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A NEW PROCEDURE FOR THE NONLINEAR ANALYSIS OF REISSNER PLATE BY BOUNDARY ELEMENT METHOD

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Abstract—Based on a variational principle, a new approach to derive the exact boundary equation for the analysis of nonlinear Reissner plates and a related existing criterion for the solution of the boundary integral equation is presented in this paper. In this procedure, the boundary and the domain of the plate are discredited to solve the nonlinear problems. All unknown variables are at the boundary. Numerical results are also presented here to illustrate the method and demonstrate its effectiveness and accuracy.

NOTATION

- C — $5Eh/12(1+\nu)$
- C_w — a part of boundary $\partial\Omega$ of the solution domain Ω on which the deflection w is prescribed; C_R *et al.* can be defined similarly
- E — Young's modulus
- G_c — core shear modulus
- M_{ij} — bending moment tensor
- n — unit outward normal to $\partial\Omega$
- N_{ij} — membrane force tensor
- q — lateral distributed load
- Q_i — transverse shear force
- s — unit tangent to the boundary $\partial\Omega$
- t — plate thickness
- u_i — in-plane displacement
- U — strain energy density
- w — lateral deflection
- ν — Poisson's ratio
- Φ_i — average rotation of the normal to mid-surface in ith direction
- $(\bar{\ })$ — over a symbol denotes prescribed value

INTRODUCTION

Recently, several researchers have investigated the finite deformation behaviour of plates, such as Kamiya *et al.* [1], Tanaka [2], Qin [3] for thin plates and Lei *et al.* [4] for moderately thick plates. In the work reported by Lei *et al.* [4], a boundary element model for analysing the finite deflection of an isotropic plate taking into account the transverse shear deformation was deduced by way of a weighted residual method. In the course of derivation, the nonlinear terms are treated as a pseudo-transverse distributed load, which means that the nonlinear terms are considered as known external loads in the analysis.

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In this study, a set of exact boundary integral equations of a nonlinear Reissner plate are presented. Different from the previous work [4], here the nonlinear terms are treated as dependent on unknown displacements and stresses, rather than the pseudo-load. Furthermore, the integral equation is derived on the basis of a variational method, not the weighted residual method. To make the derivation tractable, a modified variational functional for the analysis of a geometrically nonlinear plate and the existing criterion of variational solution for the functional is presented originally. Finally as an application of the proposed method, a numerical example is illustrated. The results are in good agreement with already existing solutions.

VARIATIONAL PRINCIPLES

Consider an isotropic plate of uniform thickness h with mid-plane coordinates x and y. Indices i, j, k take values in the range $\{1,2\}$ and m takes values in the range $\{3,4,5\}$. The governing equations [4], which include the effects of transverse shear deformation, are

$$\Omega:\quad N_{ij,j}=0 \tag{1a}$$

$$M_{ij,j}-Q_i=0 \tag{1b}$$

$$Q_{i,1}+N_{ij}w_{,ij}+q=0 \tag{1c}$$

$$N_{i,j}=Gh(u_{i,j}+u_{j,i}+\nu(2u_{k,k}+w_{,k}w_{,ij})\delta/(1-\nu)) \tag{2}$$

$$M_{ij}=(1-\nu)D(\phi_{i,j}+\phi_{j,i}+2\nu\phi_{k,k}\delta_{ij}/(1-\nu))/2 \tag{3}$$

$$Q_i=C(W_{,i}-\phi_i) \tag{4}$$

and the boundary conditions are described by

$$C_{Nn}:\ N_n = N_{ij}n_in_j = \bar{N}_n;$$

$$C_{Nns}:\ N_{ns} = N_{ij}n_is_j = \bar{N}_{ns}$$

$$C_{Mn}:\ M_n = M_{ij}n_in_j = \bar{M}_n;$$

$$C_{Mns}:\ M_{ns} = M_{ij}n_is_j = \bar{M}_{ns}$$

$$C_R:\ R_n = Q_in_i + N_nw_{,n} + N_{ns}w_{,s} = \bar{R}_n \tag{5}$$

$$C_{Un}:\ u_n = u_in_i = \bar{u}_n;\quad C_{Us}:\ u_s = u_is_i = \bar{u}_s$$

$$C\varphi_n:\ \varphi_in_i = \bar{\varphi}_n;\quad C\varphi_s:\ \varphi_s = \varphi_is_i = \bar{\varphi}_s;$$

$$C_w:\ w = \bar{w} \tag{6}$$

$$(\partial\Omega = C_{Un}\cup C_{Nn} = C_{Us}\cup C_{Ns} = C_{\varphi n}\cup C_{Mn}$$

$$= C_{\varphi s}\cup C_{Mns} = C_w\cup C_R).$$

where (1) are the equilibrium equations, (2), (3) and (4) are constitutive equations, (5) and (6) the boundary conditions and all unspecified symbols are listed in the Notation at the beginning of the paper.

BOUNDARY INTEGRAL EQUATION

In what follows we derive a set of exact boundary integral equations for a nonlinear Reissner plate by way of a modified variational principle. To start with, we construct a functional Π_1' as below

$$\Pi_1' = \Pi_1 + \int_{cU_n}(\bar{u}_n - u_n)N_n\,\mathrm{d}c + \int_{cU_s}(\bar{u}_s - u_s)N_{ns}\,\mathrm{d}c$$
$$+ \int_{c\varphi_n}(\bar{\varphi}_n - \varphi_n)M_n\,\mathrm{d}c + \int_{c\varphi_s}(\bar{\varphi}_s - \varphi_s)M_{ns}\,\mathrm{d}c$$
$$+ \int_{c_w}(\bar{w} - w)R_n\,\mathrm{d}c \tag{7}$$

in which assume that (2)–(4) are identically satisfied.

Lemma 1

If the inequality

$$\int_\Omega N_i\delta w_{,i}\delta w_{,j}\,\mathrm{d}\Omega - \int_{cU_n}\delta u_n\delta N_n\,\mathrm{d}c - \int_{cU_s}\delta u_s\delta N_{ns}\,\mathrm{d}c$$
$$- \int_{c\varphi_n}\delta\varphi_n\delta M_n\,\mathrm{d}c - \int_{c\varphi_s}\delta\varphi_s\delta M_{ns}\,\mathrm{d}c$$
$$- \int_{c_w}\delta w\,\delta R_n\,\mathrm{d}c \geq 0 \tag{8}$$

holds in the neighbourhood of the solution of (1)–(6), we have

$$\Pi_1' \geq \Pi_{1o}', \tag{9}$$

where Π_{1o}' represents the stationary value of Π_1', and the equals sign holds if and only if the arguments of the functional Π_1' are at the critical point.

Proof

Taking the variation of Π_1', we see

$$\delta\Pi_1' = \iint_\Omega\{-N_{ij,j}\delta u - (M_{ij,j} - Q)\delta\phi_i$$
$$- (Q_{i,i} + N_{ij}w_{,ij} + q)\delta w\}\,\mathrm{d}\Omega$$
$$+ \int_{cU_n}(\bar{u}_n - u_n)\delta N_n\,\mathrm{d}c + \int_{cU_s}(\bar{u}_s - u_s)\delta N_{ns}\,\mathrm{d}c$$
$$+ \int_{c_w}(\bar{w} - w)\delta R_n\,\mathrm{d}c + \int_{c\varphi_n}(\bar{\varphi}_n - \varphi_n)\delta M_n\,\mathrm{d}c$$
$$+ \int_{c\varphi_s}(\bar{\varphi}_s - \varphi_s)\delta M_{ns}\,\mathrm{d}c$$
$$+ \int_{cNn}(N_n - \bar{N}_n)\delta u_n\,\mathrm{d}c$$
$$+ \int_{cNns}(N_{ns} - \bar{N}_{ns})\delta u_{ns}\,\mathrm{d}c$$
$$+ \int_{cR}(R_n - \bar{R}_n)\delta w\,\mathrm{d}c$$
$$+ \int_{cMn}(M_n - \bar{M}_n)\delta\phi_n\,\mathrm{d}c$$
$$+ \int_{cMns}(M_{ns} - \bar{M}_{ns})\delta\phi_s\,\mathrm{d}c \tag{10}$$

$$\delta\Pi_1' \xrightarrow{(2)\ \ (3)} 0 \rightarrow (1),\ (5)\text{ and }(6), \tag{11}$$

which means that (1), (5) and (6) are the stationary conditions of Π_1'. Consequently calculating the second variation of Π_1', one obtains

$$\delta^2\Pi_1' = \int_\Omega \delta^2U\,\mathrm{d}\Omega + \text{the left hand side of inequality (8).} \tag{12}$$

It should be noted, here, that the second variation δ^2U in (12) is with respect to variable strain, not directly to the displacement. So $\delta^2U > 0$. $\delta^2\Pi_1'$ will be, then, uniformly positive if (8) holds, and lemma 1 has been proved from the sufficient condition of local extreme of a functional [5].

Based on the functional Π_1', we obtain:

Theorem 1

The solutions of (1)–(6) satisfy the following boundary integral equations

$$
\begin{aligned}
\lambda x_k = &-\int_\Omega \tfrac{1}{2} w_{,i} w_{,j} N_{ij}^{*(k)}\,\mathrm{d}\Omega \\
&+\int_{c_{Nn}} (u_n^{*(k)} \bar{N}_n - u_n N_n^{*(k)})\,\mathrm{d}c \\
&+\int_{c_{Nns}} (u_s^{*(k)} \bar{N}_{ns} - u_s N_{ns}^{*(k)})\,\mathrm{d}c \\
&+\int_{c_{Un}} (u_n^{*(k)} N_n - \bar{u}_n N_n^{*(k)})\,\mathrm{d}c \\
&+\int_{c_{Us}} (u_s^{*(k)} N_{ns} - \bar{u}_s N_{ns}^{*(k)})\,\mathrm{d}c
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
\lambda x_m = &\int_\Omega (w^{*(m)} q - N_{ij} w_{,i} w_{,j}^{*(m)})\,\mathrm{d}\Omega \\
&+\int_{c_{Mn}} (\varphi_n M_n - \varphi_n M_c)\,\mathrm{d}c \\
&+\int_{c_{Mns}} (\varphi_s^{*(m)} \bar{M}_{ns} - \varphi_s M_{ns}^{*(m)})\,\mathrm{d}c \\
&+\int_{c_R} (w_n^{*(m)} \bar{R}_n - w \bar{Q}_n^{(m)})\,\mathrm{d}c \\
&+\int_{c_{\varphi n}} (\varphi_n^{*(m)} M_n - \bar{\varphi}_n M_n^{*(m)})\,\mathrm{d}c \\
&+\int_{c_{\varphi s}} (\varphi_s^{*(m)} M_{ns} - \bar{\varphi}_s M_{ns}^{*(m)})\,\mathrm{d}c \\
&+\int_{c_w} (w^{*(m)} R_n - \bar{w} Q_n^{*(m)})\,\mathrm{d}c
\end{aligned} \tag{14}
$$

and the solution of (13) and (14) exists if inequality (8) holds, where $\{x\} = \{x_1\ x_2\ x_3\ x_4\ x_5\} = \{u_1\ u_2\ w\ \phi_1\ \phi_2\}$ is a displacement vector, λ a conventional boundary shape coefficient, and symbols $(*)^{(K)}$ and $(*)^{(m)}$ represent the related functional solution which are well-documented in [6] and [7].

Proof

Noting that the displacement vector $\{x\}$ in (10) is not constrained by the boundary condition (6), the quantity $\delta\{x\}$ can be arbitrarily assumed. Naturally let

$$\delta\{x\} = \epsilon\{x^*\}^{(p)} = (P, Q) \quad (p = 1, 2, \ldots, 5) \tag{15}$$

where ϵ is infinitesimal. The components $x_q^*(P, Q)$ of $\{x^*\}^{(p)}(P, Q)$ means the displacements (for $q = 1$, 2 and 3) or the rotations (for $q = 4, 5$) at the field point Q of an infinite plate when a unit point force (for $p = 1$, 2 and 3) or a unit point couple (for $p = 4, 5$) is applied at the source point P. For the linear plate boundary expression, the fundamental solutions have been given in [8, 9]. Obviously the membranous equations (1a) and bending ones (1b, 1c) are independent of each other in the case of linear elasticity, that is, $x_i^{*(m)} = 0$ and $x_m^{*(i)} = 0$. Thus (11) can be further transformed into (13) and (14) by using the property of the fundamental vector $\{x\}$. The solutions of (1)–(6) satisfy (13) and (14), while the existing condition for the solution of integral equations (13) and (14) can be obtained from lemma 1. This completes the proof.

BOUNDARY ELEMENT ANALYSIS

In order to obtain a solution of (13) and (14), the boundary $\partial\Omega$ and the solution domain Ω of a plate are, respectively, divided into a series of boundary elements and internal cells as in the usual BEM. After performing the discretization by the use of various kinds of boundary element (e.g., constant element, linear element or higher-order element), (13) and (14) become two sets of linear algebraic equations including the variables N_n, N_s, u_n, u_s, M_n, M_{ns}, ϕ_n, ϕ_s, Q, w. Of the ten quantities, five need to be prescribed on the boundary points and the remaining five are to be determined. Since an incremental formulation may have a wider applicability to higher nonlinear problems it is necessary to express (13) and (14) in incremental form. Denoting the incremental variable by the superimposed dot, (13) and (14) can be expressed in matrix form:

$$[Q]\{N\} + [S]\{u\} = \{R_1\} \tag{16a}$$

$$[H]\{M\} + [G]\{\varphi\} = \{R_2\}, \tag{16b}$$

where $[Q]$, $[S]$, $[H]$ and $[G]$ denote the coefficient matrices which can be calculated in a usual way; while

$$\{N\} = \{N_n\ N_{ns}\}, \quad \{u\} = \{u_n\ u_s\}$$

$$\{M\} = \{R_n\ M_n\ M_{ns}\}, \qquad \{\phi\} = \{w\ \phi_n\ \phi_s\}.$$

$\{R_1\}$ and $\{R_2\}$ contain the nonlinear and inhomogeneous terms which can be deduced from eqns (13) and (14). To compute the nonlinear terms, an iterative procedure is required. An efficient iterative scheme given by Qin and Huan [6] will be adopted in the BE analysis. For the sake of conciseness, we omit those which are straightforward.

NUMERICAL EXAMPLE

The performance of the present element model is investigated by a benchmark problem. To study the convergence properties of the present approach, 16

Table 1. Central deflection (w/h) of the sandwich plate

Q	10	20	30	40
B_9 cells	0.720	1.311	1.692	1.917
E_{E16}	0.715	1.289	1.659	1.858
M_{25}	0.713	1.282	1.647	1.841
Schmit and Manforton†	0.70	1.26	1.62	1.82

† Values obtained from Fig. 4 on p. 1458 of [8].

constant elements on the boundary and three meshes of internal cell ($3 \times 3, 4 \times 4, 5 \times 5$) are used. The convergence tolerance is $\epsilon = 0.001$.

As an application of the proposed method, consider a plate consisting of two identical facings ($E = 0.74 \times 10^6$ kg/cm^2, $\nu = 0.3$, side length $2a = 127$ cm) which is $t = 0.381$ cm thick and an aluminium honeycomb core ($G_c = 0.35 \times 10^4$ kg/cm^2) which is $h = 2.54$ cm thick, and subjected to a uniform transverse load q; the boundaries of the square plate are fully clamped so that the imposed displacement boundary conditions are

$$u_1 = u_2 = \varphi_1 = \varphi_2 = w = 0 \text{ on the whole boundary.}$$

Table 1 shows the results for central deflection versus load parameter

$$Q = 12a^3(1 - \nu^2)q/(th^2E),$$

and comparison is made with those obtained by Schmit and Manforton [8].

It can be seen from those tables that the results obtained by employing the present method agree excellently with those appearing in [8]. The numerical results seem inert to the varying internal mesh from the table. In the couse of computation, convergence was achieved with between 12 and 18 iterations for each loading increment.

CONCLUDING REMARKS

In this study, a general and effective method is presented for establishing the exact nonlinear boundary integral equation and for deriving the existing criterion of its solution. In fact, the method is based on a modified variational principle which is given in the paper. The approach shows that a boundary integral formulation can be exactly transformed from a modified variational functional. Meanwhile, it also reveals the intrinsic relations between the variational principle and the boundary integral equation. The numerical example shows that the aforementioned boundary element model is very effective for nonlinear analysis of Reissner plates.

REFERENCES

1. N. Kamiya, Y. Sawaki and Y. Nakamure, Finite and post buckling deflection of heated plates and shallow shell. In *Boundary Elements* (Edited by C. A. Brebbia, T. Futagami and M. Tanaka). Springer-Verlag, Berlin, Heidelberg (1983).
2. M. Tanaka, Large deflection analysis of thin elastic plates. In *Developments in Boundary Element Methods*, Vol. 3 (Edited by P. K. Banerjee and S. Mukhoee). Applied Science Publishers, London, New York (1984).
3. Q. H. Qin, Variational principle, finite element and boundary element analysis of finite deformation of plates and shells. Ph.D. thesis, Huazhong University of Science and Technology (1990).
4. X. Y. Lei, M. K. Huang and X. X. Wang, Geometrically nonlinear analysis of a Reissner type plate by the boundary element methods. *Comput. Struct.* **37**, 911–906 (1990).
5. H. C. Simpson and S. J. Spector, On the positive of the second variation finite elasticity. *Arch. Rational Mech. Anal.* **98**, 1 (1987).
6. Q. H. Qin and Y. Y. Huang, BEM of postbuckling analysis of thin plate. *Appl. Math. Modelling* **14**, 544 (1982).
7. F. Van der Wreen, Application of the boundary integral equation method to Reissner's plate model. *Int. J. Numer. Meth. Engng* **18**, 1–10 (1982).
8. J. L. A. Schmit and G. R. Manforton, Finite deflection discrete element analysis of sandwich plates and cylindrical shells with laminated faces. *AIAA J.* **8**, 1454 (1970).