Anti-plane shear crack in a magnetoelectroelastic layer sandwiched between dissimilar half spaces

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Abstract

The crack problem of a magnetoelectroelastic layer bonded to dissimilar half spaces under anti-plane shear and in-plane electric and magnetic loads is considered. Fourier transforms are used to reduce the mixed boundary value problems of the crack, which is assumed to be permeable, to simultaneous dual integral equations, and then expressed in terms of Fredholm integral equations of the second kind. Numerical results show that the stress intensity factors are influenced by the magnetoelectric interactions and the geometry size ratio.

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1. Introduction

Composites made of piezoelectric/piezomagnetic materials exhibit magnetoelectric effects that are not present in single-phase piezoelectric or piezomagnetic materials, and they are extensively used as electric packaging, sensors and actuators, acoustic/ultrasonic devices, hydrophones, and transducers with the responsibility of electro-magneto-mechanical energy conversion [1,2]. Studies of the properties of piezoelectric/piezomagnetic composites have been carried out by numerous investigators in recent years [3–6]. These materials can fail due to some defects, e.g. cracks, holes, etc. arising during their manufacturing process, and there is a growing interest among researchers in solving fracture mechanics problems for magnetoelectroelastic materials. Recently, Song and Sih [7] investigated crack initiation behavior in a magnetoelectroelastic composite under in-plane deformation; Gao et al. [8] studied the fracture mechanics for a mode III crack in a magnetoelectroelastic solid. The general two-dimensional solutions to the magnetoelectroelastic problem of a crack were obtained by Wang and Mai [9,10] who also considered mode III crack problems in an infinite magnetoelectroelastic medium using a complex variable technique. An exact treatment of crack problems
in a magnetoelectric solid subjected to far-field loadings was carried out by Gao et al. [11]. Qin [12] derived 2D Green’s functions of defective magnetoelectric solids under thermal loading, which can be used to establish boundary formulations and to analyze relevant fracture problems. Li [13] performed transient analysis of a cracked magnetoelectric medium under anti-plane mechanical and in-plane electric and magnetic impacts. The moving crack problem in an infinite magnetoelectric material under longitudinal shear was studied by Hu and Li [14]. The dynamic solution of two collinear interface cracks in magnetoelectric composites under harmonic anti-plane shear wave loading was given by Zhou et al. [15]. However, the anti-plane crack problem in a magnetoelectric layer bonded to dissimilar elastic half spaces, can more accurately reflect the reality as most composite structures. To the authors’ knowledge, this problem has not yet been reported in the literature.

The objective of this paper is to seek the solution of the anti-plane crack problem in a magnetoelectric layer sandwiched between dissimilar half spaces under longitudinal shear. Fourier transforms are used to reduce the problem to the solution of dual integral equations. The solution of the dual integral equations is then expressed in terms of Fredholm integral equations of the second kind. Explicit expressions of the field intensity factors are obtained, and results show that the corresponding field intensity factors are influenced by the material constants, the geometry size ratio and the mechanical loads applied. The results show that the effect of magnetoelectric interaction and the geometry size ratio on the stress intensity factor is significant.

2. Formulation of the problem

Consider a Griffith crack of length $2c$ situated in the mid-plane of a magnetoelectric layer that is sandwiched between two elastic half planes with an elastic stiffness constant $\epsilon_{44}^E$, as shown in Fig. 1. Quantities in the elastic half plane will subsequently be designated by superscript $E$. A coordinate system $(x, y, z)$ is set at the center of the crack for reference. Due to the assumed symmetry in geometry and loading conditions, it is sufficient to consider the problem for $0 \leq x < \infty$, $0 \leq y < \infty$ only.

The magnetoelectric boundary value problem is simplified considerably if we consider only the out-of-plane displacement, the in-plane electric fields and in-plane magnetic fields, i.e.,

\begin{align}
  u_x &= u_y = 0, \quad u_z = u_z(x, y) \\
  E_x &= E_x(x, y), \quad E_y = E_y(x, y), \quad E_z = 0 \\
  H_x &= H_x(x, y), \quad H_y = H_y(x, y), \quad H_z = 0 \\
  u_x^E &= u_y^E = 0, \quad u_z^E = u_z^E(x, y)
\end{align}

Fig. 1. A magnetoelectric laminate with a finite crack.
where \((u_x, u_y, u_z), (E_x, E_y, E_z)\) and \((H_x, H_y, H_z)\) are the components of displacement, electric field and magnetic field vectors, respectively. The constitutive equations for anti-plane magneto-electroelastic material take the form of [14]:

\[
\begin{pmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{zz} \\
D_x \\
D_y \\
B_z
\end{pmatrix} =
\begin{pmatrix}
c_{44} & -e_{15} & -h_{15} \\
e_{15} & \lambda_{11} & \beta_{11} \\
h_{15} & \beta_{11} & \gamma_{11}
\end{pmatrix}
\begin{pmatrix}
u_{zx} \\
u_{zy} \\
u_{zz}
\end{pmatrix}
\begin{pmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{zz} \\
D_x \\
D_y \\
B_z
\end{pmatrix} =
\begin{pmatrix}
c_{44} & -e_{15} & -h_{15} \\
e_{15} & \lambda_{11} & \beta_{11} \\
h_{15} & \beta_{11} & \gamma_{11}
\end{pmatrix}
\begin{pmatrix}
u_{zx} \\
u_{zy} \\
u_{zz}
\end{pmatrix}
\]

(5)

where \(\sigma_{xx}, \sigma_{yy}, D_x, D_y\) and \(B_z, B_y\) are the components of stress, electric displacement and magnetic induction, respectively; \(c_{44}, e_{15}, h_{15}\) and \(\beta_{11}\) are elastic, piezoelectric, piezomagnetic and electromagnetic constants, respectively; \(\lambda_{11}\) and \(\gamma_{11}\) are dielectric permittivities and magnetic permeabilities, respectively. A comma followed by \(i (i = x, y)\) denotes partial differentiation with respect to the coordinate \(i\). The gradient equations are

\[
E_i = -\phi_{,i}, \quad H_i = -\varphi_{,i} \quad (i = x, y)
\]

(6)

where \(\phi\) and \(\varphi\) are electric potential and magnetic potential, respectively.

The governing equations are:

\[
c_{44}\nabla^2 u_x + e_{15}\nabla^2 \phi + h_{15}\nabla^2 \varphi = 0
\]

\[
e_{15}\nabla^2 u_y - \lambda_{11}\nabla^2 \phi - \beta_{11}\nabla^2 \varphi = 0
\]

\[
h_{15}\nabla^2 u_z - \beta_{11}\nabla^2 \phi - \gamma_{11}\nabla^2 \varphi = 0
\]

\[
\nabla^2 u_z^E = 0
\]

(7)

(8)

where \(\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2\) is the two-dimensional Laplacian operator in the variables \(x\) and \(y\).

We will consider four possible cases of electrical and magnetic boundary conditions on the edges of the magneto-electroelastic layer:

Case 1: \(D_y(x, h) = D_0, \quad B_y(x, h) = B_0\)

Case 2: \(E_y(x, h) = E_0, \quad B_y(x, h) = B_0\)

Case 3: \(D_y(x, h) = D_0, \quad H_y(x, h) = H_0\)

Case 4: \(E_y(x, h) = E_0, \quad H_y(x, h) = H_0\)

(9)

(10)

(11)

(12)

The mechanical conditions are:

\[
\sigma_{zz}(x, 0) = 0 \quad (0 \leq x < c)
\]

\[
u_z(x, 0) = 0 \quad (c \leq x < \infty)
\]

\[
\sigma_{zz}^E(x, y) = T_{\infty} \quad (x^2 + y^2 \to \infty)
\]

\[
\sigma_{zz}(x, h) = \sigma_{zz}^E(x, h)
\]

\[
u_z(x, h) = \nu_z^E(x, h)
\]

(13)

(14)

(15)

(16)

The shear stress \(T_{\infty}\) can be expressed as

\[
T_{\infty} = \begin{cases}
\frac{\mu}{\varepsilon_{44}} T_0 + \left(\frac{e_{15} \beta_{11} + h_{15} \lambda_{11}}{\gamma_{11}}\right) B_0 + \left(\frac{h_{15} \beta_{11} - e_{15} \gamma_{11}}{\beta_{11}}\right) D_0 \\
\frac{\mu}{\varepsilon_{44}} T_0 + \left(\frac{h_{15} \beta_{11} - e_{15} \gamma_{11}}{\beta_{11}}\right) E_0 - h_{15} B_0 \\
\frac{\mu}{\varepsilon_{44}} T_0 + \left(\frac{e_{15} \beta_{11} - h_{15} \gamma_{11}}{\beta_{11}}\right) H_0 - e_{15} D_0 \\
\frac{\mu}{\varepsilon_{44}} T_0 - e_{15} E_0 - h_{15} H_0
\end{cases}
\]

\(\text{(Case 1)}\)

\(\text{(Case 2)}\)

\(\text{(Case 3)}\)

\(\text{(Case 4)}\)

(17)
where $T_0$ is a uniform shear stress at zero electrical and magnetic loads, and $\mu_j (j = 1, 2, 3, 4)$ are the magneto-electroelastic stiffened elastic constants defined as

$$
\mu_1 = c_{44} + \frac{\gamma_{11} e_{15}^2 + \lambda_{11} h_{15}^2 - 2\beta_{11} h_{15} e_{15}}{\lambda_{11} \gamma_{11} - \bar{\beta}_{11}^2}
$$

$$
\mu_2 = (c_{44} \gamma_{11} + h_{15}^2) / \gamma_{11}
$$

$$
\mu_3 = (c_{44} \lambda_{11} + e_{15}^2) / \lambda_{11}
$$

$$
\mu_4 = c_{44}
$$

The electrical and magnetic conditions for the permeable crack case can be expressed as [11,14]:

$$
D_j (x, 0^+) = D_j (x, 0^-) \quad E_j (x, 0^+) = E_j (x, 0^-) \quad (0 \leq x < c)
$$

$$
\phi (x, 0) = 0 \quad (c \leq x < \infty)
$$

$$
B_j (x, 0^+) = B_j (x, 0^-) \quad H_j (x, 0^+) = H_j (x, 0^-) \quad (0 \leq x < c)
$$

$$
\varphi (x, 0) = 0 \quad (c \leq x < \infty)
$$

3. Solution procedure

Fourier transforms are applied to Eqs. (7) and (8), and we obtain the results as

$$
u_z (x, y) = \frac{2}{\pi} \int_0^{\infty} [A_1(x) \exp(xy) + A_2(x) \exp(-xy)] \cos(zx) dx + a_0 y
$$

$$
\phi (x, y) = \frac{2}{\pi} \int_0^{\infty} [B_1(x) \exp(xy) + B_2(x) \exp(-xy)] \cos(zx) dx - b_0 y
$$

$$
\varphi (x, y) = \frac{2}{\pi} \int_0^{\infty} [C_1(x) \exp(xy) + C_2(x) \exp(-xy)] \cos(zx) dx - c_0 y
$$

$$
u_z^E (x, y) = \frac{2}{\pi} \int_0^{\infty} A_3(x) \exp(-xy) \cos(zx) dx + d_0 y + e_0
$$

where $A_j (x) (j = 1, 2, 3)$ and $B_j (x) (i = 1, 2)$ are the unknowns to be solved and $a_0, b_0, c_0, d_0$ and $e_0$ are real constants which can be determined by considering the far-field and interface conditions as given in the Appendix. A simple calculation leads to the stress, electric displacement and magnetic induction expressions:

$$
\sigma_{xy} = \frac{2}{\pi} \int_0^{\infty} \left\{ [c_{44} A_1 (x) + e_{15} B_1 (x) + h_{15} C_1 (x)] \exp(xy) - [c_{44} A_2 (x) + e_{15} B_2 (x) + h_{15} C_2 (x)] \exp(-xy) \right\} \cos(zx) dx + c_{44} a_0 - e_{15} b_0 - h_{15} c_0
$$

$$
D_y = \frac{2}{\pi} \int_0^{\infty} \left\{ [e_{15} A_1 (x) - \lambda_{11} B_1 (x) - \beta_{11} C_1 (x)] \exp(xy) - [e_{15} A_2 (x) - \lambda_{11} B_2 (x) - \beta_{11} C_2 (x)] \exp(-xy) \right\} \cos(zx) dx + e_{15} a_0 + \lambda_{11} b_0 + \beta_{11} c_0
$$

$$
B_y = \frac{2}{\pi} \int_0^{\infty} \left\{ [h_{15} A_1 (x) - \beta_{11} B_1 (x) - \gamma_{11} C_1 (x)] \exp(xy) - [h_{15} A_2 (x) - \beta_{11} B_2 (x) - \gamma_{11} C_2 (x)] \exp(-xy) \right\} \cos(zx) dx + h_{15} a_0 + \beta_{11} b_0 + \gamma_{11} c_0
$$

$$
\sigma^E_{xy} = \frac{-2}{\pi} c_{44} \int_0^{\infty} \left\{ A_3 (x) \exp(-xy) \cos(zx) dx + c_{44} d_0 \right\}
$$
Satisfaction of the boundary conditions (9)–(12), (15) and (16) leads to the result that

\[ A_1(x) = \frac{(\mu_i - c_{44}^F) \exp(-2xh)F(x)}{\Omega_i} \]  
\[ A_2(x) = \frac{(\mu_i + c_{44}^F)F(x)}{\Omega_i} \]  
\[ A_3(x) = 2\mu_i F(x)/\Omega_i \]  
\[ B_1(x) = \frac{\exp(-2xh)[G(x) + m_i F(x)]}{1 + \exp(-2xh)} \]  
\[ B_2(x) = \frac{G(x) - m_i \exp(-2xh)F(x)}{1 + \exp(-2xh)} \]  
\[ C_1(x) = \frac{\exp(-2xh)[H(x) + n_i F(x)]}{1 + \exp(-2xh)} \]  
\[ C_2(x) = \frac{H(x) - n_i \exp(-2xh)F(x)}{1 + \exp(-2xh)} \]

where \( F(x) \), \( G(x) \) and \( H(x) \) are the only unknown functions, and \( m_i, n_i \) and \( \Omega_i (i = 1, 2, 3, 4) \) are defined for Case \( i \) (1, 2, 3, 4), respectively as

\[ m_1 = \frac{2c_{44}^e(h_{15}^i - e_{11}^1)}{\Omega_i}, \quad m_2 = m_4 = 0, \quad m_3 = -\frac{2c_{44}^e e_{15}}{\lambda_1 \Omega_3} \]  
\[ n_1 = \frac{2c_{44}^e(e_{15} h_{15}^i - h_{15}^i \beta_{11}^i)}{\Omega_i(\lambda_1 \gamma_{11}^1 - \beta_{11}^i)} \quad n_2 = \frac{2c_{44}^e h_{15}^i}{\gamma_{11} \Omega_2}, \quad n_3 = n_4 = 0 \]  
\[ \Omega_i = \mu_i + c_{44}^e + (\mu_i - c_{44}^F) \exp(-2xh) \quad (i = 1, 2, 3, 4) \]

By applying the mixed boundary conditions (13), (19) and (20), we can reduce the problem to the unknowns \( F(x) \), \( G(x) \) and \( H(x) \) that satisfy the following simultaneous dual integral equations

\[ \int_0^\infty zM_i(x)F(x) \cos(xz) \, dx = \frac{\pi T}{2c_{44}} \quad (0 \leq x < c) \quad (i = 1, 2, 3, 4) \]  
\[ \int_0^\infty F(x) \cos(xz) \, dx = 0 \quad (x \geq c) \]  
\[ \int_0^\infty G(x) \sin(xz) \, dx = 0 \quad (0 \leq x < c) \]  
\[ \int_0^\infty G(x) \cos(xz) \, dx = 0 \quad (x \geq c) \]  
\[ \int_0^\infty H(x) \cos(xz) \, dx = 0 \quad (0 \leq x < c) \]  
\[ \int_0^\infty H(x) \cos(xz) \, dx = 0 \quad (x \geq c) \]

where \( M_i(x) \) were defined as

\[ M_1(x) = \frac{2}{\Omega_1} \left\{ \mu_i + c_{44}^e + \frac{2c_{44}^e \exp(-2xh)(\gamma_{11}^i e_{15}^2 + \lambda_1 h_{15}^i - 2\beta_{11}^i h_{15} e_{15})}{c_{44}(\lambda_1 \gamma_{11}^1 - \beta_{11}^i)(1 + \exp(-2xh))} \right\} - 1 \]  
\[ M_2(x) = \frac{2}{\Omega_2} \left\{ \mu_i + c_{44}^e + \frac{2c_{44}^e h_{15}^i \exp(-2xh)}{c_{44}(\gamma_{11}^i)(1 + \exp(-2xh))} \right\} - 1 \]  
\[ M_3(x) = \frac{2}{\Omega_3} \left\{ \mu_i + c_{44}^e + \frac{2c_{44}^e e_{15} \exp(-2xh)}{c_{44}(\lambda_1)(1 + \exp(-2xh))} \right\} - 1 \]  
\[ M_4(x) = \frac{2(\mu_i + c_{44}^e)}{\Omega_4} - 1 \]
Eqs. (39)–(41) can be solved by using the method of Copson [16], and the solutions are found as follows

\[ F(\xi) = \frac{\pi T \xi^2}{2c_{44}} \int_{0}^{\infty} \xi \Phi_1(\xi) J_0(\alpha \xi) d\xi \]

\[ G(\xi) = \frac{\pi T \xi^2}{2c_{44}} \int_{0}^{\infty} \xi \Phi_2(\xi) J_0(\alpha \xi) d\xi \]

\[ H(\xi) = \frac{\pi T \xi^2}{2c_{44}} \int_{0}^{\infty} \xi \Phi_3(\xi) J_0(\alpha \xi) d\xi \]

where \( J_0(\cdot) \) is the zero order Bessel function of the first kind. The function \( \Phi_i(\xi) \) should satisfy the Fredholm integral equations of the second kind in the form,

\[ \Phi_i(t) + \int_{0}^{1} \Phi_i(\eta) K_i(\eta, t) d\eta = 1 \quad (i = 1, 2, 3, 4) \]

where

\[ K_i(\eta, t) = \eta \int_{0}^{\infty} s[M_i(s/c) - 1] J_0(st) J_0(s\eta) d\eta \]

The functions \( \Psi_i(\xi) (i = 1, 2) \) are \( \Psi_i(\xi) = 0 \).

The stress intensity factor (SIF), the electric displacement intensity factor (EDIF), and the magnetic induction intensity factor (MIIF) are defined and determined, respectively as

\[ K^T = \lim_{\varepsilon \to c} \sqrt{2\pi(x-c)\tau_{23}(x,0)} = T_{\infty} \Phi_1(1) \sqrt{\pi c} \quad (i = 1, 2, 3, 4) \]

\[ K^D = \lim_{\varepsilon \to c} \sqrt{2\pi(x-c)D_3(x,0)} = \frac{e_{15}}{c_{44}} K^T \]

\[ K^B = \lim_{\varepsilon \to c} \sqrt{2\pi(x-c)B_3(x,0)} = \frac{h_{15}}{c_{44}} K^T \]

For this particular problem, the stresses, electric displacements and magnetic inductions at the crack tip show inverse square root singularities. It is clear that the SIF, EDIF and MIIF are dependent on the geometry size of the magnetoelectroelastic layer, the mechanical load conditions and the material constants.

In the case of \( \beta_{11} = 0 \) and \( h_{15} = 0 \), our results are exactly reduced to the solution of a cracked piezoelectric layer bonded to dissimilar half spaces given by Narita et al. [17]. This shows that our solutions are correct and universal.

### 4. Numerical results and discussion

From expressions (49)–(51) we know that the determination of the field intensity factors must require the solution of the function \( \Phi_i(1) = \Phi_i(t)|_{t=1} \). The solution of the Fredholm integral equation (47) can be solved by computer with the use of Gaussian quadrature formulas.

The magnetoelectroelastic layer is assumed to be a transversely isotropic material exhibiting full coupling between elastic, electric, and magnetic fields, with a unique axis along the \( z \) direction. The material constants we used in the following numerical calculation are selected as [5,8]:

\[ c_{44} = 4.53 \times 10^{10} \text{ (N/m}^2) , \quad e_{15} = 11.6 \text{ (C/m}^2) , \quad h_{15} = 550 \text{ N/A m} \]

\[ \lambda_{11} = 0.8 \times 10^{-10} \text{ (C}^2/\text{N m}^2), \quad \gamma_{11} = 5.9 \times 10^{-4} \text{ (N s}^2/\text{C}^2) \]

\[ \beta_{11} = 0.5 \times 10^{-11} \text{ (N s/V C)} \]

The material constants of the half spaces we used are that of aluminum as [17]

\[ c_{44}^E = 2.65 \times 10^{10} \text{ (N/m}^2) \quad \text{(for aluminum)} \]
To analyze the electrical and magnetic effects of the magnetoelectroelastic layer on the fracture behavior of the laminate, we define the normalized stress intensity factor $K$ as

$$K = K^T / T_0 \sqrt{\pi c} = T_\infty \Phi(1)/T_0$$

(54)

Fig. 2 shows the variation of the normalized stress intensity factor $K$ versus the layer-thickness to crack-length ratio $h/c$ at various values of the normalized electrical and magnetic loads $L_1 = c_{44}(e_{15}b_{11} - h_{15}b_{11})[D_0 + h_{15}b_{11} - e_{15}b_{11}]B_0 / T_{00}(e_{15}b_{11} - b_{11})$ (Case 1), and indicates that the SIF is always larger than the value $T_\infty \sqrt{\pi c}$ corresponding to the infinite magnetoelectroelastic material when $h \to \infty$. It is shown that the SIF increases when the thickness of the magnetoelectroelastic layer decreases, which is also seen in Figs. 3–5. When electric field and magnetic induction loads $L_2 = c_{44}[(e_{15}e_{15} - h_{15}h_{15})E_0 + h_{15}h_{15}]B_0 / T_{00}(e_{15}h_{15} - h_{15})$ are applied, the effects of the thickness of the layer on SIF were displayed in Fig. 3, for Case 2. Fig. 4 shows the variations of $h/c$ on $K$ when electric displacement and magnetic field $L_3 = c_{44}[(h_{15}h_{15} - h_{15}h_{15})H_0 + h_{15}H_0]B_0 / T_{00}(h_{15}h_{15} - h_{15})$ are applied to the magnetoelectroelastic layer. The higher ratio $h/c$, the smaller value of $K$. When electric field and magnetic field $L_4 = (e_{15}E_0 + h_{15}H_0)/T_0$ are applied, the corresponding results of $K$ versus $h/c$ are given in Fig. 5. From Figs. 2–5, we know that the magnetoelectric effects on the normalized stress intensity factors are larger for Cases 1 and 3 than that for Cases 2 and 4. For $L_i \to 1$ ($i = 1, 2, 3, 4$), $K$ approaches zero, which could be seen in Figs. 2–5.
5. Concluding remarks

The magnetoelectroelastic problem of a finite crack in a magnetoelectroelastic layer bonded to dissimilar elastic half spaces under longitudinal shear has been analyzed theoretically. A closed-form solution to the anti-plane crack problem of the laminate has been obtained. The stress intensity factor (SIF), electric displacement intensity factor (EDIF) and magnetic induction intensity factor (MIIF) was obtained, and the results were presented to study the influence of electric and magnetic loads on the fracture behavior of the magnetoelectroelastic laminate. The SIF can either be increased or decreased by varying the thickness of the magnetoelectroelastic layer. Our analysis show that, at a given mechanical load, the presence of electric and magnetic loads can retard crack growth, depending on the magnitude and type of the loads applied.

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Appendix

The constants \( a_0, b_0, c_0, d_0 \) and \( e_0 \) can be obtained by considering the far-field and interface conditions as:

Case 1:
\[
\begin{pmatrix}
  a_0 \\
  b_0 \\
  c_0
\end{pmatrix} =
\begin{pmatrix}
  c_{44} & -e_{15} & -h_{15} \\
  e_{15} & \lambda_{31} & \beta_{11} \\
  h_{15} & \beta_{11} & \gamma_{11}
\end{pmatrix}^{-1}
\begin{pmatrix}
  T_\infty \\
  D_0 \\
  B_0
\end{pmatrix}
\]

Case 2:
\[
\begin{pmatrix}
  a_0 \\
  c_0
\end{pmatrix} =
\begin{pmatrix}
  c_{44} & -h_{15} \\
  h_{15} & \gamma_{11}
\end{pmatrix}^{-1}
\begin{pmatrix}
  T_\infty + e_{15}E_0 \\
  B_0 - \beta_{11}E_0
\end{pmatrix}
\]

Case 3:
\[
\begin{pmatrix}
  a_0 \\
  b_0
\end{pmatrix} =
\begin{pmatrix}
  c_{44} & -e_{15} \\
  e_{15} & \lambda_{11}
\end{pmatrix}^{-1}
\begin{pmatrix}
  T_\infty + h_{15}H_0 \\
  D_0 - \beta_{11}H_0
\end{pmatrix}
\]

Case 4:
\[
a_0 = \frac{1}{c_{44}} (T_\infty + e_{15}E_0 + h_{15}H_0), \quad b_0 = E_0, \quad c_0 = H_0
\]
\[
d_0 = T_\infty / c_{44}^E, \quad e_0 = h(a_0 - T_\infty / c_{44}^E)
\]

References