Solving the nonlinear Poisson-type problems with F-Trefftz hybrid finite element model

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\begin{abstract}
A hybrid finite element model based on F-Trefftz kernels (fundamental solutions) is formulated for analyzing Dirichlet problems associated with two-dimensional nonlinear Poisson-type equations including nonlinear Poisson–Boltzmann equation and diffusion–reaction equation. The nonlinear force term in the Poisson-type equation is frozen by introducing the imaginary terms at each Picard iteration step, and then the induced Poisson problem is solved by the present hybrid finite element model involving element boundary integrals only, coupling with the particular solution method with radial basis function interpolation. The numerical accuracy of the present method is investigated by numerical experiments for problems with complex geometry and various nonlinear force functions.
\end{abstract}

1. Introduction

The nonlinear Poisson-type equation written by

\begin{equation}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y, u) \quad \text{in} \quad \Omega \in \mathbb{R}^2
\end{equation}

is the governing equations for many physical problems including heat conduction, mass transfer, seepage, electric and magnetic fields, and chemical reaction. In the nonlinear equation mentioned above, \(u(x, y)\) is the scalar potential at a given field point \((x, y)\), \(f\) is a generalized force function related to the unknown potential \(u\), and \(\Omega\) is the two dimensional bounded domain of the problem under consideration.

The complete boundary value problems should consist of the governing equation (1) and the following Dirichlet boundary conditions:

\begin{equation}
u = \pi \text{ on } \Gamma_u
\end{equation}

where an over bar denotes the specified potential.

Owing to the nonlinear behavior of the partial differentiable equation (PDE) under consideration, the theoretical analysis has been proved to be considerably difficult, especially for problems with irregular geometries and non-uniform boundary conditions. To study the nonlinearity in complex solution domain, it is a long history in resorting to numerical solutions. So far, different numerical techniques, including finite element method (FEM) [1], boundary element method (BEM) [2–4], finite difference method (FDM) [5], the hybrid Trefftz finite element method (HT-FEM) [6], the method of fundamental solution (MFS) [7–11], the locally boundary integral equation (LBIE) method [12], Kansa method with radial basis function (RBF) interpolation [13,14], were developed for solving the nonlinear Poisson-type potential problems consisting of Eqs. (1) and (2). Among them, the HT-FEM has recently been paid more attention in the past decades [6,15–18], because it has some advantages over the conventional FEM and boundary element method (BEM). For example, it possesses versatile element construction and can capture the variation of singular fields or stress concentration by employing suitable interpolation kernels. More discussion on this topic can be found in literatures [17,18].

As an alternative to the HT-FEM, the hybrid finite formulation with F-Trefftz functions or fundamental solutions as interpolating kernel functions within the elements has been established for analyzing heat conduction and elastic problems [19–21]. The method was known as HFS-FEM for distinguishing the presented model from HT-FEM. In the HFS-FEM, the fundamental solutions (F-Trefftz functions), instead of T-complete functions (Trefftz functions), are used to construct the interior field, and independent boundary frame field is approximated by conventional shape functions. A new variational functional was constructed to guarantee the inter-element continuity, link the two fields and establish the final force-displacement equations. The use of fundamental solutions can convert the domain integral in the variational functional to element boundary integrals. It is worth...
noting that no singular integrals are involved in the HFS-FEM, although the fundamental solutions are employed. It is because the source points used for the evaluation of fundamental solution are placed outside the element of interest as was done in the MFS. Thus, the source point and field point never overlap during the computation. Clearly, the present HFS-FEM inherits the advantages of HT-FEM and simultaneously can alleviate or remove some drawbacks of HT-FEM such as the properly selection of the number of terms of T-complete functions, the complicated coordinate transformation needed in the HT-FEM, and the relatively complicated expressions by comparison with the fundamental solutions which usually contains one term only, rather than a series containing infinite terms, and are available for most physical problems.

In this paper, we focus on the extension of the developed HFS-FEM to the nonlinear Poisson-type problems. The nonlinear term appearing in the right-hand side of the differential equation is first frozen by introducing the Picard iteration process, and then in each iteration step, radial basis functions are employed to approximate the part of particular solutions, and the HFS-FEM is formulated to determine the homogeneous potential distribution with modified boundary conditions. Both the iteration residual and the inter-iteration difference are used to assess the convergent performance.

The paper is arranged as follows. The solution procedure including the iterative method and the presented hybrid finite element formulation is stated in Section 2. In Section 3, we consider some numerical examples including nonlinear Poisson–Boltzmann equation and nonlinear diffusion–reaction equation. A comparison of the numerical results from HFS-FEM is made with those from either analytical approach or other numerical methods. Finally, some conclusions are presented in Section 4.

2. Solution methodology

In order to develop a generalized algorithm for the nonlinear boundary value problem (BVP) consisting of Eq. (1) and Dirichlet boundary condition (2), the Picard method of iteration implemented by Chen et al. [11] for the two-dimensional Dirichlet problem is employed for solving nonlinear Poisson-type equations with HFS-FEM. The desired solutions are obtained by assuming \( v \) at the step of the iterations and solving the linearizing equation at each level of iteration. For this purpose, we construct the following linear iteration process to evaluate the potential distribution at the current iteration step \( m \) by freezing the nonlinear term appeared in the right-hand term \( f \) which is evaluated using the results at previous iteration \( (m-1) \) step

\[
\frac{\partial^2 u^m}{\partial x_1^2} + \frac{\partial^2 u^m}{\partial x_2^2} = f(X_1,X_2,u^{(m-1)}) \quad \text{in } \Omega \subset \mathbb{R}^2
\] (3)

It is obvious that the sequence \( \{u^m\} \) is expressed in terms of \( \{u^{(m-1)}\} \), which is known from previous iteration step, for each \( m(m=1,2,\ldots) \), and \( u^{(0)} \) represents any initial guess. To complete the iteration and obtain convergent results, the iteration convergent criterion is set by controlling both the maximum residual related to the nonlinear governing equation

\[
\left\| \frac{\partial^2 u^m}{\partial x_1^2} + \frac{\partial^2 u^m}{\partial x_2^2} - f(X_1,X_2,u^{(m-1)}) \right\|_{\infty} \leq \varepsilon_1
\] (4)

and the inter-iteration difference

\[
\left\| u^m - u^{(m-1)} \right\|_{\infty} \leq \varepsilon_2
\] (5)

where \( \| * \|_{\infty} \) represents the infinite norm, and \( \varepsilon_1 \) and \( \varepsilon_2 \) are iteration tolerances, respectively.

Generally, the treatment of nonhomogeneous term \( f \) involves the domain integral. To remove the domain integral from the element stiffness equation, the radial basis functions are employed in this work.

Before introducing radial basis functions, it is observed that the linearity of Eq. (3) makes its solution to be divided into two major parts

\[
u = u_h + u_p
\] (6)

where the nonhomogeneous solution, also named as particular solution, \( u_p \) is required to satisfy

\[
\nabla^2 u_p = f(X_1,X_2,u^{(m-1)})
\] (7)

without any boundary conditions, and the homogeneous solution \( u_h \) is obtained by solving the following linear system with modified boundary condition:

\[
\begin{align*}
\nabla^2 u_h &= 0 \quad \text{in } \Omega \\
u_h &= \mathbf{n} \cdot u_p \quad \text{on } \Gamma_u
\end{align*}
\] (8)

2.1. Particular solution

In order to obtain the particular solution, the radial basis functions are used here to approximate the induced fictitious function \( f \), that is

\[
f(X_1,X_2,u^{(m-1)}) = \sum_{k=1}^{N_k} a_k \phi_k(X_1,X_2) = \langle \phi | \mathbf{a} \rangle
\] (9)

where \( N_k \) denotes the number of interpolation points in the domain of interest, \( a_k \) are unknown interpolating coefficients, \( \phi_k(X_1,X_2) \) be radial basis function centered at the point \( (X_1^k,X_2^k) \), and \( \{ \phi \} = \{ \varphi_1 \ \varphi_2 \ \ldots \ \varphi_{N_k} \}, \ \{ \mathbf{a} \} = \{ a_1 \ a_2 \ \ldots \ a_{N_k} \} \) are corresponding basis vector and coefficient vector, respectively.

Radial basis functions are usually expressed in terms of the Euclidian distance, so they can work well in any dimensional space and does not increase the computational difficulty when the dimension of a problem increases. In most numerical analyses, the commonly used RBFs include linear polynomial, thin plate spline (TPS) and multiquadric (MQ). Among them, the linear polynomial and thin plate spline are piecewise smooth in the space, while the MQ is infinitely smooth. Due to the high sensitivity to shape parameter in MQ, we will not employ MQ in this work.

In the standard dual reciprocity procedure, it is reasonable to assume that the particular solution is approximated by

\[
u_p(X_1,X_2) = \sum_{k=1}^{N_k} a_k \psi_k(X_1,X_2) = \langle \psi | \mathbf{a} \rangle
\] (10)

where \( r \) represents the Euclidean distance of the given point \( (X_1,X_2) \) from a fixed point \( (X_1^k,X_2^k) \) in the domain of interest

\[
r = \sqrt{(X_1 - X_1^k)^2 + (X_2 - X_2^k)^2}
\] (11)

In Eq. (13), \( \{ \psi \} = \{ \psi_1 \ \psi_2 \ \ldots \ \psi_{N_k} \} \) represents the set of approximated particular kernel.
For the TPS basis, the corresponding particular kernel \( \psi_k \) is given as [22]

\[
\psi_k = \frac{r^4 \ln r - r^4}{16} - \frac{r^4}{32}
\]  

(14)

For convenience, only interior interpolation points are used for the procedure of RBF approximation, and these points are displayed with star symbol used in each element (see Fig. 1).

2.2. Formulation of hybrid finite element model

Due to the use of the fundamental solution as intra-element trial function, the hybrid finite element approach proposed in [19–21] is effective for treating the homogeneous linear system (8) by dividing the domain under consideration into finite cells or elements.

For a typical element \( e \) shown in Fig. 1, measured in a local coordinate system \((x_1, x_2)\) whose origin locates at the center of element, we introduce two independent fields here.

2.2.1. Intra-element field defined in the element

Within the element \( e \), the linear combination of fundamental solutions with \( \mathbf{N}_e \) source points is used to approximate the homogeneous field

\[
\mathbf{u}_h(x_1, x_2) = \sum_{i=1}^{N_e} c_i \mathbf{u}^i (x_1, x_2) = [\mathbf{N}_e] \mathbf{c}_i
\]  

(15)

where the source points \((x^i_1, x^i_2)\) locate outside the element domain (see Fig. 1), and \( c_i \) represents unknown coefficient.

\[
\mathbf{u}^i (x_1, x_2) = -\frac{1}{2\pi} \ln \sqrt{(x_1 - x^i_1)^2 + (x_2 - x^i_2)^2}
\]  

(16)

and

\[
[\mathbf{N}_e] = \left\{ \mathbf{u}^1 \quad \mathbf{u}^2 \quad \cdots \quad \mathbf{u}^{N_e} \right\}, \quad [\mathbf{c}_i] = \left\{ c_1 \quad c_2 \quad \cdots \quad c_{N_e} \right\}^T
\]

The source points used in Eq. (15) can be generated by the following way [19]:

\[
x^i_j = (1 + \gamma) x^i_j, \quad i = 1 \rightarrow N_e, \quad j = 1,2
\]  

(17)

where \( \gamma \) is the dimensionless parameter and \((x^i_1, x^i_2)\) be the element boundary points.

2.2.2. Frame field defined on the element boundary

In order to enforce the conformity of the homogeneous field, i.e. \( \mathbf{u}_h = \mathbf{u}_0 \) on the interface of any two neighboring elements \( e \) and \( f \), an auxiliary frame field \( \mathbf{u}_f \) is introduced and expressed in terms of nodal degree of freedom (DOF) of the element. For instance, for the element \( e \) shown in Fig. 1 with quadratic varying potential, one can assume that the frame potential on the side the node 3–5 (see Fig. 1) is expressed as

\[
\mathbf{u}_f(x_1, x_2) = \{ \mathbf{N}_e \} \mathbf{d}_e
\]  

(18)

in which the shape function vector \([\mathbf{N}_e]\) and the nodal vector \([\mathbf{d}_e]\) are given by

\[
[\mathbf{N}_e] = \{ 0 \quad 0 \quad \tilde{N}_1 \quad \tilde{N}_2 \quad \tilde{N}_3 \quad 0 \quad 0 \quad 0 \quad 0 \}
\]

\[
[\mathbf{d}_e] = \{ d_1 \quad d_2 \quad d_3 \quad d_4 \quad d_5 \quad d_6 \quad d_7 \quad d_8 \quad d_9 \quad d_{10} \}^T
\]  

(19)

with \( \tilde{N}_i \) (i = 1,2,3) being the conventional shape function used in the FEM, and are expressed in terms of local natural coordinate \( \xi \) as

\[
\tilde{N}_1 = -\frac{\xi(1-\xi)}{2}, \quad \tilde{N}_2 = 1-\xi^2, \quad \tilde{N}_3 = -\xi(1+\xi)
\]  

(20)

To establish a relationship between the two independent fields, the hybrid variational functional in [19,21] is employed

\[
P_{me} = \frac{1}{2} \int_{\Omega_e} \left[ \left( \frac{\partial u_h}{\partial x_1} \right)^2 + \left( \frac{\partial u_h}{\partial x_2} \right)^2 \right] d\Omega + \int_{\Gamma_e} q_0 (u_h - u_0) d\Gamma
\]  

(21)

Using the divergence theorem, we have

\[
P_{me} = -\frac{1}{2} \int_{\Omega_e} q_0 u_h d\Omega + \int_{\Gamma_e} q_0 \nabla u_h \cdot \mathbf{n} d\Gamma
\]  

(22)

The substitution of Eqs. (15) and (18) into (22) yields

\[
P_{me} = -\frac{1}{2} [\mathbf{c}_i]^T [\mathbf{H}_e] [\mathbf{c}_i] + [\mathbf{c}_i]^T [\mathbf{G}_e] [\mathbf{d}_e]
\]  

(23)

in which

\[
[\mathbf{H}_e] = \int_{\Omega_e} \left[ \mathbf{Q}_i \right]^T [\mathbf{N}_e] \mathbf{n} d\Omega
\]

\[
[\mathbf{G}_e] = \int_{\Gamma_e} \left[ \mathbf{Q}_i \right]^T [\mathbf{N}_e] \mathbf{n} d\Gamma
\]  

(24)

with

\[
[\mathbf{Q}_i] = \frac{\partial [\mathbf{N}_e]}{\partial n} = \begin{bmatrix} n_1 \quad n_2 \end{bmatrix} \begin{bmatrix} \frac{\partial N_{e1}}{\partial n} \\ \frac{\partial N_{e2}}{\partial n} \end{bmatrix}
\]  

(25)

The minimization of the functional \( P_{me} \) yields

\[
\frac{\partial P_{me}}{\partial [\mathbf{d}_e]} = [\mathbf{G}_e]^T [\mathbf{c}_i] = 0
\]

\[
\frac{\partial P_{me}}{\partial [\mathbf{c}_i]} = [\mathbf{H}_e]^{-1} [\mathbf{G}_e] [\mathbf{d}_e] = 0
\]  

(26)

from which the optional relationship between \([\mathbf{c}_i]\) and \([\mathbf{d}_e]\)

\[
[\mathbf{c}_i] = [\mathbf{H}_e]^{-1} [\mathbf{G}_e] [\mathbf{d}_e]
\]  

(27)

and the stiffness equation

\[
[\mathbf{K}_e][\mathbf{d}_e] = [\mathbf{0}]
\]  

(28)

are produced.

Once the homogeneous solution and the particular solution in each iteration step are evaluated using the hybrid approach and
the particular solution method, the complete solution can be obtained from Eq. (6). The main steps of the iteration strategy are described below:

1. **Initial guess:** \( u^{(0)} \).

2. **Iteration loop.**
   - For each iteration step:
     - (a) Calculate the right-hand term \( f(X_1X_2u^{(n-1)}) \).
     - (b) Solve Eq. (9) for the unknown interpolating coefficients \( \{a\} \).
     - (c) Calculate the approximating particular solutions \( u_p \) at boundary nodes by means of Eq. (12).
     - (d) Modify the specified Dirichlet boundary condition by setting \( u_b = \pi - u_p \).
     - (e) Solve the homogeneous system (8) with the present HFS-FEM.
     - (f) Calculate the solution \( u \) at the specified points.
     - (g) Check the convergence using Eqs. (4) and (5); if both Eqs. (4) and (5) are satisfied, go to next iterative step; otherwise go to (a).

3. **Numerical verification**

To assess the proposed computational model, three numerical examples are taken into consideration in this section. Because of the feature of the algorithm involving boundary integrals only, arbitrary shaped elements, which can model complex boundary geometry, can be easily constructed and used in the present algorithm. However, for simplicity, only 8-node parabolic quadrilateral elements are involved in the following numerical calculations, and in each element, 8 interior points which are the midpoints of the line connecting the element centroid and nodes, are chosen to perform RBF interpolation, as illustrated in Fig. 1. Moreover, the results from MATLAB® PDE Toolbox are provided for the purpose of comparison. In the calculation with MATLAB PDE Toolbox, to ensure mesh quality and validity of solution, the mesh was refined until there was less than 0.5% difference in the solution between refinements.

### 3.1. Nonlinear Poisson–Boltzmann equation in rectangular domain

The nonlinear Poisson–Boltzmann equation is often applied in bio-molecular process and electro-static interactions between colloidal particles. Here, we begin with solving this equation in a rectangular domain \([-1, 1] \times [-1, 1]\) to demonstrate the efficiency of the proposed method.

For \( f = 4 + \sinh(u) - \sinh(X_1^2 + X_2^2 + e^h \cos X_2) \), an exact solution is obtained for validating the proposed numerical method

\[
u = X_1^2 + X_2^2 + e^h \cos X_2
\]

which is also used to generate the Dirichlet boundary conditions along all outer boundaries.

Besides, an average relative error on potential \( u \) (\( \text{Arerr}(u) \)) defined as

\[
\text{Arerr}(u) = \sqrt{\frac{\sum_{i=1}^{N} (u_{\text{HFS-FEM}} - u_{\text{exact}})^2}{\sum_{i=1}^{N} (u_{\text{exact}})^2}}
\]

is introduced to study the accuracy and convergent behavior of the present algorithm, where \( u_{\text{HFS-FEM}} \) and \( u_{\text{exact}} \) denote the numerical and exact results of potential field \( u \), respectively.

In the practical computation, firstly \( 2 \times 2 \) elements with 21 nodes are used to model the entire domain. The initial guess of potential \( u^{(0)} \) is assumed to be zero. Fig. 2 plots the convergent history of the newly developed algorithm with \( \gamma = 5 \), whose reasonability is explained late. It is found that after 57 nonlinear iterations, the residual and the inter-iteration difference decrease rapidly to the machine accuracy, it is about \( 10^{-12} \). During iteration, there is about 14 orders of magnitude descent for the two norms, while the inter-iteration difference seems to have about one order lower than the residual. So, to achieve the desired accuracy, it is necessary to control the residual of governing equation during computation. In the following computation, the convergent tolerance of the residual is taken to be \( 10^{-6} \) to ensure the accuracy of solution.

On the other hand, the choice of the dimensionless parameter \( \gamma \) is another important factor to control the location of source points, which may affect the accuracy and convergence of the procedure. To clarify it, the variation of \( \text{Arerr}(u) \) is displayed against different values of \( \gamma \) in Fig. 3, from which it can be seen that small values of \( \gamma \) may cause larger error and vibration. It is because the distance between source point and field point is close to zero for this case, thus singular disturbance of fundamental solutions makes the numerical accuracy of element boundary integrals decrease and become unstable. Conversely, from the
point of view of numerical computation, the round-off error in floating point algorithm may lead to another problem when the source points are far from the element boundary, which corresponds to larger values of $g$. In this case, it is not beneficial to the inverse operation of square matrix $[H_{e}]$. It is found that the results are not sensitive to the parameter $g$ if it is within a range from 3 to 10 (see Fig. 3), so we will take $g=5$ in the following practical computation. Besides, refined meshes in both the transverse and longitudinal directions are also investigated for the convergent performance (see Fig. 3). For all three meshing schemes, the number of iterations is 29. As expected, improved numerical accuracy is observed in Fig. 3 along with an increase in the number of elements.

Finally, the contour line of potential field $u$ is shown in Fig. 4 with $4 \times 4$ elements, from which a good agreement of numerical results with the exact solution are observed. Additionally, the three-dimensional mesh plot of the potential is given in Fig. 5.

### 3.2. Nonlinear Poisson–Boltzmann equation in multi-connected domain

Having validated the procedure from example 1, in this example, we conduct numerical experiments to investigate the interaction of circular electrostatic particles, depicted by the nonlinear Poisson–Boltzmann equation with $f = \sin h(u)$, as shown in Fig. 6. In the model, the radii of two circles centered at $(\pm 1,0)$ are 0.4. The potentials $u=2$ and 1 are imposed on the outer and interior boundaries, respectively. Fig. 7 displays the mesh division in the domain with 224 quadrilateral elements and 743 nodes. After 96 iterations with initial guess $u^{(0)}=1$, convergent results are obtained with the presented approach, and the potential variation along the middle line $X_1=0$ is plotted in Fig. 8 to verify the numerical accuracy, by comparing with the results from MATLAB PDE Toolbox with similar or more nodal degrees of freedom (DOFs). The Newton iteration parameters used in MATLAB PDE Toolbox are set as $10^{-6}$ (nonlinear tolerance), 1 (initial solution), and fixed (Jacobian). It can be found that the results from HFS-FEM with the low level of discretization can match well with that of MATLAB with more nodal DOFs. However, the iteration number of MATLAB PDE Toolbox is 10 only. The main reason is that MATLAB PDE Toolbox based on the conventional finite element technology can conveniently treat the nonlinear effect in a uniform form, while in the present computational model, the problem must be split into two parts which may affect the convergent rate, but it is easy to understand and to use by practical researchers and engineers. The radial basis function approximation is resorted to determine the particular solutions related to the nonlinearity and the HFS-FEM is employed to evaluate the homogeneous solutions. Such strategy is also found in other fundamental solutions based methods, like the MFS and...
3.3. Oval of Cassini

To demonstrate the efficacy of the proposed method for irregular geometries, we consider the oval of Cassini, whose parametric equation is given by (see Fig. 11)

\[
X_1 = r(\theta)\cos(\theta) \\
X_2 = r(\theta)\sin(\theta) \\
r(\theta) = \sqrt{\cos(2\theta) + 1.1 - \sin^2(2\theta)} \quad (0 \leq \theta \leq 2\pi) \quad (31)
\]

The force function in the Poisson-type equation is chosen as \( f = 5u^n \) \( (n=1,2,3) \), which corresponds to the reaction–diffusion problem and the potential \( u \) represents the concentration of the diffusing species. Dirichlet boundary conditions \( u = 1 \) are applied on the entire outer boundary.

Due to symmetry in terms of geometry and boundary conditions, only one half of the domain is modeled with the finite element mesh as shown in Fig. 12, including 58 quadrilateral elements with 203 nodes. The initial guess of the concentration is 1. To illustrate numerical accuracy of the proposed HFS-FEM, Table 1 shows the comparison of the HFS-FEM, MFS, the constant element of dual reciprocity boundary element method (DRM) and

<table>
<thead>
<tr>
<th>( f = 5u )</th>
<th>( f = 5u^2 )</th>
<th>( f = 5u^3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>HFS-FEM</td>
<td>0.8366</td>
<td>0.8571</td>
</tr>
<tr>
<td>MFS [8]</td>
<td>0.794</td>
<td>*</td>
</tr>
<tr>
<td>DRM [8]</td>
<td>0.786</td>
<td>*</td>
</tr>
<tr>
<td>MATLAB</td>
<td>0.8370</td>
<td>0.8574</td>
</tr>
</tbody>
</table>

Note: the symbol * denotes the numerical results that are unavailable.
MATLAB solutions at the origin of the coordinate system for different values of $n$. Among them, the results of MFS and DRM are taken from Ref. [8], while results of MATLAB are obtained by means of the inherent PDE Toolbox with 1448 triangular finite elements and 787 nodes, based on conventional finite element techniques and Newton iteration. It can be clearly seen that the results of the proposed method are very close to those of MATLAB, while there are larger difference between MFS, DRM, and MATLAB (about 5% relative error in concentration). Hence, the presented approach has higher accuracy at the low level of discretization (230 nodal DOFs).

Besides, to investigate the effect of nonlinear forcing function $f$ to potential distribution, the potential variation along the $X_1$ axis is plotted in Fig. 13, from which it is found that the minimum value of the potential $u$ in the domain becomes larger, as the order $n$ of chemical reaction increases. As an example, in order to study the potential distribution in the entire domain, the two- and three-dimensional isotherms are presented in Figs. 14 and 15 for the case of $f=5u^3$. It can be seen that not only the proposed method can capture the variation of concentration profiles but also it preserves the symmetry of the concentration profiles about the horizontal axis.

4. Conclusions

In this paper, applicability of the fundamental-solution-based hybrid finite element model with radial basis function approximation has been demonstrated to solve nonlinear Poisson-type problems, such as the electrostatic Poisson–Boltzmann problems and chemical diffusion–reaction problems. In particular, the nonlinearity in the force function is removed by introducing the Picard iteration procedure, and then the radial basis function and the HFS-FEM are, respectively, formulated to approximate the particular solutions and homogeneous solutions of the induced Poisson problems in current iteration step. In the procedure, only boundary integrals are involved. Numerical experiments are carried out for accuracy and convergence investigations. A comparison of the numerical results evaluated with the proposed hybrid finite element model against the exact solutions or other numerical methods indicates that the proposed method is accurate and efficient in dealing with complicated geometry and strong nonlinearity.

References


