A thermo-viscoelastic analysis using the eigenvector expansion method

W.X. Zhang, a,∗ Q.H. Qin, a,b F. Yuan a

a School of Civil Engineering and Architecture, Henan University of Technology, Zhengzhou 450052, China
b Research School of Engineering, Australian National University, Canberra, ACT 2601, Australia

A R T I C L E   I N F O

Article history:
Received 7 February 2014
Received in revised form 11 March 2014
Accepted 24 March 2014
Available online 31 March 2014

Keywords:
Eigenvector
Viscoelastic
Symplectic

A B S T R A C T

A novel eigenvector expansion method under the symplectic system is introduced for boundary condition problems of thermo-viscoelastic materials. On the basis of the state space formalism and the use of the Laplace integral transform, the general solutions of the governing equations, zero and non-zero-eigenvalue eigenvectors, are obtained analytically. Since the eigenvectors are expressed in concise analytical forms, the adjoint symplectic relation in the Laplace domain is generalized to the time domain. Using this method, various boundary conditions and the particular solution of non-homogeneous governing equations can be conveniently described by combinations of the eigenvectors.

© 2014 Elsevier Ltd. All rights reserved.

1. Introduction

Many materials, such as fibrous polymeric composites and engineering polymers, exhibit a time dependent behaviour which can be modelled in the framework of viscoelasticity. Since the time-dependent behaviour of the materials is known to be very sensitive to changes of temperature, thermo-viscoelastic problems have gained much attention in the recent years (Khan et al., 2006). However, it is difficult to find analytical solutions because of the complexity of the constitutive relations. Therefore, the numerical method is taken into account, especially the finite element method. Cao and Chen (2012) investigated the indentation of viscoelastic composites, and performed a theoretical analysis which is followed by finite element analysis. Levin and Markov (2011) employed a self-consistent scheme named the effective field method for the calculation of the velocities and quality factors of elastic waves propagating in double-porosity media, and analyzed the effective viscoelastic properties of the matrix with the primary small-scale pores. The other important approach is the boundary element method. Compared with the finite element method, this method can reduce the dimension of the problem and provide an attractive idea for the thermo-viscoelastic research. Zhang et al. (2010) discussed crack problems in linear viscoelastic materials by generalizing the Heaviside function to represent the displacement discontinuity across the crack surface. Based on differential constitutive relations, Mesquita and Coda (2007) provided the important algebraic equations and presented a method for the treatment of two dimensional coupling problems between the finite element method and the boundary element method by discussing Kelvin and Boltzmann models in a two-dimensional boundary element atmosphere. It should be pointed out that the Laplace transform is an effective method for viscoelasticity since the problem can be transformed from the time domain to the Laplace domain (Schanz et al., 2005). The mathematical integral transform method is applied in most of the above mentioned researches. However, it is difficult and important to solve Laplace inverse transforms. A lot of inverse transforms cannot be solved analytically, therefore the numerical method of Laplace inverse transformation had to be developed and applied (Syngellakis, 2003).

Taking original variables (displacements) and their dual variables as the basic variables, Zhong (2004) developed the symplectic approach for elasticity problems on the basis of the mathematical theory of symplectic geometry. In contrast to the well-known semi-inverse method, the symplectic method takes original variables and their dual variables as the basic variables in the discussion of the governing equations (Lim et al., 2009; Xu et al., 2008; Wang and Qin, 2007). However, this method cannot be applied directly into thermo-viscoelastic problems because of the energy non-conservation property of the materials. Noticing that, by using the Laplace integral transform, viscoelastic constitutive equations can be transformed into a set of corresponding elastic ones, the symplectic method can be introduced into viscoelastic researches (Zhang et al., 2012). In this paper, the novel symplectic method is
further extended to the deformation of thermo-viscoelastic solids under certain boundary conditions. Since the general solutions in the Laplace domain are expressed in concise analytical form, their corresponding expressions in the time domain can be obtained easily. Using this method, the particular solution and the eigenvector expansion method can be discussed directly in the eigenvector space of the time domain or in the Laplace domain. Accordingly, various boundary conditions and the particular solution of non-homogeneous governing equations can be conveniently described by combinations of the eigenvectors. The numerical results we have obtained are proved to be accurate enough because the results agree with the exact ones.

2. Governing equations in the Hamiltonian system

Consider a homogeneous viscoelastic strip plane-domain shown in Fig. 1 in the Cartesian coordinates \((x, z)\). The origin is located at the central point of the domain, and the width and the length are \(2h\) and \(2l\), respectively. The strain energy density in the Laplace domain is (Zhong, 2004)

\[
\bar{L} = \frac{\bar{E}}{4(1 + \nu)(1 - 2\nu)} \left\{ 2(1 - \nu)\left( \partial_x \bar{\varepsilon}^2 + \bar{\varepsilon} \right) + 4\nu(1 - \nu)\partial_z \bar{\varepsilon} \bar{\varepsilon} + (1 - 2\nu)\left( \partial_x \bar{\varepsilon}^2 + \bar{\varepsilon} \right) \right\} - \theta T (\bar{\varepsilon} + \partial_x \bar{\varepsilon})
\]

where a bar over a variable represents its Laplace transform, \(\bar{\varepsilon}_x = \partial_x \bar{\varepsilon}_x, \bar{\varepsilon}_z = \partial_z \bar{\varepsilon}_z\), \(\bar{\varepsilon} = \bar{\varepsilon}_x + \bar{\varepsilon}_z\) are displacements along \(x\) and \(z\) directions respectively, \(\bar{E}\) and \(\bar{\varepsilon}\) are Young’s modulus and Poisson’s ratio of the material, \(T\) is the temperature, \(\theta\) is the coefficient of the thermostress, and an overdot on a variable denotes its differentiation with respect to the coordinate \(z\).

In the Laplace domain, the viscoelastic stress–strain relations can be expressed as

\[
\bar{s}_{ij}(r, s) = 2\bar{G}(s)\bar{\varepsilon}_{ij}(r, s)
\]

\[
\bar{\sigma}_{mn}(r, s) = 3\bar{K}(s)\bar{\varepsilon}_{mn}(r, s) - \theta T
\]

where \(s\) is the Laplace transform parameter, \(r\) is the position vector, \(\bar{G}(s)\) and \(\bar{K}(s)\) are relaxation functions of shear modulus and bulk modulus, \(\bar{\sigma}_{mn}\) and \(\bar{\varepsilon}_{mn}\) are volumetric stress and strain, and the deviatoric components of the stress and strain tensors \(\bar{s}_{ij} = \bar{\sigma}_{ij} - \delta_{ij} \bar{\sigma}_{nn}/3\) and \(\bar{\varepsilon}_{ij} = \bar{\varepsilon}_{ij} - \delta_{ij} \bar{\varepsilon}_{nn}/3\). The Young’s modulus of \(\bar{E}(s)\) and Poisson’s ratio \(\bar{\varepsilon}(s)\) can be obtained by the relaxation moduli \(\bar{G}(s)\) and \(\bar{K}(s)\). In the differential approach, viscoelastic constitutive models are constructed by various combinations of linear springs and dashpots. Fundamental models include the Kelvin type and the Maxwell type. Certainly, a very comprehensive and accurate model may be constructed by combining a generalized Maxwell model and a generalized Kelvin model. Irrespective of how we combine springs and dashpots, one can easily write their relaxation moduli in analytical forms with the use of Laplace transform. For the convenience of discussion, we only discuss the generalized Maxwell model (shown in Fig. 2) in this paper. Its Young’s modulus can be written as

\[
\bar{E}(s) = \sum_i \frac{E_i}{s + \bar{E}_i/\eta_i}
\]

In order to reduce the complexity of the mathematical derivations, we assume that the Poisson’s ratio is independent of time. Write the displacements in vector form

\[
\bar{q} = [\bar{u} \bar{v}]^T
\]

The dual vector of the displacement vector can be obtained by

\[
\bar{p} = \bar{\partial}_q \bar{L} = \begin{bmatrix} \bar{\partial}_z \bar{\varepsilon} \bar{u} + \frac{\bar{E}}{2(1 + \nu)}(\bar{u} + \bar{\varepsilon}) \\ \bar{\partial}_z \bar{\varepsilon} \bar{v} + \frac{\bar{E}}{2(1 + \nu)}(\bar{v} + \bar{\varepsilon}) \end{bmatrix}
\]

Using the principle of minimum total potential energy:

\[
\delta \int_{-l}^{l} \int_{-h}^{h} \bar{L} dz dx = 0
\]

we get the governing equations in the Hamiltonian system

\[
\bar{\psi} = \bar{H}\bar{\psi} + \bar{\Phi}
\]

where

\[
\bar{H} = \begin{bmatrix} 0 & -\beta_3 \bar{\varepsilon} & \beta_2 & 0 \\ -\beta_1 & 0 & 0 & \beta_1 \\ 0 & 0 & 0 & -\partial_x T \\ 0 & -\beta_2 \bar{\varepsilon} & -\beta_3 \bar{\varepsilon} & 0 \end{bmatrix}, \quad \bar{\Phi} = \begin{bmatrix} \beta_2 \partial_T \\ 0 \\ 0 \\ \partial_x T \end{bmatrix}
\]

in which \(\beta_1 = 2(1 + \nu)/\bar{E}, \quad \beta_2 = (1 + \nu)(1 - 2\nu)/\bar{E}(1 - \nu), \quad \beta_3 = \nu/(1 - \nu), \quad \beta_4 = \bar{E}/(1 - \nu^2).

3. General solutions

Applying the method of separation of variables, we write the solution as

\[
\hat{\psi}(z, x) = \hat{\phi}(z)\hat{\psi}(x)
\]

Substituting Eq. (9) into the corresponding homogeneous form of Eq. (7) gives \(\hat{\phi}(z) = e^{\mu z}\), and

\[
\bar{H}\hat{\psi}(x) = \mu \hat{\psi}(x)
\]
where $\mu$ is the eigenvalue, and $\varphi$ is the eigenvector. The characteristic equation is

$$
\begin{vmatrix}
-\mu - \nu \lambda & (1 - \nu^2) / \bar{E} & 0 \\
-\lambda & -\mu & 2(1 + \nu) / \bar{E} \\
0 & 0 & -\mu \\
0 & -\bar{E} \lambda^2 & -\nu \lambda & -\mu
\end{vmatrix} = 0
$$

(11)

The solutions are

$$
\lambda_{1,2} = \pm (1 + \kappa)i, \quad \lambda_{3,4} = \pm (1 - \kappa)i
$$

(12)

Since four different characteristic values are obtained, the general solution can be expressed as

$$
\varphi = \begin{bmatrix}
\bar{A}_1 \cos(\mu x) + \bar{B}_1 \sin(\mu x) + \bar{C}_1 \sin(\mu x) + \bar{D}_1 x \cos(\mu x) \\
\bar{A}_2 \sin(\mu x) + \bar{B}_2 x \cos(\mu x) + \bar{C}_2 \cos(\mu x) + \bar{D}_2 x \sin(\mu x) \\
\bar{A}_3 \cos(\mu x) + \bar{B}_3 x \sin(\mu x) + \bar{C}_3 \sin(\mu x) + \bar{D}_3 x \cos(\mu x) \\
\bar{A}_4 \sin(\mu x) + \bar{B}_4 x \cos(\mu x) + \bar{C}_4 \cos(\mu x) + \bar{D}_4 x \sin(\mu x)
\end{bmatrix}
$$

(13)

For brevity, we divide the solutions into two groups: symmetric solutions and anti-symmetric solutions. The symmetrical solutions are

$$
\varphi_s = \begin{bmatrix}
\bar{c}_{11} \sin(\mu x) + \bar{c}_{12} x \sin(\mu x) \\
\bar{c}_{21} \sin(\mu x) + \bar{c}_{22} x \cos(\mu x) \\
\bar{c}_{31} \cos(\mu x) + \bar{c}_{32} x \sin(\mu x) \\
\bar{c}_{41} \sin(\mu x) + \bar{c}_{42} x \cos(\mu x)
\end{bmatrix}
$$

(14)

Substituting Eq. (14) into Eq. (10), we get the constants:

$$
\bar{c}_{11} = -[\cos^2(\mu h) - (1 - \nu)(1 + \nu)], \quad \bar{c}_{12} = \bar{c}_{21} = \mu, \quad \bar{c}_{22} = -\bar{E} \mu [1 + \cos^2(\mu h)] / (1 + \nu),
$$

$$
\bar{c}_{31} = -[\cos^2(\mu h) - (1 + \nu) / (1 + \nu)], \quad \bar{c}_{32} = \bar{E} \mu^2 / (1 + \nu), \quad \bar{c}_{41} = \bar{E} \mu \cos^2(\mu h) / (1 + \nu). \quad \text{The anti-symmetrical one is}
$$

$$
\varphi_a = \begin{bmatrix}
\bar{e}_{11} \sin(\mu x) + \bar{e}_{12} x \sin(\mu x) \\
\bar{e}_{21} \cos(\mu x) + \bar{e}_{22} x \cos(\mu x) \\
\bar{e}_{31} \sin(\mu x) + \bar{e}_{32} x \sin(\mu x) \\
\bar{e}_{41} \cos(\mu x) + \bar{e}_{42} x \sin(\mu x)
\end{bmatrix}
$$

(15)

in which $\bar{e}_{11} = -\sin^2(\mu h) - (1 - \nu) / (1 + \nu)$, $\bar{e}_{12} = \bar{e}_{22} = -\mu$, $\bar{e}_{21} = -\sin^2(\mu h) + 2(1 + \nu)$, $\bar{e}_{21} = -\bar{E} \mu [1 + \sin^2(\mu h)] / (1 + \nu)$, $\bar{e}_{32} = \bar{E} \mu^2 / (1 + \nu)$, $\bar{e}_{41} = -\bar{E} \mu \sin^2(\mu h) / (1 + \nu)$. For the case of $\mu = 0$, it is not difficult to find the solutions by solving the governing equations directly. For brevity, these solutions are displayed as

$$
\begin{bmatrix}
\varphi_{T1} \\
\varphi_{T2} \\
\varphi_{T3} \\
\varphi_{T4} \\
\varphi_{T5} \\
\varphi_{T6}
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
-\kappa & 0 & 0 & 0 \\
0 & -\nu y & \bar{E} & 0 \\
0 & \nu x^2 & -2\bar{E}x & 0 \\
(u + 2)\kappa^3 - 6(1 + \nu)h^2 x & 0 & 0 & 3\bar{E}(x^2 - h^2)
\end{bmatrix}
$$

(16)

Eq. (13) is the solution for the conventional Saint-Venant problem. Generally, the traditional tension, pure bending and shear bending problems can be constituted by the combination of the eigenvectors.

### 4. Boundary conditions

It is well known that boundary conditions can be displacement conditions or stress conditions, and they also can be the mixed conditions of displacement and stress. In the symplectic system, the boundary conditions just correspond with the fundamental variables, which makes it convenient in the discussion of boundary condition problems.

#### 4.1. Lateral boundary conditions

Suppose lateral boundary conditions are

$$
\bar{\sigma}_x = \beta_3 \bar{\sigma}_x + \beta_4 \bar{\sigma}_n \bar{v} + (\beta_3 - 1) \bar{\theta} = \bar{\sigma}_0^T
$$

(17)

$$
\bar{\tau}_{xy} = \bar{\tau}_0^T
$$

(18)

in order to homogenize boundary condition (17), let

$$
\bar{\psi} = \frac{1}{2\pi} \begin{bmatrix} 0 \\
0 \\
\beta_5 [\bar{\sigma}_0^T(h + x) + \bar{\sigma}_0^T(h - x)] + \beta_6 \bar{T} \\
\bar{T}_0^T(h + x) + \bar{T}_0^T(h - x)
\end{bmatrix}
$$

(19)

in which $\beta_5 = 1 / \beta_3$, $\beta_6 = (1 - 2\nu) / \nu$. By introduce the new variables $\bar{\psi}^* = \bar{\psi} - \bar{\psi}_0$, Eq. (17) is transformed into homogeneous one. However, the governing equations should be rewritten as

$$
\bar{\psi}^* = H \bar{\psi}^0 + \bar{f}^*
$$

(20)

in which the inhomogeneous term $\bar{f}^* = f + H \bar{\psi}^0 - \bar{\psi}_0^*$. Thus, the problem to be solved is finding a particular solution of Eq. (19). For this purpose, we define the integral product:

$$
\langle \bar{\psi}_1, \bar{\psi}_2 \rangle = \int_{-h}^{h} \bar{\psi}_1 \bar{J} \bar{\psi}_2 dx
$$

(21)

where $\bar{\psi}_1$ and $\bar{\psi}_2$ are arbitrary eigenvectors, and $\bar{J}$ is a unit rotational matrix. The eigenvectors satisfy the adjoint symplectic relationships:

$$
\langle \bar{\psi}_1, \bar{\psi}_2 \rangle = -\langle \bar{\psi}_1, \bar{\psi}_2 \rangle = \delta_{ij}
$$

(22)

$$
\langle \bar{\psi}_1, \bar{\psi}_2 \rangle = \langle \bar{\psi}_1, \bar{\psi}_2 \rangle = 0
$$

where $\bar{\psi}_1 = \bar{\psi}(\mu = \mu_i) \bar{\psi}_2 = \bar{\psi}(\mu = -\mu_i)$. Accordingly, an arbitrary solution vector can always be expanded by the combination of these eigenvectors

$$
\bar{\psi} = \sum_n (\bar{u}_n \bar{\psi}_n + \bar{\psi}_n \bar{\psi}_n)
$$

(23)

The coefficients are determined by the adjoint symplectic relationships. As a result, they are

$$
\bar{a}_{n} = \pm \langle \bar{\psi}_n, \bar{\psi}_{\pm n} \rangle e^{\pm \bar{u}_{n} x}
$$

(24)

It should be mentioned that the Laplace domain adjoint symplectic relationships can be easily generalized to the time domain and the eigenvectors are expressed in concise analytical forms. The nonhomogeneous term and the particular solution of Eq. (16) can be developed as

$$
\bar{f}^* = \sum_n (\bar{c}_{n} \bar{\psi}_n + \bar{\psi}_n \bar{\psi}_n)
$$

(25)
and
\[ \Psi_p = \sum_n \left( \hat{d}_{an} \Phi_n + \hat{d}_{an} \Phi_{-n} \right) \]  
(25)
respectively, in which the coefficients \( \hat{c}_{an} \) can be obtained from Eq. (23), and \( \hat{d}_{an} \) are coefficients to be determined. Substituting Eqs. (24) and (25) into Eq. (19), we get differential equations about the coefficients
\[ \hat{d}_{an} \bar{\Phi}_n(x) = \pm \mu n \hat{d}_{an} \Phi_n(x) + \hat{c}_{an} \Phi_n(x) \]  
(26)
The solutions are
\[ \hat{d}_{an} = \int_0^2 \hat{c}_{an} e^{\pm \mu n(z-x)} \, dx \]  
(27)

4.2. End conditions

Consider the boundary conditions at the ends:
\[ \mathbf{q}_0 = [u_0 \, v_0]^T \quad (x = 0) \]  
(28)
\[ \mathbf{p}_0 = [\sigma_{0x} \, \sigma_{0y}]^T \quad (x = l) \]  
(29)
The complete solution is composed of the general solution and a particular solution
\[ \Psi = \Psi_\sigma + \Psi_p \]  
(30)
Here, only the general solution \( \Psi_\sigma \) is unknown. Write the complete solution as
\[ \Psi = \sum_n \left( a_n \Phi_n + a_{-n} \Phi_{-n} \right) + \Psi_p \]  
(31)
where \( a_n \) are coefficients to be determined. Based on Eq. (30), the end condition (28) can be described by the eigenvectors as
\[ \mathbf{q}_0 = \sum_n \left( a_n \mathbf{q}_n + a_{-n} \mathbf{q}_{-n} \right) + \mathbf{q}_p \quad (x = 0) \]  
(32)
\[ \mathbf{p}_0 = \sum_n \left( a_n \mathbf{p}_n + a_{-n} \mathbf{p}_{-n} \right) + \mathbf{p}_p \quad (x = l) \]  
(33)
where \( \mathbf{q}_p \) and \( \mathbf{p}_p \) are the displacement component and the stress displacement respectively. Due to the adjoint symplectic relationships between the eigenvectors and the completeness of the eigenvector space, we obtain
\[ \int_0^x \mathbf{q}_0 \cdot \mathbf{p}_j \, dx = \int_0^x \sum_{n=0}^{\infty} \left[ a_n \mathbf{q}_n + a_{-n} \mathbf{q}_{-n} \right] \cdot \mathbf{p}_j \, dx \]  
(34)
\[ \int_0^x \mathbf{p}_0 \cdot \mathbf{q}_j \, dx = \int_0^x \sum_{n=0}^{\infty} \left[ a_n \mathbf{p}_n + a_{-n} \mathbf{p}_{-n} \right] \cdot \mathbf{q}_j \, dx \]  
(35)

In numerical calculations, we can take \( N \) terms in the eigenvector expansion. Accordingly \( 2N \) algebraic equations including \( 2N \) unknown coefficients are established.

5. Numerical calculations

In this section, we consider the generalized Maxwell viscoelastic model including four spring-dashpot analogues. The moduli of the materials are selected as: \( E_1 = E_2 = E_3 = E_4 = E \), \( E_1 = E_2 = E_3 = E_4 = \eta_1 = \eta_2 = \eta_3 = \eta_4 = \eta \). Here we let all the moduli be equal to simplify the mathematical derivation. For the cases when they are not equal, the eigenvector expansion method also can be applied directly in the time domain without any more difficulties. The reason is that the time domain moduli and the general solutions of governing equations can be obtained easily by using the inverse Laplace transform.

5.1. Stress and displacement boundary conditions

Suppose both of ends are subjected to external stress boundary conditions
\[ \sigma_z = \frac{x^2}{h^2}, \quad \tau_{zx} = 0 \quad (z = \pm l) \]  
(36)
The geometrical data is taken to be \( l = h \). Using the eigenvector expansion method, the solution is obtained numerically. According to Saint-Venant principle, the stress effects should be confined in the regain very near the boundary, and cannot be transferred far away. This point of view is well verified by Fig. 3. It can be seen from the figure that the stress boundary effect is seriously confined to the region near the loaded ends, and falls off at a rapid rate along the coordinate \( z \). At the location about \( z/h = 0.3 \), the curves turn to be nearly horizontal. The figure also shows a good agreement between the numerical result and the exact analytical ones at the free end \( z/h = 1 \), which demonstrates that the numerical result is enough accurate.

5.2. Mixed boundary conditions

In this section, the geometrical data is taken to be \( l = h = 5 \), and the temperature conditions are supposed as:
\[ T = \frac{x^2}{h^2} + 1 \quad (x = -l) \]  
(37)
\[ \phi = -\kappa x T = 1 - \frac{x^2}{h^2} \quad (x = l) \]  
(38)
where \( \phi \) is the heat flux function along coordinate \( z \), and \( \kappa \) is the coefficient of heat conduction. The heat flux function in \( x \) direction along the lateral boundary is supposed to be adiabatic (\( \partial_x T = 0 \)). According to the heat conduction equations, the temperature field can be calculated. The boundary conditions at both ends are given as
\[ \sigma_z = p, \quad \tau_{zx} = 0 \quad (z = -l) \]  
(39)
\[ \sigma_z = p, \quad \tau_{zx} = 0 \quad (z = l) \]  
(40)
Figs. 4 and 5 show the distributions of the normal strain \( \varepsilon_z/\eta \) and the shear strain \( \varepsilon_{zx}/\eta \) at the initial time respectively. In Fig. 4, the curves are almost straight parallel the lines in the region far from the end boundary, which indicates that the local effect of normal strain only appear near the ends due to the restraints of
6. Conclusion

Thermo-viscoelastic problems belong to non-conservative system because of the existence of the energy consumption. However, we can discuss the problems in the Laplace domain in which the symplectic method is applicable. Using this method, all the general solutions of the governing equations, including zero and non-zero-eigenvalue eigenvectors, are obtained analytically. These general solutions which satisfy the adjoint symplectic relationships are expressed in concise analytical form, and therefore the time domain expressions are not difficult to be found. Moreover, the problems of non-homogeneous lateral boundary conditions, the end boundary conditions, and the non-homogeneous governing equations are discussed in detail with the use of the eigenvector expansion method.

Acknowledgement

The support of National Natural Science Foundation of China (No. 11372100) is gratefully acknowledged.

References