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A VARIATIONAL PRINCIPLE AND HYBRID TREFFTZ FINITE ELEMENT FOR THE ANALYSIS OF REISSNER PLATES

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Abstract—A modified variational functional for the analysis of Reissner plates with the Trefftz method is presented. The existence theorem of the variational solution for the functional has also been deduced. It is found that the new functional is nonconvex. Finally a hybrid Trefftz finite element formulation is developed based on the functional. Two numerical examples of rectangular plate are considered and their results are in good agreement with the known ones.

NOTATION

- C — $5Eh/12(1+\nu)$, a transverse shear rigidity
- C_w — a part of boundary $\partial\Omega$ of the solution domain Ω, on which deflection w is prescribed; C_{M_n}, C_Q, etc. are defined similarly
- D — $Eh^3/12(1-\nu^2)$, the plate flexural rigidity
- E — modulus of elasticity
- h — thickness of plate
- M_{ii} — bending moment
- M_{ij} — twisting moment $(i \neq j)$
- n_i — component of the outward normal to the boundary $\partial\Omega$
- q — lateral distributed load
- Q_i — transverse shear force
- r — $(x^2+y^2)^{1/2}$
- s_i — component of the tangent to boundary $\partial\Omega$
- U — $0.5[b_{ijkl}\psi_{i,j}\psi_{k,l}+f_{ij}(w_{,i}-\psi_i)(w_{,j}-\psi_j)]$, strain energy density
- w — lateral deflection
- δ — variational symbol
- δ_{ij} — Kronecker delta
- θ — $\arctan(y/x)$
- λ — $\sqrt{10}/h$
- ν — Poisson's ratio
- ψ_i — average rotations of the normals to the plate mid-surface
- $(\bar{\ })$ — over a symbol denotes prescribed value

1. INTRODUCTION

Elastic moderately thick plates occur in numerous technological applications, and the solution of plate bending problems has received a great deal of attention. The application of the finite element method to the analysis of moderately thick plates can be traced back to the late 1960s [1]. Reddy studied the Mindlin plate elements using the Lagrange multiplier and penalty method [2], Hinton and Huang [3] investigated Mindlin plate elements with substitute shear strain fields. Later, Qin [4] and Jin and Qin [5] proposed a family of variational functional and related hybrid and boundary element models which are suitable for the nonlinear analysis of Reissner plates. An extensive list of references on the subject may also be found in Ref. [6].

On the other hand, the applications of the hybrid Trefftz finite element (HTFE) can be traced to the late 1970s [7]. The HT approach has been applied to plane elasticity [8], to isotropic [9] and orthotropic [10] Kirchoff plates, thin shells with constant principal curvatures [11] and to Poisson's equation [12, 13].

The present study deals with the derivation to analyze the bending problems of Reissner plates with the Trefftz method. To this end, a modified variational functional is proposed, and the sufficient condition for the local extreme of the functional is derived. A quadrilateral HTFE model is established based on the functional. The numerical results demonstrate good performance of the proposed techniques.

2. VARIATIONAL PRINCIPLE

Consider an anisotropic Reissner plate of thickness h, occupying a two-dimensional arbitrary shaped region Ω, bounded by a curve boundary $\partial\Omega$. Indices i and j take values in the range $(1, 2)$. The bending behavior of the plate is governed by the differential equations and boundary conditions as [14]

$$b_{ijkl}\psi_{j,kl} - f_{ij}(\psi_j - w_{,j}) = 0 \quad \text{in } \Omega \tag{1a}$$

$$f_{ij}(\psi_{i,j} - w_{,ij}) = q \quad \text{in } \Omega \tag{1b}$$

$$M_n = M_{ij}n_in_j = \bar{M}_n \quad (\text{on } C_{M_n});$$

$$M_{ns} = M_{ij}n_is_j = \bar{M}_{ns} \quad (\text{on } C_{M_{ns}}) \qquad (2a, b)$$

$$Q_n = Q_in_i = \bar{Q} \quad (\text{on } C_Q) \qquad (2c)$$

$$\psi_n = \psi_in_i = \bar{\psi}_n \quad (\text{on } C_{\psi_n}),$$

$$\psi_s = \psi_is_i = \bar{\psi}_s \quad (\text{on } C_{\psi_s}) \qquad (2d, e)$$

$$w = \bar{w} \quad (\text{on } C_w) \qquad (2f)$$

$$(\partial\Omega = C_{\psi_n} \cup C_{M_n} = C_{\psi_s} \cup C_{M_{ns}} = C_w \cup C_Q),$$

where b_{ijkl} and f_{ij} are the components of the tensor of elasticity defined, for example for isotropic materials, by the following formulae

$$b_{ijkl} = D((1-v)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})/2 + v\delta_{ij}\delta_{kl}),$$

$$i, j, k, l = 1, 2$$

$$f_{ij} = b_{i3j3} = C\delta_{ij}.$$

In the present study, we assume that following constitutive relationships are identically satisfied

$$M_{ij} = -b_{ijkl}\psi_{k,l}, \qquad Q_i = f_{ij}(w_{,j} - \psi_j). \quad (3a, b)$$

For the boundary-value problem (1)–(3), a modified functional used for deriving HTFE model can be, then, given by

$$\Gamma = \iint_\Omega U\,d\Omega + \int_{C_{\psi_n}} M_n\bar{\psi}_n\,dc + \int_{C_{\psi_s}} M_{ns}\bar{\psi}_s\,dc + \int_{C_w} Q_n\bar{w}\,dc - \int_{C_{Mn}} \psi_n(\bar{M}_n - M_n)\,dc - \int_{C_{M_{ns}}} \psi_s(\bar{M}_{ns} - M_{ns})\,dc - \int_{C_Q} w(\bar{Q}_n - Q_n)\,dc \qquad (4)$$

in which we assume that eqns (1) are satisfied in advance.

For the functional (4), we obtain the following.

Theorem 1

If the expression below

$$\iint_\Omega \delta^2 U\,d\Omega + \int_{C_{M_n}} \delta\psi_n\delta M_n\,dc + \int_{C_{M_{ns}}} \delta\psi_s\delta M_{ns}\,dc + \int_{C_Q} \delta w\delta Q_n\,dc \qquad (5)$$

is uniformly positive (or negative) around the exact solution of (1) and (2), one has

$$\Gamma \geqslant \Gamma_0 \quad (\text{or } \Gamma \leqslant \Gamma_0), \qquad (6)$$

where Γ_0 represents the stationary value at (ψ_{i0}, w_0), where ψ_{i0} and w_0 are the exact solution of the boundary-value problem (1), (2), and the equal sign in (6) holds if the arguments ψ_i and w are at the critic point.

Proof

From the first we prove that solution of the boundary-value problem (1)–(3) is the stationary conditions of Γ. In doing this, taking variation of Γ and using (1), we see

$$\delta\Gamma \overset{(1)}{=} -\int_{C_{\psi_n}} (\psi_n - \bar{\psi}_n)\delta M_n\,dc - \int_{C_{\psi_s}} (\psi_s - \bar{\psi}_s)\delta M_{ns}\,dc - \int_{C_w} (w - \bar{w})\delta Q_n\,dc - \int_{C_{M_n}} (\bar{M}_n - M_n)\delta\psi_n\,dc - \int_{C_{M_{ns}}} (\bar{M}_{ns} - M_{ns})\delta\psi_s\,dc - \int_{C_Q} (\bar{Q}_n - Q_n)\delta w\,dc \qquad (7)$$

$$= 0 \Rightarrow (2), \qquad (8)$$

where the symbol '$\overset{(1)}{=}$' means that the constraint equations (1a, b) are satisfied, *a priori*. Therefore the exact solution of (1) and (2) is the stationary condition of the functional (4).

Second, attention will be focused on proving the property of Γ at the neighborhood of critical point. Taking the variation of $\delta\Gamma$ [eqn (7)] in this connection, we obtain

$$\delta^2\Gamma \overset{(1)}{=} \text{expression (5)}. \qquad (9)$$

Thus the theorem is proved by using the sufficient condition on the existence of local extreme value of a functional [15].

3. FINITE ELEMENT FORMULATION

3.1. *Complete systems*

In the Trefftz methods, one needs to choose from the trial and/or weighted functions in such a manner that the differential equations are satisfied, throughout the domain Ω *a priori*. Usually they are the terms chosen from the complete solutions of the homogeneous differential equations. The complete system of Reissner plate bending equations was obtained by Qin *et al.* [16] through introducing two

Table 1. A suitable match of *i*s and *j*s

k	1	2	3	4	5	6	7	8	9	10	11	etc.
i	1	0	2	3	0	4	5	0	6	7	0	etc.
j	0	1	0	0	2	0	0	3	0	0	4	etc.

In the table: $g_0=0, g_1=r^2, g_2=r^2\cos(2\theta), g_3=r^2\sin(2\theta), \ldots, f_0=0, f_1=I_0(\lambda r), f_2=I_1(\lambda r)\cos\theta, f_3=I_1(\lambda r)\sin\theta, \ldots$.

auxiliary functions, g and f, which have the following relations

$$\psi_x = g_{,x} + f_{,y}, \qquad \psi_y = g_{,y} - f_{,x}. \tag{10}$$

Thus the resulting governing equation for an isotropic Reissner plate becomes [16]

$$\nabla^2 f - \lambda^2 f = 0 \tag{11}$$

$$D\nabla^4 g = q \tag{12}$$

together with the relationships

$$w = g - D\nabla^2 g/C; \qquad \psi_x = g_{,x} + f_{,y};$$

$$\psi_y = g_{,y} - f_{,x}, \tag{13}$$

where eqns (11) and (12) are, respectively, known as the modified Bessel equation and a biharmonic equation. Their homogeneous solutions may be formed by [16]

$$I_k(\lambda r)\cos(k\theta); \qquad I_k(\lambda r)\sin(k\theta), \quad k = 0, 1, 2, \ldots \tag{14}$$

and

$$r^2 Z^k \text{ and } Z^{k+2}, \quad k = 0, 1, 2, \ldots \tag{15}$$

where $r^2 = x^2 + y^2$; $I_k(\cdot)$ is a modified Bessel function of first kind with order k; $Z = x + \mathrm{i}y$ and $\mathrm{i} = \sqrt{-1}$ is an imaginary number.

3.2. *Modification of the basic functional Γ*

As indicated in [7], the basic variational functional Γ cannot directly be used as the basis for a finite element formulation since the approximate displacement fields (ψ_i, w), if defined independently for each element

$$u = \mathring{u} + [N]\{e\} \tag{16}$$

violates the interelement continuity requirements in general, where $u = \{w, \psi_x, \psi_y\}$ and $\mathring{u} = \{\mathring{w}, \mathring{\psi}_x, \mathring{\psi}_y\}$ are a set of particular solutions which satisfy the following relationships:

$$\nabla^4 \mathring{g} = q/D \tag{17a}$$

$$\mathring{w} = \mathring{g} - D\nabla^2 \mathring{g}/C \qquad \mathring{\psi}_x = \mathring{g}_{,x}, \qquad \mathring{\psi}_y = \mathring{g}_{,y} \tag{17b--d}$$

whilst

$$[N]\{e\} = [N_1\, N_2 \cdots N_m]\{e_1\, e_2 \cdots e_m\}^T \tag{18}$$

with

$$N_k = \{g_i - D\nabla^2 g_i/C, \quad g_{i,x} + f_{j,y}, \quad g_{i,y} - f_{j,x}\}^T, \tag{19}$$

where g_i and f_j are chosen from the complete systems (14) and (15).

Generally speaking, indexes *i*s and *j*s in expression (19) may be independently selected. In the paper we match them in the way shown in Table 1.

The advantage of the above-mentioned ordering is to make it easy to preserve the invariment properties of element under the rotation of its coordinate axis when the set is truncated.

To enforce the compatibility (in a variational sense), an additional boundary displacement vector, $\tilde{u} = \{\tilde{w}, \tilde{\psi}_x, \tilde{\psi}_y\}$, is independently assumed along each particular element boundary ∂A_e. Hence, the basic variational functional Γ modified for relaxed element compatibility condition can be given in the form

$$\Gamma^m = \sum_e \left\{ \Gamma_e - \int_{\partial A_e} [M_n \tilde{\psi}_n + M_{ns}\tilde{\psi}_s + Q_n \tilde{w}]\,\mathrm{d}c \right\}. \tag{20}$$

For a particular element j, a conforming frame displacement vector $\tilde{u}_j$ can be expressed in terms of generalized nodal parameters. The customary form is

$$\tilde{u}_j = \tilde{N}_j d_j, \tag{21}$$

where d_j is the vector of generalized nodal displacements of the element j and $\tilde{N}_j$ is a matrix of conventional FE shape functions. For example, along the

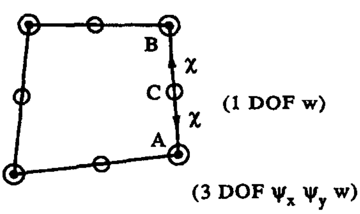

Fig. 1. A typical element j.

Table 2. Uniformly loaded, simply supported square plate

	QUAD4S (ref. [3])		HTFE (present)	
Mesh	β	M_0	β	M_0
2 × 2	0.421	0.477	0.422	0.478
3 × 3	0.425	0.479	0.425	0.479
4 × 4	0.426	0.479	0.427	0.479
Exact	0.427	0.479		

side A–C–B of a typical HT element j (Fig. 1), the frame displacement can be further given in the form

$$\tilde{w} = N_1 w_A + N_2 w_B + N_3 w_c$$

$$\tilde{\psi}_x = N_4 \psi_{xA} + N_5 \psi_{xB}$$

$$\tilde{\psi}_y = N_4 \psi_{yA} + N_5 \psi_{yB}, \qquad (22a–c)$$

where $N_1 = \xi(\xi - 1)/2$, $N_2 = \xi(\xi + 1)/2$, $N_3 = 1 - \xi^2$, $N_4 = (1 - \xi)/2$ and $N_5 = (1 + \xi)/2$.

Finally substituting eqns (16) and (21) into eqn (20), and by taking variation of Γ^m with respect to $\{e\}$, one can express $\{e\}$ in terms of $\{d\}$, therefore Γ^m depends only on variable $\{d\}$. Again for $\partial\Gamma^m/\partial\{d\} = 0$, we obtain

$$\sum K^e d_e = G, \qquad (23)$$

where K^e is the element stiffness matrix and G the equivalent nodal force.

4. NUMERICAL EXAMPLES

Example 1

Consider a simply supported square plate under uniform load q with a span/thickness ratio of $b/h = 10$ and $v = 0.3$, where b is the side length of the plate.

A symmetric quadrant of a uniformly loaded, simply supported, square plate is modeled by a series of meshes of $N \times N$ elements. Table 2 compares results of central deflection coefficient β ($W_c = \beta 10^{-2} qb^4/D$) and the central moment coefficient M_0 ($M = 0.1 M_0 qb^2$) obtained using the HTFE and the approaches given in [3].

Example 2

As the second example we consider a clamped square plate with $v = 0.3$ which is subjected to a uniform load.

With the same way as in Example 1, a quadrant of the plate will be modelled by a series of meshes of $N \times N$ elements for a span/thickness ratio of $b/h = 10$ in the HTFE analysis.

Table 3 shows the results of central deflection coefficient β ($w_c = \beta \times 10^{-2} qb^4/D$) and central moment one M_0 ($M = 0.1 M_0 qb^2$), and comparison is made with those obtained in Ref. [3].

Table 3. Uniformly loaded, clamped supported square plate

	QUAD4S (ref. [3])		HTFE (present)	
Mesh	β	M_0	β	M_0
2 × 2	0.146	0.226	0.147	0.227
3 × 3	0.148	0.241	0.148	0.229
4 × 4	0.149	0.234	0.149	0.230
Exact	0.150		0.231	

5. CONCLUDING REMARKS

This paper presents a modified functional for the analysis of Reissner plates with the Trefftz method and the sufficient condition of the local extreme for the functional is originally given. The study indicates that the functional is nonconvex. The numerical results obtained here show that the HTFE model is very effective for thick plate bending problems.

It should be mentioned that the important advantage of the approach is ease of handling the local effects, such as load-dependent singularity, singular corners, and so on, by using various special-purpose elements, these special elements will be reported in another paper [16].

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