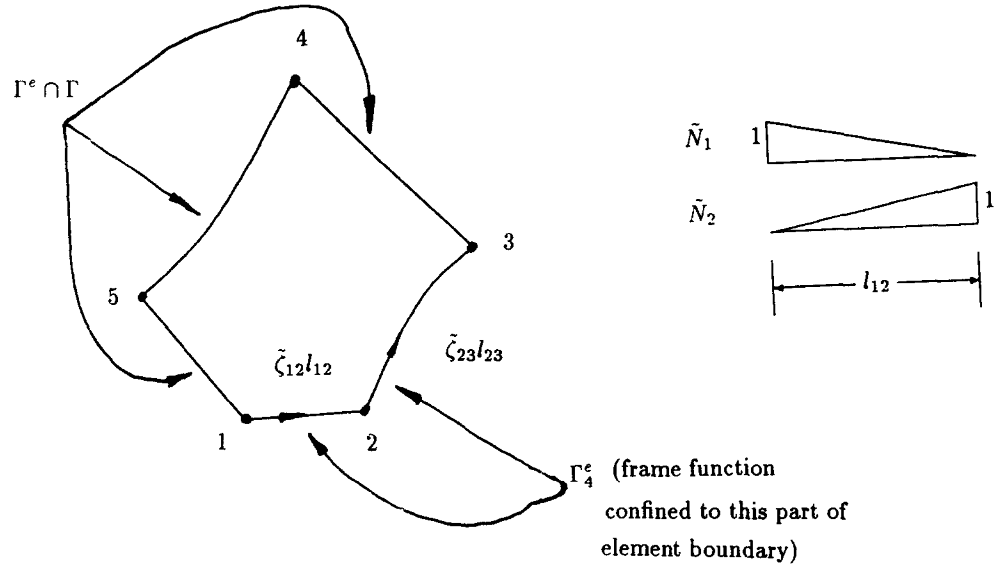

Fig. 1. Typical HT element with linear frame function.

and along a particular element side situated on Γ_4^e, for example, the side 1–2, we have simply:

$$\tilde{u} = \tilde{N}_1 \tilde{u}_1 + \tilde{N}_2 \tilde{u}_2, \tag{20}$$

where

$$\tilde{N}_1 = 1 - \tilde{\zeta}_{12}, \quad \tilde{N}_2 = \tilde{\zeta}_{12}. \tag{21}$$

There are no degrees of freedom at nodes four and five situated on $\Gamma^e \cap \Gamma$ (Γ = domain boundary).

To enforce the boundary conditions [eqn (9)] and the interelement continuity on u, we minimize for each element the following least square functional:

$$\int_{\Gamma_1^e} (u - \bar{u})^{\mathrm{T}}(u - \bar{u})\,\mathrm{d}\Gamma + d^2 \int_{\Gamma_2^e} (p - \bar{p})^{\mathrm{T}}(p - \bar{p})\,\mathrm{d}\Gamma + d^2 \int_{\Gamma_3^e} (s - \bar{s})^{\mathrm{T}}(s - \bar{s})\,\mathrm{d}\Gamma + \int_{\Gamma_4^e} (u - \tilde{u})^{\mathrm{T}}(u - \tilde{u})\,\mathrm{d}\Gamma = \mathrm{Min} \tag{22}$$

where $d > 0$ is an arbitrary chosen length (in this paper the average distance between the element center and element corners)

$$d = \frac{1}{n} \sum_{i=1}^{n} \sqrt{x_i^2 + y_i^2} \tag{23}$$

which serves the purpose of getting a physically meaningful functional (homogeneity of physical units!). The least square statement (22) yields for the internal parameters $\mathbf{c}$ the following system of linear equations:

$$\mathbf{A}\mathbf{c} = \mathbf{a} + \mathbf{Q}\mathbf{d}, \tag{24}$$

where

$$\mathbf{A} = \int_{\Gamma_1^e \cup \Gamma_4^e} \mathbf{N}^{\mathrm{T}}\mathbf{N}\,\mathrm{d}\Gamma + d^2 \int_{\Gamma_2^e} \mathbf{P}^{\mathrm{T}}\mathbf{P}\,\mathrm{d}\Gamma + d^2 \int_{\Gamma_3^e} \mathbf{S}^{\mathrm{T}}\mathbf{S}\,\mathrm{d}\Gamma \tag{25}$$

$$\mathbf{a} = \int_{\Gamma_1^e} \mathbf{N}^{\mathrm{T}}(\bar{u} - \mathring{u})\,\mathrm{d}\Gamma + d^2 \int_{\Gamma_2^e} \mathbf{P}^{\mathrm{T}}(\bar{p} - \mathring{p})\,\mathrm{d}\Gamma + d^2 \int_{\Gamma_3^e} \mathbf{S}^{\mathrm{T}}(\bar{s} - \mathring{s})\,\mathrm{d}\Gamma \tag{26}$$

$$\mathbf{Q} = \int_{\Gamma_4^e} \mathbf{N}^{\mathrm{T}}\tilde{\mathbf{N}}\,\mathrm{d}\Gamma. \tag{27}$$

From eqns (24)–(27), the internal coefficients $\mathbf{c}$ are readily expressed in terms of the nodel parameters $\mathbf{d}$:

$$\mathbf{c} = \mathring{\mathbf{c}} + \mathbf{C}\mathbf{d}, \tag{28}$$

where

$$\mathring{\mathbf{c}} = \mathbf{A}^{-1}\mathbf{a}, \quad \mathbf{C} = \mathbf{A}^{-1}\mathbf{Q}. \tag{29}$$

To ensure a good numerical conditioning during the inversion of the matrix $\mathbf{A}$, the homogeneous solutions N_j in eqn (12) have to be expressed in terms of suitably scaled local coordinates originated at the element centroid C,

$$\xi = \frac{x - x_C}{d}, \quad \eta = \frac{y - y_C}{d},$$

where d has been defined in eqn (23) rather than in terms of the global coordinates x, y.

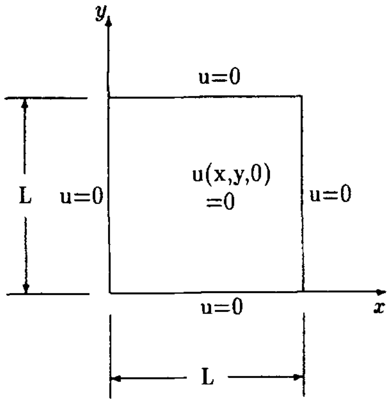

Fig. 2. Boundary and initial conditions for example 1.

With this comment in mind, we now turn the attention to the evaluation of element matrices. In order to enforce the "traction reciprocity"

$$\partial u^e/\partial n^e + \partial u^f/\partial n^f = 0 \quad \text{on } \Gamma^e \cap \Gamma^f$$

and to obtain a symmetric positive definite stiffness matrix, we set, in a similar way as in Jirousek [26],

$$K\int_{\Gamma^e} \delta u^{\mathrm{T}}\, \partial u/\partial n \,\mathrm{d}\Gamma = \int_{\Gamma_2^e} \delta u^{\mathrm{T}} \bar{p}\,\mathrm{d}\Gamma + \int_{\Gamma_3^e} \delta u^{\mathrm{T}} \bar{s}\,\mathrm{d}\Gamma - h\int_{\Gamma_3^e} \delta u^{\mathrm{T}} u\,\mathrm{d}\Gamma + K\delta\mathbf{d}^{\mathrm{T}}\mathbf{r}, \quad (30)$$

where $\mathbf{r}$ stands for vector of the fictitious equivalent nodal forces conjugate to nodal displacement $\mathbf{d}$. This leads to the customary "force–displacement" relationship

$$\mathbf{r} = \mathring{\mathbf{r}} + \mathbf{K}\mathbf{d} \quad (31)$$

where

$$\mathring{\mathbf{r}} = \mathbf{C}^{\mathrm{T}}(\mathbf{H}\mathring{\mathbf{c}} + \mathbf{h}) \quad \text{and} \quad \mathbf{K} = \mathbf{C}^{\mathrm{T}}\mathbf{H}\mathbf{C}. \quad (32)$$

The auxiliary matrices $\mathbf{h}$ and $\mathbf{H}$ are calculated by setting

$$\partial u/\partial n = \partial(\mathring{u} + \mathbf{N}\mathbf{c})/\partial n = \mathring{t} + \mathbf{T}\mathbf{c} \quad (33)$$

$$\mathring{t} = \partial\mathring{u}/\partial n \quad \text{and} \quad \mathbf{T} = \partial\mathbf{N}/\partial n \quad (34)$$

and then performing the following boundary integrals:

$$\mathbf{h} = \int_{\Gamma^e} \mathbf{N}^{\mathrm{T}}\mathring{t}\,\mathrm{d}\Gamma - \frac{t}{K}\left\{\int_{\Gamma_2^e} \mathbf{N}^{\mathrm{T}}\bar{p}\,\mathrm{d}\Gamma + \int_{\Gamma_3^e} \mathbf{N}^{\mathrm{T}}(\bar{s} - h\mathring{u})\,\mathrm{d}\Gamma\right\} \quad (35)$$

$$\mathbf{H} = \int_{\Gamma^e} \mathbf{N}^{\mathrm{T}}\mathbf{T}\,\mathrm{d}\Gamma + \frac{h}{K}\int_{\Gamma_3^e} \mathbf{N}^{\mathrm{T}}\mathbf{N}\,\mathrm{d}\Gamma. \quad (36)$$

Through integration by parts, it is easy to show that the first integral in eqn (36) may be written as

$$\int_{\Gamma^e} \mathbf{N}^{\mathrm{T}}\mathbf{N}\,\mathrm{d}\Gamma = \int_{\Omega^e} \mathbf{B}^{\mathrm{T}}\mathbf{B}\,\mathrm{d}\Omega, \quad (37)$$

where

$$\mathbf{B} = \{\partial/\partial x, \partial/\partial y\}^{\mathrm{T}}\mathbf{N}. \quad (38)$$

As a consequence, $\mathbf{H}$ is a symmetric matrix.

2.2. *Generation of the intraelement field*

2.2.1. *Homogeneous solutions N_j*. The eqn (8) is the modified Helmholtz equation, for which a T-complete system of homogeneous solution can be expressed, in polar coordinates r and θ, as

$$\mathbf{N} = [I_0(r\sqrt{a})I_1(r\sqrt{a})\cos\theta\, I_1(r\sqrt{a})\sin\theta \ldots, \text{etc.}], \quad (39)$$

where $I_m(\,)$ stands for modified Bessel function of the first kind with order m.

2.2.2. *Particular solution.* The particular solution $\mathring{u}$ of eqn (13), for any right-hand side b, can be obtained by integration of the source (or Green's) function

$$G(P, R) = K_0(r_{PR}\sqrt{a})/2\pi, \quad (40)$$

where P designates the point under consideration, R stands for the source point, $K_0(\,)$ is the modified Bessel function of second kind with zero order and

$$r_{PR} = \sqrt{(x_R - x_P)^2 + (y_R - y_P)^2}.$$

Table 1. Example 1 with 5×5 elements over symmetric quadrant: distribution of temperature along $y = 1.5$ at $t = 1.2$ h

	x	0.300	0.600	0.900	1.20	1.500
Ref. [8]	$\Delta t = 0.05$ h	0.599	1.139	1.568	1.843	1.938
	$\Delta t = 0.10$ h	0.652	1.239	1.706	2.005	2.108
Present HT FE	$\Delta t = 0.05$ h	0.578	1.103	1.518	1.797	1.894
	$\Delta t = 0.10$ h	0.588	1.126	1.553	1.835	1.929
Exact		0.560	1.065	1.466	1.723	1.812

Table 2. Temperature vs N (mesh density) and Δt ($\{x, y, t\} = \{1.5, 1.5, 1.2\text{ h}\}$), for HT-element results of example 1

N	5	10	15
$\Delta t = 0.01$ h	1.852	1.821	1.816
$\Delta t = 0.05$ h	1.894	1.848	1.829
$\Delta t = 0.10$ h	1.929	1.891	1.874
$\Delta t = 0.15$ h	2.012	1.943	1.902
Exact		1.812	

As a consequence the particular solution $\mathring{u}$ of eqn (13) can be expressed as

$$\mathring{u}(P) = \frac{1}{2\pi}\int_{\Omega^e} b(R)K_0(r_{PR}\sqrt{a})\,\mathrm{d}\Omega(R). \tag{41}$$

The area integration in eqn (41) will be performed by numerical quadrature using the Gauss–Legendre rule.

3. NUMERICAL EXAMPLES

Since the main purpose of this paper is to outline the basic principles of the proposed method, the assessment has been limited to two simple examples. The domain in these two examples is a square

$$\Omega = \{(x, y), 0 < x < L, 0 < y < L\}$$

with

$$\rho c = 1, K = 1.25, L = 3, Q = h = 0$$

as in Bruch and Zyvoloski [8], who have used the prismatic elements in a space–time domain and based the solution on a weighted residual procedure. For comparison, we adopt the same space discretization and the same DOF (one DOF at element corners) as in Ref. [8]. Furthermore, the linear interpolation of the time variable used in Ref. [8] corresponds to the single-step method of eqns (8) and (11). In eqn (12), $m = 6$.

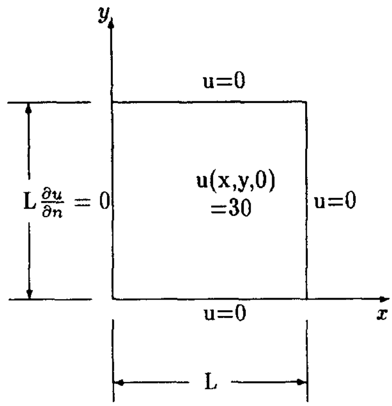

Fig. 3. Boundary and initial conditions for example 2.

In the first example, the governing eqn (8) is to be solved, subject to the following boundary and initial conditions (Fig. 2).

$$u(0, y, t) = u(x, 0, t) = u(L, y, t) = u(x, L, t) = 0,$$

$$u_0(x, y) = 30.$$

The analytical solution of this problem is

$$u(x, y, t) = \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} A_n \sin\frac{n\pi x}{L}\sin\frac{j\pi y}{L} \times \exp\left[\frac{-K\pi^2(n^2 + j^2)t}{L^2}\right]$$

Table 3. Example 1: comparison of u for distorted (e and z as shown in Fig. 4) and undistorted 5 × 5 HT-element meshes over the symmetric quadrant ($y = 1.5$, $t = 1.2$ h, $\Delta t = 0.05$ h)

x	0.300	0.600	0.900	1.200	1.500
Distorted for $e = 0.4z$	0.580	1.107	1.527	1.809	1.911
Distorted for $e = 0.3z$	0.585	1.113	1.538	1.821	1.925
Exact	0.560	1.065	1.466	1.723	1.812

Table 4. Example 2 with 10 × 5 elements over symmetric half of domain: distribution of temperature along $x = 1.5$ at $t = 1.2$ h

	y	0.300	0.600	0.900	1.20	1.500
Ref. [8]	$\Delta t = 0.05$ h	1.418	2.697	3.713	4.364	4.589
	$\Delta t = 0.10$ h	1.480	2.815	3.874	4.554	4.788
Present HT FE	$\Delta t = 0.05$ h	1.393	2.651	3.649	4.298	4.524
	$\Delta t = 0.10$ h	1.415	2.678	3.685	4.324	4.570
	Exact	1.377	2.618	3.604	4.237	4.455

Table 5. Temperature vs N and Δt ($\{x, y, t\} = \{1.5, 1.5, 1.2\,\text{h}\}$), for HT-element results of example 2

N	5	10	15
$\Delta t = 0.01$ h	4.502	4.475	4.461
$\Delta t = 0.05$ h	4.524	4.491	4.473
$\Delta t = 0.10$ h	4.570	4.512	4.484
$\Delta t = 0.15$ h	4.641	4.575	4.509
Exact		4.455	

where

$$A_n = 4 \times 30 \times \frac{[(-1)^n - 1][(-1)^j - 1]}{nj\pi^2}.$$

Owing to the symmetry of the problem, only one quadrant of the solution domain has been discretized by $N \times N$ HT elements with linear frame functions. Some results obtained by the approach are displayed in Tables 1–3 along with the results of Ref. [8].

The second example differs from the previous one only by the boundary conditions, which now read (Fig. 3)

$$u(L, y, t) = u(x, 0, t) = u(x, L, t) = 0,$$

$$\partial u(0, y, t) \partial x = 0.$$

The analytical solution of this problem is equal to

$$u(x, y, t) = \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} B_n \sin\frac{(2n-1)\pi x}{2L} \sin\frac{j\pi y}{L} \times \exp\left[\frac{-K\pi^2((n-0.5)^2 + j^2)t}{L^2}\right],$$

where

$$B_n = 8 \times 30 \times (-1)^{n+2} \frac{(-1)^j - 1}{j\pi^2(2n-1)}.$$

Taking into account the symmetry about $y = 1.5$, only one half of the domain has been discretized by

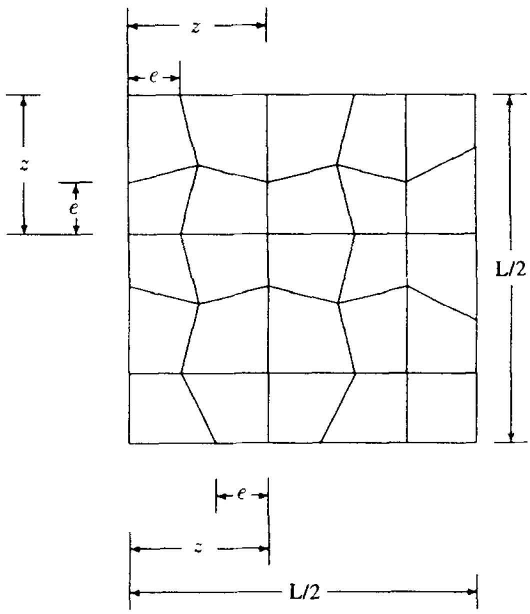

Fig. 4. Distorted mesh in Examples 1 and 2. $z = 0.2\,L$.

using a regular $2N \times N$ mesh of HT elements. As in example 1, Table 4 gives the predicted temperature along $x = 1.5$ at $t = 1.2$ h and compares it with that of Ref. [8]. Table 5 shows the variation of the results as a function of the mesh density N and time step Δt. Table 6 exhibits the study of sensitivity to the mesh distortion.

The results displayed in tables 1–6 show that, for the same element size, time step Δt and number of DOF as in Ref. [8], the results are more accurate. Furthermore, the results exhibit remarkable insensitivity to the mesh distortion.

4. CONCLUSIONS

A HT finite element model has been presented for analysis of transient heat conduction. Although the proposed formulation and the numerical examples have been confined to a single step scheme and simple quadrilateral element with a single DOF at corner nodes, some extensions are possible. For example, the extension to a multistep algorithm for time discretization and to a HT-p element form for space discretization is under way.

Table 6. Example 2: comparison of u for distorted (e and z as shown in Fig. 4) and undistorted 10×5 HT-element meshes over the symmetric half of domain ($x = 1.5$, $t = 1.2$ h, $\Delta t = 0.05$ h)

y	0.300	0.600	0.900	1.200	1.500
Distorted for $e = 0.4z$	1.398	2.662	3.668	4.310	4.539
Distorted for $e = 0.3z$	1.404	2.670	3.678	4.321	4.550
Exact	1.377	2.618	3.604	4.237	4.455

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