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A NOVEL BOUNDARY-INTEGRAL BASED FINITE ELEMENT METHOD FOR 2D AND 3D THERMO-ELASTICITY PROBLEMS

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In this article a new hybrid boundary integral-based (HBI) finite element method (FEM) is presented for analyzing two-dimensional (2D) and three-dimensional (3D) thermoelastic problems with arbitrary distribution of body force and temperature changes. The method of particular solution is used to decompose the displacement field into homogeneous part and particular part. The homogeneous solution is obtained by using the HBI-FEM with fundamental solutions, yet the particular solution related to the body force and temperature change is approximated by radial basis function (RBF). The detailed formulation for both 2D and 3D HBI-FEM for thermoelastic problems are given, and two different approaches for treating the inhomogenous terms are presented and compared. Five numerical examples are presented to demonstrate the accuracy and performance of the proposed method. When compared with the existing analytical solutions or ABAQUS results, it is found that the proposed method works well for thermoelastic problems and also when using a very coarse mesh, results with satisfactory accuracy can be obtained.

Keywords: Body force; Fundamental solution; Hybrid FEM; Particular solution; Radial basis function; Temperature; Thermoelasticity.

INTRODUCTION

Problems of thermoelasticity arise in many practical designs and structures such as the design of steam and gas turbines, jet engines, rocket motors and nuclear reactors. Thermal stresses induced in these structures are one of the important concerns in product design and analysis. A general thermoelastic problem is governed by two time-dependent coupled differential equations: the heat conduction equation and the Navier-Cauchy equation with body force induced by temperature change [1]. In most engineering applications, the coupling term of the heat equation and the inertia term in the Navier-Cauchy equation are generally negligible [1].

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Consequently, most of the analyses employ the uncoupled thermoelasticity theory, which is adopted in this work.

Currently, numerical methods such as the finite element method (FEM) have been widely employed to study thermoelasticity problems [2–6]. Despite many attractive features of FEM, it is still a non-trivial and time consuming task for problems with complicated domains, particularly for three-dimensional (3D) problems. One alternative to alleviate this difficulty is to use the boundary element method (BEM), which requires only boundary discretization rather than the domain discretization [7–9]. However, the treatment of singular or near-singular boundary integrals is usually quite tedious and inefficient, and an extra boundary integral equation is also required to evaluate the interior fields inside the domain. In addition, for multi-domain problems, the implementation of the BEM becomes quite complicated and the nonsymmetrical coefficient matrix of the resulting equations may weaken the advantages of BEM [10–13]. Recently, the combination of fundamental solution (MFS) and dual reciprocity method (DRM) was also utilized to solve two-dimensional thermoelasticity with general body forces [14] and three-dimensional thermoelasticity [15].

On the other hand, hybrid Trefftz FEM (HT-FEM) as an important alternative to traditional FEM, has become popular over the past three decades and has been increasingly used to analyze various engineering problems [16–24]. In contrast to conventional FEM, HT-FEM is based on a hybrid method, which includes the use of an independent auxiliary inter-element frame field defined on element boundary and an independent internal field chosen as a prior satisfying the homogeneous governing differential equations by means of a suitable truncated T-complete function set of homogeneous solutions. The property of nonsingular element boundary integral in HT-FEM, enables us to easily construct an arbitrarily shaped element. However, the terms of truncated T-complete functions should be carefully selected when achieving desired results, and the T-complete functions for some physical problems are difficult to develop [12, 25]. To remove the drawback of HT-FEM, a novel hybrid boundary integral-based FEM (HBI-FEM) was developed for solving two dimensional heat conduction problems in single and multilayer-materials [26, 27] and isotropic and orthotropic elastic problems [28–30]. In this approach, the intra-element field is approximated by a linear combination of the fundamental solutions, analytically satisfying the related governing equations. The independent frame field defined along the element boundary and the corresponding variational functional are employed to guarantee the inter-element continuity, generate the stiffness equation and establish the linkage between the boundary frame field and internal field at the element level. In the HBI-FEM, the domain integrals in the hybrid functional can be directly converted into boundary integrals without any appreciable increase in computational effort, and no singular integrals are involved by locating the source point outside the element of interest and do not overlap with field point during the computation [28]. Moreover, the features of two independent interpolation fields and element boundary integral in the HBI-FEM make the algorithm have potential applications in the aspect of mesh reduction by constructing special-purpose elements such as functionally graded element, hole element, crack element, and so on [23, 31, 32].

In this article, a new solution procedure based on the HBI-FEM is proposed to solve 2D and 3D thermoelastic problems with arbitrary body forces and
temperature changes. The method of particular solution is used to decompose
the displacement solution into two parts: homogeneous solution and particular
solution. The homogeneous solution is obtained by using the HBI-FEM with
elastic fundamental solutions, yet the particular solution associated with the body
force and temperature effects is approximated by using the radial basis function
interpolation. Five numerical examples are presented to demonstrate the accuracy
and versatility of the proposed method. Compared with the existing closed-form
solutions or ABAQUS results, it is shown that even with a very coarse mesh,
relatively accurate results can still be obtained by the proposed method.

The article is organized as follows; first, we review the basic equations of
the thermoelasticity, then describe the methods of particular solution and radial
basis function approximation which are employed to deal with the temperature
change and the body force. Two different approaches are presented to treat this
problem, followed by the detailed derivation of the HBI-FEM formulations. Five
numerical examples are presented to demonstrate the validity and performance of
the approach, and conclusions are presented.

BASIC EQUATIONS FOR THERMOELASTICITY

Consider a finite isotropic material in domain $\Omega$ (see Figure 1), and let $(x_1,
x_2, x_3)$ denote the coordinates in Cartesian coordinate system. The equilibrium
governing equation of the thermoelasticity with the body force $b_i$ is expressed as

$$\sigma_{ij,j} = -b_i$$  (1)

where $\sigma_{ij}$ is the stress tensor, $b_i$ the body force vector and $i, j = 1, 2, 3$. The
generalized thermoelastic stress-strain relations and the generalized kinematical
relation are given as

$$\sigma_{ij,j} = \frac{2G_i}{1-2\nu} \delta_{ij}e_j + 2G e_{ij} - m \delta_{ij}T$$  (2)

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$  (3)

Figure 1 Geometrical definitions and boundary conditions for general 3D problems. (color figure
available online.)
in which $e_{ij}$ is the strain tensor, $u_i$ the displacement vector, $T$ the temperature change, $G$ the shear modulus, $\nu$ the Poisson’s ratio, $\delta_{ij}$ the Kronecker delta, and

$$m = 2G\alpha(1 + \nu)/(1 - 2\nu)$$  \hspace{1cm} (4)

is the thermal constant with $\alpha$ being the coefficient of linear thermal expansion. Substituting Eqs. (2) and (3) into Eq. (1), the equilibrium equations may be rewritten in terms of displacements as

$$Gu_{i,jj} + \frac{G}{1 - 2\nu}u_{j,\beta} = mT_i - b_i$$  \hspace{1cm} (5)

For a well-posed boundary value problem, the following boundary conditions, either displacement or traction boundary condition, are given as

$$u_i = \bar{u}_i \text{ on } \Gamma_u,$$  \hspace{1cm} (6)

$$t_i = \bar{t}_i \text{ on } \Gamma_t,$$  \hspace{1cm} (7)

where $\Gamma_u \cup \Gamma_t = \Gamma$ is the boundary of the solution domain $\Omega$, $\bar{u}_i$ and $\bar{t}_i$ are the prescribed boundary values, and

$$t_i = \sigma_{ij}n_j,$$  \hspace{1cm} (8)

is the boundary traction, in which $n_j$ denotes the boundary outward normal.

**THE METHOD OF PARTICULAR SOLUTION**

It is important to evaluate the inhomogeneous term, $mT_i - b_i$ of Eq. (5). A widely used approach is to evaluate it by employing the method of particular solution [1, 20, 28, 33]. Using superposition principle, we can decompose the displacement $u_i$ into two major parts, the homogeneous solution $u^h_i$ and the particular solution $u^p_i$ as follows

$$u_i = u^h_i + u^p_i$$  \hspace{1cm} (9)

in which the particular solution $u^p_i$ satisfies the governing equation

$$Gu^p_{i,jj} + \frac{G}{1 - 2\nu}u^p_{j,\beta} = mT_i - b_i$$  \hspace{1cm} (10)

but does not necessarily satisfy any boundary conditions. It should be pointed out that its solution is not unique and can be obtained by various numerical techniques. However, the homogeneous solution should satisfy

$$Gu^h_{i,jj} + \frac{G}{1 - 2\nu}u^h_{j,\beta} = 0$$  \hspace{1cm} (11)

with modified boundary conditions

$$u^h_i = \bar{u}_i - u^p_i, \text{ on } \Gamma_u$$  \hspace{1cm} (12)

$$t^h_i = \bar{t}_i + mTn_i - t^p_i, \text{ on } \Gamma_t$$  \hspace{1cm} (13)
It can be seen in Eqs. (12) and (13) that once the particular solution $u_p^i$ is known, the homogeneous solution $u_h^i$ defined in Eqs. (11)–(13) can be determined. In the following two sections, radial basis function approximation is introduced to obtain the particular solution, and the corresponding HBI-FEM is derived for solving the boundary-value problem defined by Eqs. (11)–(13).

**RADIAL BASIS FUNCTION APPROXIMATION**

It is usually impossible to find an analytical solution for Eq. (10) except for some special cases. As a consequence, Radial Basis Functions (RBF) [34, 35] will be used in our analysis to approximate the body force $b_i$ and the temperature field $T$ in order to obtain a particular solution. There are two different ways to implement this approximation: one is to treat body force $b_i$ and the temperature field $T$ separately as done by Tsai [15]. The other is to treat $mT_i - b_i$ as a whole. It is found that the performance of the two approaches is different from one another which will be discussed in the numerical examples presented later.

**Approach 1**

The body force $b_i$ and temperature $T$ are assumed in the following forms

\[ b_i = \sum_{j=1}^{N} \varphi_j i (i = 1, 2 \text{ in } \mathbb{R}^2 \text{ and } i = 1, 2, 3 \text{ in } \mathbb{R}^3) \]  
(14)

and

\[ T = \sum_{j=1}^{N} \beta^j \varphi^j \]  
(15)

where $N$ is the number of interpolation points, $\mathbb{R}^2$ and $\mathbb{R}^3$ denote 2D and 3D real space, respectively, $\varphi^j$ are the basis functions, and $\varphi^i$ and $\beta^i$ are the coefficients to be determined by collocation. Subsequently, the approximate particular solution can be written as follows

\[ u_p^i = \sum_{k=1}^{3} \sum_{j=1}^{N} \varphi^j \Phi_{ik}^j + \sum_{j=1}^{N} \beta^j \Psi_{ik}^j \]  
(16)

where $\Phi_{ik}^j$ and $\Psi_{ik}^j$ is the approximated particular solution kernels. Once the RBF is selected, the problem of finding a particular solution will be reduced to solve the following equations

\[ G \Phi_{ik}^j + \frac{G}{1 - 2v} \Phi_{ik,ki}^j = -\delta_{ii} \varphi \]  
(17)

\[ G \Psi_{ik}^j + \frac{G}{1 - 2v} \Psi_{ik,ki}^j = m \varphi_{,i} \]  
(18)
To solve Eq. (17), the displacement is expressed in terms of the Galerkin-Papkovich vectors as [34, 36]

\[ \Phi_{ik} = \frac{1}{G} F_{ik,mm} - \frac{1}{2G} F_{mk,mi} \]  

(19)

Substituting Eq. (19) into Eq. (17), yields following bi-harmonic equation

\[ \nabla^4 F_{ii} = -\frac{1}{1-v} \delta_{ij} \phi \]  

(20)

If the Spline Type RBF \( \varphi = r^{2n-1} \) is used, we can obtain following solutions

\[ F_{ii} = -\frac{\delta_{ii}}{1 - v (2n + 1)^2 (2n + 3)^2} \left( \varphi^2 \right) \text{ for } n = 1, 2, 3 \ldots \]  

(21)

\[ \Phi_{ii} = A_0 (A_1 \delta_{ii} + A_2 r_{j,j}) \left( \varphi^2 \right) \text{ for } n = 1, 2, 3 \ldots \]  

(22)

\[ A_0 = -\frac{1}{2G(1 - v) (2n + 1)^2 (2n + 3)} \]  

\[ A_1 = 5 + 4n - 2v(2n + 3) \]  

\[ A_2 = -(2n + 1) \]  

(23)

for a two-dimensional problem and

\[ F_{ii} = -\frac{\delta_{ii}}{1 - v (2n + 1)(2n + 2)(2n + 3)(2n + 4)} \left( \varphi^3 \right) \text{ for } n = 1, 2, 3 \ldots \]  

(24)

\[ \Phi_{ii} = A_0 (A_1 \delta_{ii} + A_2 r_{j,j}) \left( \varphi^3 \right) \text{ for } n = 1, 2, 3 \ldots \]  

(25)

\[ A_0 = -\frac{1}{8G(1 - v) (n + 1)(n + 2)(2n + 1)} \]  

\[ A_1 = 7 + 4n - 4v(n + 2) \]  

\[ A_2 = -(2n + 1) \]  

(26)

for a three-dimensional problem, where \( r_j \) represents the Euclidean distance between a field point \((x, y)\) and a given point \((x_j, y_j)\) in the solution domain. The corresponding particular stress solution can be obtained as

\[ S_{ij} = G(\Phi_{ii,j} + \Phi_{ij,i}) + \lambda \delta_{ij} \Phi_{ik,k} \]  

(27)

where \( \lambda = \frac{2n}{3 - 2v} G \). Substituting Eq. (25) into (27), we obtain

\[ S_{ij} = B_0 \left\{ B_1 (r_{i,j} \delta_{ii} + r_{j,i} \delta_{jj}) + B_2 \delta_{ij} r_{j,i} + B_3 r_{i,j} r_{j,i} \right\} \left( \varphi^2 \right) \text{ for } n = 1, 2, 3 \ldots \]  

(28)
\[ B_2 = v(2n+3) - 1 \]
\[ B_3 = 1 - 2n \]

for a two-dimensional problem, one can obtain
\[ S_{ij} = B_0 \left\{ B_1 (r_i \delta_{ij} + r_j \delta_{ij}) + B_2 r_i r_j + B_3 r_i r_j r_i \right\} \quad (9b) \quad \text{for } n = 1, 2, 3 \ldots \]
\[ B_0 = -\frac{1}{4(1-v)(n+1)(n+2)} r^{2n} \]
\[ B_1 = 3 + 2n - 2v(n+2) \]
\[ B_2 = 2v(n+2) - 1 \]
\[ B_3 = 1 - 2n \]

for a the three-dimensional problem.

To solve Eq. (18), a function \( \Psi_i \) is introduced as the gradient of a scalar function
\[ \Psi_i = U_{,i} \quad (30) \]

Substituting Eq. (30) into Eq. (18) yields Poisson’s equation
\[ \nabla^2 U = \frac{m(1 - 2v)}{2G(1 - v)} \phi \quad (31) \]

Thus, assuming \( \phi = r^{2n-1} \), its particular solution can be obtained as
\[ U = \frac{m(1 - 2v)}{2G(1 - v)} \frac{r^{2n+1}}{(2n+1)(2n+2)} \quad (9b) \quad \text{for } n = 1, 2, 3 \ldots \]
\[ U = \frac{m(1 - 2v)}{2G(1 - v)} \frac{r^{2n+1}}{(2n+1)(2n+2)} \quad (9b) \quad \text{for } n = 1, 2, 3 \ldots \]

Then from Eq. (30) \( \Psi_i \) is obtained as \[ 36 \]
\[ \Psi_i = \frac{m(1 - 2v)}{2G(1 - v)} \frac{r^{2n}}{2n+1} \quad (9a) \quad \text{for } n = 1, 2, 3 \ldots \]
\[ \Psi_i = \frac{m(1 - 2v)}{2G(1 - v)} \frac{r^{2n}}{2n+2} \quad (9a) \quad \text{for } n = 1, 2, 3 \ldots \]

The corresponding particular stress solution can be obtained as
\[ S_{ij} = G(\Psi_{,ij} + \Psi_{,i}) + \lambda \delta_{ij} \Psi_{,k} \quad (36) \]

Then we have
\[ S_{ij} = \frac{m r^{2n-1}}{(1 - v)(2n+1)} \left\{ (1 + 2nv) \delta_{ij} + (1 - 2v)(2n - 1) r_i r_j \right\} \quad (9a) \quad \text{for } n = 1, 2, 3 \ldots \]
\[ S_{ij} = \frac{m r^{2n-1}}{(1-v)(2n+2)} \left\{ (1-2v)(2n-1)r_i r_j + \delta_{ij}(1+2nv) \right\} \]
\[(38)\]

**Approach 2**

Considering the temperature gradient as a part of the body force, \( mT_i - b_i \) can be evaluated by the following equation

\[ mT_i - b_i = \sum_{j=1}^{N} x_j \varphi_j^i \]
\[(39)\]

Thus, the particular solution can be written as

\[ u_p^i = \sum_{k=1}^{3} \sum_{j=1}^{N} x_j^i \Phi_k^j \]
\[(40)\]

Consequently, the follow-up treatment is the same as that in Approach 1, i.e., making use of Eq. (17) and Eqs. (19)–(28) to obtain the desired \( \Phi_k^j \) and \( S_{ij} \), which are the same as that of body force \( b_i \) only. Once the particular solutions of Eq. (9) are established, they can be used to obtain the modified boundary conditions in Eq. (12) to solve the homogeneous solution of Eq. (11). In the following section, we will employ the proposed HBI-FEM to determine the homogeneous part of the solution.

**FORMULATIONS OF THE HBI-FEM**

**Assumed Fields for 2D Problems**

In the HBI-FEM approach, two different assumed fields are employed: intra-element field and frame field. The intra-element continuity is enforced on a nonconforming internal displacement field chosen to be the fundamental solution of the problem [28]. In this approach, the intra-element displacement field is approximated in terms of a linear combination of fundamental solutions of the problem as

\[ \mathbf{u}(\mathbf{x}) = \begin{bmatrix} u_1(\mathbf{x}) \\ u_2(\mathbf{x}) \end{bmatrix} = \sum_{j=1}^{n_s} \begin{bmatrix} u_{11}(\mathbf{x}, y_j) & u_{12}(\mathbf{x}, y_j) \\ u_{21}(\mathbf{x}, y_j) & u_{22}(\mathbf{x}, y_j) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \mathbf{N}_c \mathbf{c}_e \quad (\mathbf{x} \in \Omega, y_j \notin \Omega_e) \]
\[(41)\]

where \( n_s \) is the number of source points outside the element domain, which is equal to the number of nodes of an element in the present work, \( \mathbf{c}_e \) an unknown coefficient vector (not nodal displacements), \( \mathbf{N}_c \) the fundamental solution matrix, which is written as

\[ \mathbf{N}_c = \begin{bmatrix} u_{11}^*(\mathbf{x}, y_1) & u_{12}^*(\mathbf{x}, y_1) & \cdots & u_{11}^*(\mathbf{x}, y_{n_s}) & u_{12}^*(\mathbf{x}, y_{n_s}) \\ u_{21}^*(\mathbf{x}, y_1) & u_{22}^*(\mathbf{x}, y_1) & \cdots & u_{21}^*(\mathbf{x}, y_{n_s}) & u_{22}^*(\mathbf{x}, y_{n_s}) \end{bmatrix} \]
\[(42)\]
\[ c_e = [c_{11}, c_{21}, \ldots, c_{1n}, c_{2n}]^T \] (43)

where \( \mathbf{x} \) and \( \mathbf{y}_{ij} \) are, respectively, the field point and source point in the local coordinate system \((X_1, X_2)\). The component \( u^*_i(x, y_{ij}) \) is the fundamental solution, i.e., induced displacement component in \( i \)-direction at the field point \( \mathbf{x} \) due to a unit point load applied in \( j \)-direction at the source point \( \mathbf{y}_{ij} \), which are given by [13]

\[
u^*_i(x, y_{ij}) = \frac{-1}{8\pi(1-v)G} \left\{ (3 - 4v)\delta_{ij} \ln r - r_i r_j \right\} \] (44)

where \( r_i = x_i - x_{is}, r = \sqrt{r_1^2 + r_2^2} \).

In our analysis, the number of source points is seen to be the same as the number of element nodes, which is free of spurious energy modes and keep the matrix \( \mathbf{H} \) (see Eq. (73)) in full rank, as indicated in [12]. The source point \( y_{ij} (j = 1, 2, \ldots, n_e) \) can be generated by means of the following procedure [28]

\[
u_j = x_0 + \gamma(x_0 - x_c) \] (45)

where \( \gamma \) is a dimensionless coefficient, \( x_0 \) is the related nodal point on the element boundary and \( x_c \) the geometrical centroid of the element (see Figure 2). Determination of \( \gamma \) was already discussed in [27, 28] and \( \gamma = 5 \) is used in the following analysis.

Making use of the assumption of intra-element field (41), the corresponding stress fields can be obtained by the constitutive Eq. (1) as

\[
u(x) = [\sigma_{11}, \sigma_{22}, \sigma_{12}]^T = \mathbf{T}_e c_e \] (46)

Figure 2 Intra-element field and frame field of an HBI-FEM element for 2D thermoelastic problems. (color figure available online.)
where

\[
T_e = \begin{bmatrix}
\sigma_{111}(x, y_1) & \sigma_{211}(x, y_1) & \cdots & \sigma_{111}(x, y_n) & \sigma_{211}(x, y_n) \\
\sigma_{122}(x, y_1) & \sigma_{222}(x, y_1) & \cdots & \sigma_{122}(x, y_n) & \sigma_{222}(x, y_n) \\
\sigma_{112}(x, y_1) & \sigma_{212}(x, y_1) & \cdots & \sigma_{112}(x, y_n) & \sigma_{212}(x, y_n)
\end{bmatrix}
\]  

(47)

As a consequence, the traction is written as

\[
\begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = n\sigma = Q_c c_e
\]  

(48)

in which

\[
Q_c = nT_e, \quad n = \begin{bmatrix} n_1 & 0 & n_2 \\ 0 & n_2 & n_1 \end{bmatrix}
\]  

(49)

The components \(\sigma_{ij}^*(x, y)\) for plain strain problems is given as

\[
\sigma_{ij}^*(x, y) = \frac{-1}{4\pi(1-\nu)^2} \left[ (1-2\nu)(r_{ij}\delta_{ij} + r_{ij}\delta_{ik} - r_{ij}\delta_{jk}) + 2r_{ij}r_{ik}r_{jk} \right]
\]  

(50)

The unknown \(c_e\) in Eq. (41) can be calculated using a hybrid technique [27], in which the elements are linked through an auxiliary conforming displacement frame which has the same form as in conventional FEM (see Figure 3). This means that in the HBI-FEM, a conforming displacement field should be independently defined on the element boundary to enforce the field continuity between elements and also to establish a relationship between the unknown \(c\) and the nodal displacement \(d_e\).

Thus, the frame field is defined as

\[
\tilde{u}(x) = \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix} = \begin{bmatrix} \tilde{N}_1 \\ \tilde{N}_2 \end{bmatrix} d_e = \tilde{N}_e d_e, \quad (x \in \Gamma_c)
\]  

(51)

Figure 3 Typical quadratic interpolation for the frame fields. (color figure available online.)
where the symbol “~” is used to specify that the field is defined on the element boundary only, $\tilde{N}_e$ is the matrix of shape functions, $d_e$ is the nodal displacements of the element under consideration. Taking the side 3-4-5 of a particular 8-node quadrilateral element (see Figure 2) as an example, $\tilde{N}_e$ and $d_e$ can be expressed as

$$\tilde{N}_e = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}$$

(52)

$$d_e = \begin{bmatrix}
u_{11} \\
u_{21} \\
u_{12} \\
u_{22} \\
u_{13} \\
u_{23} \\
u_{14} \\
u_{24} \\
\end{bmatrix}^T$$

(53)

where $\tilde{N}_1$, $\tilde{N}_2$, and $\tilde{N}_3$ are expressed by natural coordinate as

$$\tilde{N}_1 = \frac{-\xi(1 - \xi)}{2}, \quad \tilde{N}_2 = 1 - \xi^2, \quad \tilde{N}_3 = \frac{\xi(1 + \xi)}{2} (\xi \in [-1, 1])$$

(54)

**Assumed Fields for 3D Problems**

In the case of a three-dimensional problem, the intra-element displacement fields are approximated by

$$u(x) = \begin{bmatrix}
u_1(x) \\
u_2(x) \\
u_3(x) \\
\end{bmatrix} = N_e c_e \quad (x \in \Omega_e, y_{ij} \notin \Omega_e)$$

(55)

where the matrix $N_e$ and vector $c_e$ for 3D elements can be written as

$$N_e = \begin{bmatrix}
u_{11}(x, y_{ij}) & \nu_{12}(x, y_{ij}) & \nu_{13}(x, y_{ij}) & \cdots \\
u_{12}(x, y_{ij}) & \nu_{12}(x, y_{ij}) & \nu_{23}(x, y_{ij}) & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}$$

(56)

$$c_e = \begin{bmatrix}
c_{i1} & c_{21} & c_{31} & \cdots & c_{in} & c_{2n} & c_{3n} \\
\end{bmatrix}^T$$

(57)

in which the fundamental solution components $\nu_{ij}^*(x, y_{ij})$ are given by

$$\nu_{ij}^*(x, y_{ij}) = \frac{1}{16\pi(1 - \nu)}Gr \left\{ (3 - 4\nu)\delta_{ij} + r_{ij} \right\}$$

(58)

where $r_i = x_i - x_{ij}$, $r = \sqrt{r_1^2 + r_2^2 + r_3^2}$. In 3D analysis, the number of source points is also assumed to be the same as the number of element nodes as was done in the 2D case.

The corresponding stress fields in 3D coordinate system can be expressed as

$$\sigma(x) = \begin{bmatrix}
\sigma_{11} & \sigma_{22} & \sigma_{33} & \sigma_{23} & \sigma_{31} & \sigma_{12} \\
\end{bmatrix}^T = T_e c_e$$

(59)
where

$$\mathbf{T}_e = \begin{bmatrix}
\sigma_{111}(\mathbf{x}, \mathbf{y}) & \sigma_{211}(\mathbf{x}, \mathbf{y}) & \sigma_{311}(\mathbf{x}, \mathbf{y}) & \cdots \\
\sigma_{122}(\mathbf{x}, \mathbf{y}) & \sigma_{222}(\mathbf{x}, \mathbf{y}) & \sigma_{322}(\mathbf{x}, \mathbf{y}) & \cdots \\
\sigma_{133}(\mathbf{x}, \mathbf{y}) & \sigma_{233}(\mathbf{x}, \mathbf{y}) & \sigma_{333}(\mathbf{x}, \mathbf{y}) & \cdots \\
\sigma_{112}(\mathbf{x}, \mathbf{y}) & \sigma_{212}(\mathbf{x}, \mathbf{y}) & \sigma_{312}(\mathbf{x}, \mathbf{y}) & \cdots \\
\sigma_{111}(\mathbf{x}, \mathbf{y}_n) & \sigma_{211}(\mathbf{x}, \mathbf{y}_n) & \sigma_{311}(\mathbf{x}, \mathbf{y}_n) & \\
\sigma_{122}(\mathbf{x}, \mathbf{y}_n) & \sigma_{222}(\mathbf{x}, \mathbf{y}_n) & \sigma_{322}(\mathbf{x}, \mathbf{y}_n) & \\
\sigma_{133}(\mathbf{x}, \mathbf{y}_n) & \sigma_{233}(\mathbf{x}, \mathbf{y}_n) & \sigma_{333}(\mathbf{x}, \mathbf{y}_n) & \\
\sigma_{112}(\mathbf{x}, \mathbf{y}_n) & \sigma_{212}(\mathbf{x}, \mathbf{y}_n) & \sigma_{312}(\mathbf{x}, \mathbf{y}_n) & \\
\end{bmatrix}$$

(60)

in which the components $\sigma_{ijk}(\mathbf{x}, \mathbf{y})$ is given as

$$\sigma_{ijk}(\mathbf{x}, \mathbf{y}) = \frac{-1}{8\pi(1 - \nu)^2} \left\{ (1 - 2\nu) (r_j \delta_{ij} + r_j \delta_{ki} - r_j \delta_{kj}) + 3r_j r_i r_k \right\}$$

(61)

The corresponding traction is expressed as

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \mathbf{n} \sigma = \mathbf{Q}_e \mathbf{c}_e$$

(62)

where

$$\mathbf{Q}_e = \mathbf{n} \mathbf{T}_e, \quad \mathbf{n} = \begin{bmatrix} n_1 & 0 & 0 & n_3 & n_2 \\ 0 & n_2 & 0 & n_3 & n_1 \\ 0 & 0 & n_3 & n_2 & n_1 \end{bmatrix}$$

(63)
The frame field of the element is defined as
\[
\tilde{\mathbf{u}}(\mathbf{x}) = \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_3 \end{bmatrix} = \begin{bmatrix} \tilde{N}_1 \\ \tilde{N}_2 \\ \tilde{N}_3 \end{bmatrix} \mathbf{d}_e = \tilde{\mathbf{N}}_e \mathbf{d}_e, \quad (\mathbf{x} \in \Gamma_e)
\] (64)

For the face 1-2-3-10-15-14-13-9 of a particular 20-node 3D brick element (Figure 3), \(\tilde{\mathbf{N}}_e\) and \(\mathbf{d}_e\) can be written as
\[
\tilde{\mathbf{N}}_e = \begin{bmatrix} \tilde{N}_1 & \tilde{N}_2 & \tilde{N}_3 & \ldots & \tilde{N}_9 & \tilde{N}_{10} & \tilde{N}_{11} & \tilde{N}_{12} & \ldots & \tilde{N}_{18} & \tilde{N}_{19} & \tilde{N}_{20} \end{bmatrix}
\] (65)
\[
\mathbf{d}_e = \begin{bmatrix} u_{11} & u_{21} & u_{31} & \ldots & u_{12} & u_{22} & u_{32} & \ldots & u_{120} & u_{220} & u_{320} \end{bmatrix}^T
\] (66)

where the shape functions are defined as
\[
\tilde{\mathbf{N}}_i = \begin{bmatrix} \tilde{N}_i & 0 & 0 \\ 0 & \tilde{N}_i & 0 \\ 0 & 0 & \tilde{N}_i \end{bmatrix}, \quad \mathbf{0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\] (67)

and
\[
\tilde{\mathbf{N}}_i = \begin{cases} \frac{1}{4} (1 + \xi_i \xi) (1 + \eta, \eta) (\xi_i \xi + \eta \eta - 1) & (i = 1, 3, 5, 7) \\ \frac{1}{2} (1 - \xi_i^2) (1 + \eta, \eta) & (i = 2, 6) \\ \frac{1}{2} (1 - \eta^2) (1 + \xi, \xi_i) & (i = 4, 8) \xi_i, \eta \in [-1, 1] \end{cases}
\] (68)

where \((\xi_i, \eta_i)\) is the natural coordinate of the \(i\)-node of the element (Fig. 5).

Figure 5  Typical quadratic interpolation for the frame fields. (color figure available online.)
Modified Functional for Hybrid Finite Element Method

The HBI-FEM formulations for 3D thermoelastic problem can be established by the variational approach [28]. In the absence of body forces, the variational functional $\Pi_{me}$ used for deriving the present HBI-FEM can be constructed as [25]

$$\Pi_{me} = \frac{1}{2} \int_{\Omega_e} \sigma_{ij} e_{ij} d\Omega - \int_{\Gamma} \tilde{t}_i \tilde{u}_i d\Gamma + \int_{\Gamma} t_i (\tilde{u}_i - u_i) d\Gamma$$  (69)

where $u_i$ and $\tilde{u}_i$ are the intra-element displacement field defined within the element and the frame displacement field defined on the element boundary, respectively. $\Omega_e$ and $\Gamma_e$ are the element domain and element boundary, respectively. $\Gamma_n$, $\Gamma_d$, and $\Gamma_I$ stands respectively for the specified traction boundary, specified displacement boundary and inter-element boundary ($\Gamma_e = \Gamma_n + \Gamma_d + \Gamma_I$).

Compared to the functional employed in the conventional FEM, the present hybrid variational functional is constructed by adding a hybrid integral term related to the intra-element and frame displacement fields to guarantee the satisfaction of displacement and traction continuity conditions on the common boundary of two adjacent elements. By applying the Gaussian theorem, Eq. (69) can be simplified as

$$\Pi_{me} = \frac{1}{2} \int_{\Gamma} t_i u_i d\Gamma - \int_{\Omega_e} \sigma_{ij} e_{ij} d\Omega - \int_{\Gamma} \tilde{t}_i \tilde{u}_i d\Gamma + \int_{\Gamma} t_i (\tilde{u}_i - u_i) d\Gamma$$  (70)

Considering the equilibrium equation with the constructed intra-element fields, Eq (70) can be simplified as

$$\Pi_{me} = -\frac{1}{2} \int_{\Gamma} t_i u_i d\Gamma + \int_{\Gamma} t_i \tilde{u}_i d\Gamma - \int_{\Gamma} \tilde{t}_i \tilde{u}_i d\Gamma$$  (71)

The functional (71) contains boundary integrals only and will be used to derive HBI-FEM formulations for 2D and 3D thermoelastic problems.

Element Stiffness Matrix

The element stiffness equation can be generated by setting $\delta \Pi_{me} = 0$. Substitute Eqs. (55), (62) and (64) into the functional (71), we have

$$\Pi_{me} = -\frac{1}{2} c^T e \mathbf{H}_e c_e + c^T e \mathbf{G}_e d_e - d^T g_e$$  (72)

where

$$\mathbf{H}_e = \int_{\Gamma_e} \mathbf{Q}_e^T \mathbf{N}_e d\Gamma, \quad \mathbf{G}_e = \int_{\Gamma_e} \mathbf{Q}_e^T \tilde{\mathbf{N}}_e d\Gamma, \quad \mathbf{g}_e = \int_{\Gamma_e} \tilde{\mathbf{N}}_e^T \mathbf{t} d\Gamma$$  (73)
To enforce inter-element continuity on the common element boundary, the unknown vector $c_e$ should be expressed in terms of nodal DOF $d_e$. The stationary condition of the functional $\Pi_{me}$ with respect to $c_e$ and $d_e$ yields, respectively,

$$\frac{\partial \Pi_{me}}{\partial c_e} = -H_e c_e + G_e d_e = 0$$

(74)

$$\frac{\partial \Pi_{me}}{\partial d_e} = G_e^T c_e - g_e = 0$$

(75)

Therefore, the relationship between $c_e$ and $d_e$, and the stiffness equation can be obtained as follows

$$c_e = H_e^{-1} G_e d_e$$

(76)

$$K_e d_e = g_e$$

(77)

where $K_e = G_e^T H_e^{-1} G_e$ is the element stiffness matrix. Consequently, the homogeneous solution can be obtained by Eq. (41) for 2D problems and Eq. (55) for 3D problems.

**Recovery of Rigid-body Motion Terms**

It is noted that when the procedure described above is used to obtain $u_e$, the rigid body motion modes are not included so as to prevent the singularity of the inversion of matrix $H_e$. Therefore, it is necessary to reintroduce the discarded rigid-body motion terms after we have obtained the internal field of an element. The least squares method can be employed for this purpose and the missing terms can be easily recovered by setting for the augmented internal field [30]

$$u_e = N_e c_e + \begin{bmatrix} 1 & 0 & x_2 \\ 0 & 1 & -x_1 \end{bmatrix} c_0 \quad \text{(for } \mathbb{R}^2)$$

(78)

$$u_e = N_e c_e + \begin{bmatrix} 1 & 0 & 0 & 0 & x_3 & -x_2 \\ 0 & 1 & 0 & -x_3 & 0 & x_1 \\ 0 & 0 & 1 & x_2 & -x_1 & 0 \end{bmatrix} c_0 \quad \text{(for } \mathbb{R}^3)$$

(79)

The detailed formulations can be found in some previous papers [28, 30].

**NUMERICAL RESULTS AND DISCUSSION**

In this section, we presented five different numerical examples to assess the performance of the proposed methodology, in which 2D and 3D elastic and thermoelastic problems with body force and temperature change are included. The first three examples are given to investigate the capability of the method to treat the 2D problems with the temperature change, body forces, or the combination of both. The latter two examples are given to show its ability to deal with 3D thermoelasticity with arbitrary body force and temperature. The analytical results or numerical results from ABAQUS with fine meshes are presented to assess the accuracy of the method.
Example 1. Circular cylinder with axisymmetric temperature change: In this example, a long circular cylinder with axisymmetric temperature change in the domain is considered to show the performance of the proposed method. Both inside and outside surfaces of the cylinder are assumed to be free from traction. The temperature $T$ changes logarithmically along the radial direction. With the symmetry condition of the problem, only one quarter of the cylinder is modeled. The geometrical configurations and the boundary conditions of the cylinder are shown in Figure 6. Under the assumption of plane strain, the analytical solutions for stress components without body forces are given as [33]

$$
\sigma_r = -\frac{EzT_0}{2(1-v)} \left[ \frac{\ln(b/r)}{\ln(b/a)} - \frac{(b/r)^2 - 1}{(b/a)^2 - 1} \right]
$$

$$
\sigma_\theta = -\frac{EzT_0}{2(1-v)} \left[ \frac{\ln(b/r) - 1}{\ln(b/a)} + \frac{(b/r)^2 + 1}{(b/a)^2 - 1} \right]
$$

$$
T = \frac{\ln(r/b)}{\ln(a/b)} T_0
$$

which is taken as a reference. In our computation, the parameters $a = 5$, $b = 20$, $E = 1000$, $v = 0.3$, $v = 0.001$, and $T_0 = 10$ are employed. As shown in Figure 7, two different meshes with 168-node elements and 128-node elements are, respectively, employed to show the influence of the mesh density. The two approaches listed previously for approximating the body force and temperature are discussed and analyzed in this example.

Figures 8 and 9 present the variation of the radial and the circumferential thermal stresses with the cylinder radius for both coarse mesh and fine mesh,
respectively, in which the theoretical values are given for comparison. It is seen from these two figures that the results from Method 2 are much better than those obtained from Method 1 for both coarse mesh and fine mesh. When using coarse mesh with Approach 2, the radial stress close to the outer surface of the cylinder shows very large errors, yet when refining the mesh the error decreases dramatically and the results agree well with the analytical findings. It can be inferred that the error may be to a large extent due to the RBF interpolation, for which the number of interpolation points has a significant influence on its accuracy. However, it is also found that there is no significant improvement observed for Method 1 when refining the mesh to 128 individual 8-node elements.

The circumferential thermal stresses obtained from coarse mesh (left) and fine mesh (right) in Figure 7 using Approach 1 show a small error compared to the radial stresses. However, these errors are still larger than those in Approach 2, which can be seen clearly from Figure 9. Figure 10 displays the contour plots of (a) radial and (b) circumferential thermal stresses (The meshes used for contour plot is different from that employed above due to the use of quadratic elements). It demonstrates that treating temperature gradient and body force together is better than dealing with them separately. Consequently, in the latter examples Approach 2 will be
employed in our analysis. It should be pointed out that Approach 1 applies when the temperature change is in discrete distribution or the gradient of the temperature field is not available.

**Example 2.** Long beam under gravity: In the second example, we consider a long beam with rectangular cross-section subjected to gravity. The geometry of the rectangular cross-section and the corresponding boundary conditions are shown in Figure 11, where $g$ denotes the gravity accelerator, and $\rho$ is the density of material. The problem can be viewed as a plane strain problem, and the analytical solutions of displacements and stresses are given by

\[
\begin{align*}
    u_1 &= 0, \\
    u_2 &= -\frac{(1 + v)(1 - 2v)\rho g}{2E(1 - v)} \left[ L^2 - (x_2 - L)^2 \right] \\
    \sigma_{11} &= \frac{\rho g v}{1 - v} (x_2 - L), \\
    \sigma_{22} &= \rho g (x_2 - L), \\
    \sigma_{12} &= 0
\end{align*}
\]

Figure 9 Variation of the circumferential thermal stresses with the cylinder radius using (a) coarse mesh and (b) fine mesh. (color figure available online.)

Figure 10 Contour plots of (a) radial and (b) circumferential thermal stresses. (color figure available online.)
Let $L = 20$, $E = 1000$, $v = 0.25$, $\alpha = 0.001$ and $\rho g = 9.8$, and four 8-node quadratic elements are used to model the entire square cross-section domain (see Figure 12). The numerical results for the distribution of nodal displacement component $u_2$ along $x_1 = 10$ are shown in Figure 13, which shows that the numerical results agree with the analytical solutions. Additionally, the stress results at element nodes along $x_1 = 10$ are also compared with the exact solutions in Figure 13, where it is obvious that the numerical results obtained from HBI-FEM with RBF interpolation have good accuracy compared with the analytical results. Figure 14 gives the stress contour plots of the cross-section beam under gravity. It can be seen that the stresses change linearly with $x_2$ coordinates and independent from the $x_1$ coordinate, which is consistent with the analytical solution.

**Example 3.** T-shape domain with body force and temperature change: In this example, the proposed numerical method is used to model a T-shape domain with both the temperature change and body force. The boundary condition and
temperature distribution are shown in Figure 15. The material properties are $E = 30000$, $\nu = 0.15$, $\alpha = 0.001$, $\rho = 2.3$ and $g = 9.8$, respectively.

In this example, 22 separate 8-node quadrilateral elements are utilized to model the T-shape domain by the HBI-FEM. Because there is no analytical
solution for comparison, the results from ABAQUS using 352 CPE8R elements are employed as reference. Figure 17 shows the thermal stresses $\sigma_{11}$ along the line $x_2 = 14$ and $\sigma_{22}$ along line $x_1 = 0$. It is obvious that the results from the HBI-FEM by coarse mesh shown in Figure 16 agree with the results from ABAQUS with fine mesh (see Fig. 16). It should be mentioned that in order to obtain the meaningful results at the two sharp corners of the T-shape domain, a refined mesh is necessary, due to the stress concentration at the points near the corner, which is also applied to the traditional FEM (ABAQUS). The contour plots of both the displacement components and stresses components are presented in Figure 18.
**Figure 17** The variation of the thermal stresses of T-Shape domain: (a) $\sigma_{11}$ along line $x_2 = 14$ and (b) $\sigma_{22}$ along line $x_1 = 0$.

**Figure 18** Contour plots of the displacement and stresses of T-shaped domain under temperature change and body force. (color figure available online.)
Example 4. A 3D cube under arbitrary temperature distribution and body force: In this example, a three-dimensional cube of $1 \times 1 \times 1$ with center located at $(0.5, 0.5, 0.5)$ is considered, which is shown in Figure 19. The material properties of the cube are: Young’s modulus $E = 5000$, Poisson’s ratio $v = 0.3$, and linear thermal expansion coefficient $\alpha = 0.001$. The bottom surface is fixed on the ground. The temperature distribution and body force are assumed to be

$$T = 30x_1, \quad b_1 = -2000[(x-0.5)^2 + (y-0.5)^2]$$

Because there is no analytical solution available, the results from ABAQUS herein are employed for comparison. The meshes used by the HBI-FEM and ABAQUS are given in Figure 20, in which the coarse mesh consists of 125 individual 20-node brick elements and the fine mesh for ABAQUS is 8000 C3D20R elements.

Figure 21 presents the displacement $u_3$ and stress $\sigma_{33}$ along an edge of the cube which coincides with $x_3$ axis. It can be seen that the results from the HBI-FEM again agree with those by ABAQUS. It is demonstrated that the proposed procedure based on the HBI-FEM predicts the response of 3D thermoelastic problems under arbitrary temperature and body force. It is also shown that the HBI-FEM with RBF interpolation gives satisfactory results using very coarse meshes.

Example 5. A heated hollow ball: Finally, we consider a heated hollow ball to demonstrate the capability of the method to solve 3D thermal stress problems with complex geometry. As shown in Figure 22, the radius of inner hole is $a$ and the radius of outer surface of the ball is $b$. The temperature distribution is given by

$$T = T_0(2 - 10/r)$$
Figure 20 Mesh configurations of the 3D cube in Example 4: (a) HBI-FEM (125 elements); (b) ABAQUS (8000 element).

Figure 21 Displacement $u_3$ and stress $\sigma_{33}$ along one cube edge when subjected to specified temperature and body force.

Figure 22 Schematic and dimension of the heated hollow ball. (color figure available online.)
In the calculation, \( a = 5 \), \( b = 10 \), \( T_0 = 20 \) are assumed. Considering the symmetry property of the problem, only one-eighth of the ball is modeled. The mesh used for the HBI-FEM is given in Figure 23, in which a total of 864 20-node brick elements are employed. As a reference, the results from ABAQUS are calculated using a much finer mesh with 11088 C3D20R elements (see Figure 23).

Figure 24 shows the radial displacement \( u_r \) and von Mises stress of the hollow ball along its radial direction. The radial displacement increases with the radius from 0.071 to 0.143 with a slightly reduction when \( 5 < r < 6.3 \). It can be seen from Figure 24 that the inner surface of the hollow ball suffers maximum Von Mises stress of up to 20 and this value is dramatically reduced to about 0.15 at \( r = 6.5 \), then it experiences a moderate increasing to about 8.2 at the outer surface. It is obvious from Figure 24 that the results obtained by the HBI-FEM display a good agreement with the results from ABAQUS. This again demonstrates a good performance of the proposed model in predicting the thermoelastic response of 3D problems with complex geometry.

Figure 24 Radial displacement and Von Mises stress along radius of the heated hollow ball subjected to temperature change.
CONCLUSIONS

In this article a new solution procedure based on the HBI-FEM is proposed to solve two-dimensional and three-dimensional thermoelastic problems with arbitrary body forces and temperature changes. The body force and temperature change are handled by the method of particular solution, in which the homogeneous solution is obtained by using the HBI-FEM with elastic fundamental solutions, while the particular solution is approximated by the radial basis function. It is found that treating body force and temperature change as a whole is superior to approximating them separately. The five numerical examples presented in this article show that the proposed method can predict the thermoelastic response of 2D and 3D thermoelasticity problems with complex geometry, arbitrary body force and arbitrary temperature changes. It is a promising methodology for mesh reduction, which is capable of obtaining satisfying results with much coarser meshes than the traditional FEM. It is also possible to improve the results by only increasing the interpolation points while keeping the meshes at a lower density and the study is underway.

REFERENCES
