Thermoelectroelastic solutions for surface bone remodeling under axial and transverse loads

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Abstract

Theoretical prediction of surface bone remodeling in the diaphysis of the long bone under various external loads are made within the framework of adaptive elastic theory. These loads include external lateral pressure, electric and thermal loads. Two solutions are presented for analyzing thermoelectroelastic problems of surface bone remodeling. The analytical solution that gives explicit formulation is capable of modeling homogeneous bone materials, while the semi-analytical solution is suitable for analyzing inhomogeneous cases. Numerical results are presented to verify the proposed formulation and to show the effects of mechanical, thermal and electric loads on surface bone remodeling process.

Keywords: Bone surface remodeling; Piezoelectric; Thermal field

1. Introduction

The investigation of remodeling properties and smart behavior of bone tissue and their applications to biomedical engineering by considering its coupled elastic, magnetic, electric behavior has received a considerable interest among the scientists of different fields in the past decades [1–5]. Early in 1950s, Fukada and Yasuda [1, 2] found that some living bone and collagen have piezoelectricity. Later on, Gjelsvik [6] presented a physical description of the remodeling of bone tissue, in terms of very simplified form of the linear theory of piezoelectricity. Williams and Breger [7] explored the applicability of stress gradient theory for explaining the experimental data for a cantilever bone beam subjected to constant end load and showed that the approximate gradient theory is in good agreement with the experimental data. Guzelsu [8] presented a piezoelectric model for analyzing cantilever dry bone beam subjected to a vertical end load. Johnson et al. [9] further addressed the problem of dry bone beam by presenting some theoretical expressions for piezoelectric response to cantilever bending of the beam. Demiray [10] gave some theoretical descriptions on electro-mechanical remodeling models of bones. Aschero et al. [11] investigated converse piezoelectric effect of fresh bone by using a high-sensitive dilatometer. They provided further investigations on piezoelectric properties of bone and presented a set of repeated measurements of coefficient $d_{23}$ on 25 cow bone samples [12]. Fotiadis et al. [13] studied the wave propagation in a long cortical piezoelectric bone with arbitrary cross-section. El-Naggar et al. [14] and Ahmed et al. [15] further obtained an analytical solution on wave propagation in long cylindrical bones with and without cavity. Silva et al. [16] explored physico-chemical, dielectric and piezoelectric properties of anionic collagen...
and collagen–hydroxyapatite composites. Recently, Qin and Ye [4] presented a thermoelastoelectric solution for internal bone remodeling. Tsutada and Adachi [17] studied spatial and temporal regulation process of cancellous bone using computer simulation. Most developments in this area can also be found in [3,18]. It can be seen from the discussion above that bone remodeling is a highly organized process, but the mechanisms which determine where, when, and how remodeling occurs are still open questions.

In this work, two solutions for thermodistoelectric problems of surface bone remodeling, based on the theory of adaptive elasticity [19], are presented to study the effects of mechanical, thermal and electric loads on surface bone remodeling process. The analytical solution is used for investigating surface bone remodeling process on the basis of assuming a homogeneous bone material [4], while the semi-analytical solution is developed for analyzing bone materials that are assumed radially inhomogeneous. Numerical results are presented to show applicability of the proposed solutions and the effect of thermal and electric loads on the bone surface remodeling process.

2. Solution of surface modeling for a homogeneous hollow circular cylindrical bone

2.1. Equation for surface bone remodeling

The equations of the theory of adaptive elasticity of Cowin and vanBuskirk [19] are used and extended to include piezoelectric effect in this study. The remodeling rate equation in cylindrical coordinates is

$$U = C_{ij}(n,Q) [s_{ij}(Q) - s_{ij}^0(Q)] + C_i [E_i(Q) - E_i^0(Q)]$$

$$= C_{rr} s_{rr} + C_{\theta\theta} s_{\theta\theta} + C_{zz} s_{zz} + C_{rz} s_{rz} + C_{\epsilon r},$$

$$+ C_1 E_z - C_0,$$  \( \text{(1)} \)

where \( C_0 = C_{rr}^0 + C_{zz}^0 + C_{\theta\theta}^0 + C_{rz}^0 + C_{\epsilon r}^0 + C_1 E_z^0; \) \( U \) denotes the velocity of the remodeling normal to the surface; \( C_{ij} \) and \( C_i \) are surface remodeling coefficients.

2.2. Analytical solution

We now consider a hollow circular cylinder of bone, subjected to an external temperature change \( T_0, \) a quasi-static axial load \( P, \) an external pressure \( p \) and an electric potential load \( \phi_a (\text{or } \phi_b). \) The boundary conditions are

\[
\begin{align*}
T &= 0, & \sigma_{rr} = \sigma_{\theta\theta} = \sigma_{rz} &= 0, & \phi = \phi_a & \text{at } r = a, \\
T &= T_0, & \sigma_{rr} = -P, & \sigma_{\theta\theta} = \sigma_{rz} = 0, & \phi = \phi_b & \text{at } r = b.
\end{align*}
\]

\( \int_S \sigma_{zz} dS = -P, \)  \( \text{(3)} \)

where \( a \) and \( b \) denote, respectively, the inner and outer radii of the bone, and \( S \) is the cross-sectional area. For a long bone, it is assumed that except the axial displacement \( u_z, \) all displacements, temperature and electrical potential is independent of the \( z \) coordinate and that \( u_z \) may have linear dependence on \( z. \) The solution of displacements \( u_r, u_z \) and electric potential \( \phi \) to the problem above has been discussed elsewhere [4]. For the reader’s convenience we list them in Appendix A at the end of this paper.

The strains and electric field intensity can be found by introducing Eqs. (A.7)–(A.10) into (A.2) [the solution given in appendix]. They are, respectively,

\[
s_{rr} = \frac{1}{F_3} \left[ c_{33} \beta_1 [\beta_2^2 T_0 + p(t)] + \frac{c_{33} \beta_2}{c_{11}} \right],
\]

\[
+ \frac{F_2 T_0 + P(t)}{\pi (b^2 - a^2)} C_{13} - F_1 T_0 C_{13} \right]
\]

\[
- \frac{a^2 \beta_1 \beta_2^2 T_0 + p(t)}{r^2 (c_{11} - c_{12})} + \frac{\ln (r/a)}{c_{11}},
\]

\( \text{(4)} \)

\[
s_{\theta\theta} = \frac{1}{F_3} \left[ c_{33} \beta_1 [\beta_2^2 T_0 + p(t)] + \frac{c_{33} \beta_2}{c_{11}} \right]
\]

\[
+ \frac{F_2 T_0 + P(t)}{\pi (b^2 - a^2)} C_{13} - F_1 T_0 C_{13} \right]
\]

\[
+ \frac{a^2 \beta_1 \beta_2^2 T_0 + p(t)}{r^2 (c_{11} - c_{12})} + \frac{\ln (r/a) - 1}{c_{11}},
\]

\( \text{(5)} \)

\[
s_{zz} = \frac{1}{F_3} \left[ F_1 T_0 - F_2 T_0 + P(t) \right] \left[ c_{11} + c_{12} \right]
\]

\[
- 2C_1 C_3 \beta_1 \beta_2^2 T_0 + p(t) - 2 \frac{c_{11} c_{12} \beta_2}{c_{11}},
\]

\( \text{(6)} \)

\[
s_{rz} = \frac{e_1 (\phi_a - \phi_b)}{r c_{44} \ln (b/a)},
\]

\( \text{(7)} \)

\[
E_r = \frac{\Phi_b - \Phi_a}{r \ln (b/a)}.
\]

\( \text{(8)} \)

Substituting (4)–(8) into (1) yields

\[
E_v = N_{c0}^v \frac{b^2}{b^2 - a^2} + N_{e0}^v \frac{1}{\ln (b/a)} + N_{e0}^v \frac{1}{b^2 - a^2}
\]

\[
+ N_{e0}^v \frac{1}{a \ln (b/a)} - C_{0},
\]

\( \text{(9)} \)
where

\[
N_1^p = \frac{1}{F_3} \left\{ c_{13} \left( \frac{c_{13}}{c_{11}} \beta_1 - \beta_3 \right) T_0 + \frac{c_{13}}{c_{11}} \beta_1 \left( \frac{c_{12}}{c_{11}} - 1 \right) T_0 + p(t) \right\} \left( c_{11} - c_{12} \right)
\]

\[
N_2^p = \frac{1}{F_3} \left\{ c_{15} \frac{c_{13} c_{33}}{c_{11}} \beta_1 T_0 + \frac{c_{13}}{c_{11}} \left( \frac{c_{12}}{c_{11}} - 1 \right) \left( \frac{c_{13}}{c_{11}} \beta_1 \right) T_0 + p(t) \right\} \left( c_{11} - c_{12} \right)
\]

and the subscripts, p and e, refer to periosteal and endosteal, respectively. Since \( U_e \) and \( U_p \) are the velocities normal to the inner and outer surfaces of the cylinders, respectively, they are calculated as

\[
U_e = -\frac{da}{dt}, \quad U_p = \frac{db}{dt}
\]

where the minus sign appearing in the expression for \( U_e \) denotes that the outward normal of the endosteal surface is in the negative coordinate direction. Thus, Eq. (9) can be written as

\[
\frac{da}{dt} = N_1^p \frac{b^2}{b^2 - a^2} + N_2^p \frac{1}{\ln(b/a)} + N_3^p \frac{1}{b^2 - a^2}
\]

\[
\frac{db}{dt} = N_1^p \frac{b^2}{b^2 - a^2} + N_2^p \frac{1}{b^2 - a^2} + N_3^p \frac{1}{b \ln(b/a)}
\]

3. Approximation for small changes in radii

It is apparent that Eq. (20) are non-linear and cannot be solved analytically. However, the equations can be linearized approximately when they are applied to solve problems with small changes in radii. In a bone surface remodeling process, we can assume that the radii of the inner and outer surface of the bone change very little compared to their original values. This means that the changes in \( a(t) \) and \( b(t) \) are small. This is a reasonable assumption from the viewpoint of physics of the problem. To introduce the approximation the following non-dimensional parameters:

\[
\varepsilon = \frac{a}{a_0} - 1, \quad \eta = \frac{b}{b_0} - 1
\]

are adopted in the following calculations. As a result, \( a(t) \) and \( b(t) \) can be written as \( a(t) = (1 + \varepsilon(t))a_0, \quad b(t) = (1 + \eta(t))b_0 \). Since both \( \varepsilon \) and \( \eta \) are far smaller than one, their squares can be ignored from the equations. Consequently, we can have the following approximations:

\[
\frac{b^2}{b^2 - a^2} \approx \frac{L_0 + 2L_2 \frac{a^2}{b_0^2} (\varepsilon - \eta)}{a^2 - a^2}
\]

\[
\frac{a^2}{b^2 - a^2} \approx \frac{L_0 + 2L_2 \frac{b^2}{a_0^2} (\varepsilon - \eta)}{a^2 - a^2}
\]
\[
\frac{1}{b^*-a^*} \approx L_2 + 2L_2^2(a_0^2c - b_0^2\eta),
\]
(24)
\[
\frac{1}{\ln(b/a)} \approx L_1 + L_1^2(\varepsilon - \eta),
\]
(25)
\[
\frac{1}{a \ln(b/a)} \approx \frac{1}{a_0} L_1(1 - \varepsilon) + \frac{1}{a_0} L_1^2(\varepsilon - \eta),
\]
(26)
\[
\frac{1}{b \ln(b/a)} \approx \frac{1}{b_0} L_1(1 - \eta) + \frac{1}{b_0} L_1^2(\varepsilon - \eta),
\]
(27)

where
\[
L_0 = \frac{b_0^2}{b_0^2 - a_0^2},
\]
(28)
\[
L_0' = \frac{a_0^2}{b_0^2 - a_0^2},
\]
(29)
\[
L_1 = \frac{1}{\ln(b_0/a_0)},
\]
(30)
\[
L_2 = \frac{1}{1 - a_0^2/b_0^2}
\]
(31)

Thus, Eq. (20) can be approximately represented in terms of \(\varepsilon\) and \(\eta\), as follows:
\[
\frac{d\varepsilon}{dt} = B_1\varepsilon + B_2\eta + B_3,
\]
\[
\frac{d\eta}{dt} = B'_1\varepsilon + B'_2\eta + B'_3,
\]
(32)

where
\[
B_1 = -\frac{1}{a_0} \left(2L_2^2\frac{a_0^2}{b_0^2} N_1^e + L_1^2 N_2^e - \frac{L_1 N_4^p}{a_0} + \frac{L_1^2 N_5^p}{a_0}
+ 2L_2^2\frac{a_0^2}{b_0^2} N_3^e\right),
\]
(33)
\[
B_2 = \frac{1}{a_0} \left(2L_2^2\frac{a_0^2}{b_0^2} N_1^e + L_1^2 N_2^e + \frac{L_1 N_4^p}{a_0} + 2L_2^2\frac{a_0^2}{b_0^2} N_3^e\right),
\]
(34)
\[
B_3 = -\frac{1}{a_0} \left(L_0 N_1^e + L_1 N_2^e + L_2 N_3^e + \frac{L_1 N_4^p}{a_0} - C_0^e\right),
\]
(35)
\[
B'_1 = \frac{1}{b_0} \left(2L_2^2\frac{b_0^2}{b_0^2} N_1^p + L_1^2 N_2^p + \frac{L_1 N_4^p}{b_0} + 2L_2^2\frac{b_0^2}{a_0^2} N_3^p
+ 2L_2^2\frac{b_0^2}{a_0^2} N_1^p\right).
\]
(36)
\[
B'_2 = -\frac{1}{b_0} \left(2L_2^2\frac{a_0^2}{b_0^2} N_1^p + L_1^2 N_2^p - \frac{L_1 N_4^p}{b_0} + \frac{L_1^2 N_5^p}{b_0}
+ 2L_2^2\frac{b_0^2}{a_0^2} N_3^p + 2L_2^2\frac{a_0^2}{b_0^2} N_1^p\right),
\]
(37)
\[
B'_3 = \frac{1}{b_0} \left(L_0 N_1^p + L_1 N_2^p + L_2 N_3^p + \frac{L_1 N_4^p}{b_0}
+ L'_0 N_1^p - C_1^p\right).
\]
(38)

4. Analytical solution of surface remodeling

An analytical solution of Eq. (32) can be obtained if smeared homogeneous property is assumed for a bone material. In such a case, the inhomogeneous linear differential equations system (32) can be converted into the following homogeneous one:
\[
\frac{d\varepsilon'}{dt} = B_1\varepsilon' + B_2\eta',
\]
\[
\frac{d\eta'}{dt} = B'_1\varepsilon' + B'_2\eta',
\]
(39)

where
\[
\varepsilon' = \varepsilon - \varepsilon_\infty,
\]
\[
\eta' = \eta - \eta_\infty,
\]
(40)
\[
\varepsilon_\infty = \frac{1}{\det M} (B_1 B_2 - B_3 B_2'),
\]
\[
\eta_\infty = \frac{1}{\det M} (B_3 B_1' - B_3' B_1),
\]
(41)
\[
M = \begin{pmatrix} B_1 & B_2 \\ B'_1 & B'_2 \end{pmatrix},
\]
(42)

\[
\det M = B_1 B_2' - B_1' B_2.
\]
(43)

The solution of Eq. (39) subject to the initial conditions that \(\varepsilon(0) = 0\) and \(\eta(0) = 0\) can be expressed in four possible forms that fulfill the physics of the problem, i.e., when \(t \to \infty\), \(\varepsilon\) and \(\eta\) must be limited quantities, \(a < b\) and the solution must be stable. The form of the solution depends on the roots of the following quadratic equation:
\[
s^2 - \text{tr} M s + \det M = 0,
\]
(44)

where
\[
\text{tr} M = B_1 + B_2' = s_1 + s_2.
\]
(45)

All the valid solutions are showed as follows:

Case A: When \((B_1 - B_2')^2 + 4B_2 B'_1 > 0\), \(B_1 + B_2' < 0\) and \(B_1 B'_2 - B_2' B_1 > 0\), Eq. (44) has two different roots: \(s_1\) and \(s_2\), both of which are real and distinct. Then the
solutions of the equations are
\[
\dot{e}' = \frac{1}{s_1 - s_2} \left[ (s_2 e_\infty - B_3) e^{-s_2 t} + (B_3 - s_1 e_\infty) e^{-s_1 t} \right]
\]
\[
\eta' = \frac{1}{s_1 - s_2} \left[ (s_2 \eta_\infty - B_3') e^{-s_2 t} + (B_3' - s_1 \eta_\infty) e^{-s_1 t} \right],
\]
which can also be written as
\[
\dot{e}(t) = e_\infty + \frac{1}{s_1 - s_2} \left[ (s_2 e_\infty - B_3) e^{-s_1 t} + (B_3 - s_1 e_\infty) e^{-s_2 t} \right],
\]
\[
\eta(t) = \eta_\infty + \frac{1}{s_1 - s_2} \left[ (s_2 \eta_\infty - B_3') e^{-s_1 t} + (B_3' - s_1 \eta_\infty) e^{-s_2 t} \right].
\]

The formulae for the variation of the radii, i.e., \(a(t)\) and \(b(t)\), with time can be obtained by substituting (47) into (21). Thus
\[
a(t) = a_0 + a_0 e_\infty + \frac{a_0}{s_1 - s_2} \left[ (s_2 e_\infty - B_3) e^{-s_2 t} + (B_3 - s_1 e_\infty) e^{-s_1 t} \right],
\]
\[
b(t) = b_0 + b_0 \eta_\infty + \frac{b_0}{s_1 - s_2} \left[ (s_2 \eta_\infty - B_3') e^{-s_2 t} + (B_3' - s_1 \eta_\infty) e^{-s_1 t} \right].
\]

The final radii of the cylinder are then
\[
a_\infty = \lim_{t \to \infty} a(t) = a_0 (1 + e_\infty),
\]
\[
b_\infty = \lim_{t \to \infty} b(t) = b_0 (1 + \eta_\infty).
\]

Case B: When \((B_1 - B_2')^2 + 4B_2B_1' = 0\), \(B_1 \neq B_2'\) and \(B_1 + B_2' < 0\), Eq. (44) has two equal roots: \(B_2' + B_1\). The solutions of the equations are
\[
\dot{e}' = - \left\{ e_\infty + \frac{B_1 - B_2'}{2} e_\infty + B_2 \eta_\infty \right\} t e^{\frac{B_1 - B_2'}{2} t},
\]
\[
\eta' = - \left\{ \eta_\infty - \frac{(B_2 - B_1)^2}{4B_2} e_\infty - \frac{B_2' - B_1}{2} \eta_\infty \right\} t e^{\frac{B_2 - B_1}{2} t},
\]
which can also be written as
\[
\dot{e}(t) = e_\infty - \left\{ e_\infty + \frac{B_1 - B_2'}{2} e_\infty + B_2 \eta_\infty \right\} t e^{\frac{B_1 - B_2'}{2} t},
\]
\[
\eta(t) = \eta_\infty - \left\{ \eta_\infty - \frac{(B_2 - B_1)^2}{4B_2} e_\infty - \frac{B_2' - B_1}{2} \eta_\infty \right\} t e^{\frac{B_2 - B_1}{2} t}.
\]

The formulae for the variation of \(a(t)\) and \(b(t)\) with time can be obtained by substituting (51) into (21) as
\[
a(t) = a_0 + a_0 e_\infty - a_0 \left\{ e_\infty + \frac{B_1 - B_2'}{2} e_\infty + B_2 \eta_\infty \right\} t e^{\frac{B_1 - B_2'}{2} t}
\]
\[
+ B_2 \eta_\infty \right\} t e^{\frac{B_1 - B_2'}{2} t},
\]
\[
b(t) = b_0 + b_0 \eta_\infty - b_0 \left\{ \eta_\infty - \frac{(B_2 - B_1)^2}{4B_2} e_\infty - \frac{B_2' - B_1}{2} \eta_\infty \right\} t e^{\frac{B_2 - B_1}{2} t}.
\]

Case C: When \(B_1 = B_2' < 0\) and \(B_2 = 0\), the solutions of the equations are
\[
\dot{e}' = - e_\infty e^{B_1 t},
\]
\[
\eta' = - \left( B_1 e_\infty t + \eta_\infty \right) e^{B_1 t},
\]
which can also be written as
\[
\dot{e}(t) = e_\infty - e_\infty e^{B_1 t},
\]
\[
\eta(t) = \eta_\infty - \left( B_1 e_\infty t + \eta_\infty \right) e^{B_1 t}.
\]

The formulae for the variation of \(a(t)\) and \(b(t)\) with time can be obtained by substituting (55) into (21), as follows:
\[
a(t) = a_0 + a_0 e_\infty \left( 1 - e^{B_1 t} \right),
\]
\[
b(t) = b_0 + b_0 \eta_\infty - b_0 \left( B_1 e_\infty t + \eta_\infty \right) e^{B_1 t}.
\]

Case D: When \(B_1 = B_2' < 0\) and \(B_1' = 0\), the solutions of the equations are
\[
\dot{e}' = - (B_2 \eta_\infty t + e_\infty) e^{B_2' t},
\]
\[
\eta' = - \eta_\infty e^{B_2' t},
\]
which can also be written as
\[
\dot{e}(t) = e_\infty - (B_2 \eta_\infty t + e_\infty) e^{B_2' t},
\]
\[
\eta(t) = \eta_\infty - \eta_\infty e^{B_2' t}.
\]

The formulae for the variation if the radii with time can be obtained by substituting (59) into (21). Thus
\[
a(t) = a_0 + a_0 e_\infty - a_0 \left( B_2 \eta_\infty t + e_\infty \right) e^{B_2' t},
\]
\[
b(t) = b_0 + b_0 \eta_\infty \left( 1 - e^{B_2' t} \right).
\]
The final radii of the cylinder are then
\[
\begin{align*}
\alpha_\infty &= \lim_{t \to \infty} a(t) = a_0(1 + \epsilon_\infty), \\
b_\infty &= \lim_{t \to \infty} b(t) = b_0(1 + \eta_\infty).
\end{align*}
\] (61)

All the above solutions are theoretically valid. However, only the first one is the most possible solution to the problem, as it is physically possible when \( t \to \infty \). Therefore, it can be used to calculate the bone surface remodeling.

5. Semi-analytical solution for inhomogeneous cylindrical bone layers

The solution obtained in the previous section is suitable for analyzing bone cylinders if they are assumed to be homogenous [19]. It can be useful if explicit expressions and a simple analysis are required. It is a fact, however, that all bone materials exhibit inhomogeneity. In particular, for a hollow bone cylinder, the volume friction of bone matrix materials varies from inner surface to outer surface. To solve this problem we present a semi-analytical model as below.

Considering Eqs. (A.1)–(A.3) and assuming a constant longitudinal strain, the following first-order differential equations can be obtained [4]:

\[
\frac{\partial}{\partial r} \begin{pmatrix} u_r \\ \sigma_r \end{pmatrix} = \begin{pmatrix} \frac{1}{\psi} & \frac{1}{c_1} \\ \frac{c_1}{\psi} & \frac{1}{c_1} \end{pmatrix} \begin{pmatrix} u_r \\ \sigma_r \end{pmatrix} + \begin{pmatrix} \frac{1}{c_1(1-c_2/c_1)} \\ \frac{c_1(1-c_2/c_1)}{\psi} \end{pmatrix} e_z + \begin{pmatrix} \frac{1}{c_1(1-c_2/c_1)R} \\ \frac{1}{c_1(1-c_2/c_1)R} \end{pmatrix} T,
\] (62)

where \( \psi = c_{11} - c_{12}^2/c_{11} \). In the above equation, the effect of electrical potential is absent. This is because it is independent of \( u_r \) and \( \sigma_r \). The contribution of electrical field can be calculated separately as described in the previous section and then included in the remodeling rate equation.

Assuming that a bone layer is sufficiently thin, we can replace \( r \) with its mean value \( R \), and let \( r = a + z \), where \( 0 \leq z \leq h \), \( a \) and \( h \) are the inner radius and the thickness of the thin bone layer, respectively. Thus, Eq. (62) is reduced to

\[
\frac{\partial}{\partial z} \begin{pmatrix} u_r \\ \sigma_r \end{pmatrix} = \begin{pmatrix} \frac{1}{\psi} & \frac{1}{c_1} \\ \frac{c_1}{\psi} & \frac{1}{c_1} \end{pmatrix} \begin{pmatrix} u_r \\ \sigma_r \end{pmatrix} + \begin{pmatrix} \frac{1}{c_1(1-c_2/c_1)} \\ \frac{1}{c_1(1-c_2/c_1)R} \end{pmatrix} e_z + \begin{pmatrix} \frac{1}{c_1(1-c_2/c_1)R} \\ \frac{1}{c_1(1-c_2/c_1)R} \end{pmatrix} T.
\] (63)

The above equation can be written symbolically as

\[
\frac{\partial}{\partial z}[\mathbf{F}] = [\mathbf{G}][\mathbf{F}] + \{\mathbf{H}_L\} + \{\mathbf{H}_T\},
\] (64)

where \([\mathbf{G}]\), \([\mathbf{H}_L]\) and \([\mathbf{H}_T]\) are all constant matrices.

Eq. (63) can be solved analytically and the solution is [20]

\[
\begin{pmatrix} u_r(z) \\ \sigma(z) \end{pmatrix} = e^{Gz} \begin{pmatrix} u_r(0) \\ \sigma(0) \end{pmatrix} + \int_0^b e^{G(z-t)}[\mathbf{H}_L]dr
\] + \int_0^b e^{G(z-t)}[\mathbf{H}_T]dr,
\] (65)

where \( u_r(0) \) and \( \sigma_r(0) \) are, respectively, the displacement and stress at bottom surface of the layer. Rewrite Eq. (65) as

\[
\{\mathbf{F}(z)\} = [\mathbf{D}(z)][\mathbf{F}(0)] + \{\mathbf{D}_L\} + \{\mathbf{D}_T\}.
\] (66)

The exponential matrix can be calculated as follows:

\[
[\mathbf{D}(z)] = e^{Gz} = z_0(z)I + x_1(z)[\mathbf{G}],
\] (67)

where \( z_0(z) \) and \( x_1(z) \) can be solved from\n
\[
x_0(z) + x_1(z)\lambda_1 = e^{Gz},
\] (68)

\[
x_0(z) + x_1(z)\lambda_2 = e^{Gz}.
\] (69)

In Eq. (68) \( \lambda_1 \) and \( \lambda_2 \) are two eigenvalues of \([\mathbf{G}]\), which are given by

\[
\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = -\frac{1}{2R} \pm \frac{1}{2R} \sqrt{5 - 4 \frac{c_{12}}{c_{11}}}.
\] (70)

The axial stress applied at the end of the bone can be found as

\[
\sigma_r = c_{13} \left( 1 - \frac{c_{12}}{c_{11}} \right) \frac{u}{R} + \left( c_{33} - \frac{c_{12}^2}{c_{11}} \right) e_z + \frac{c_{13}}{c_{11}} \sigma_r + \frac{c_{13}}{c_{11}} \sigma_r
\] + \left( \frac{c_{13}}{c_{11}} \beta_1 - \beta_3 \right) T.
\] (71)

The stress problem can be solved by introducing the boundary conditions described on the top and bottom surfaces into Eq. (70) and

\[
\int_S \{c_{13}(1 - \frac{c_{13}}{c_{11}}) \frac{u}{R} + (c_{33} - \frac{c_{12}^2}{c_{11}}) e_z + \frac{c_{13}}{c_{11}} \sigma_r
\] + \left( \frac{c_{13}}{c_{11}} \beta_1 - \beta_3 \right) T\}dS = -P(t).
\] (72)

For a thick-walled bone section or a section with variable volume fraction in the radial direction, we can divide the bone into a number of sub-layers, each of which is sufficiently thin and is assumed to be composed of a homogeneous material. Within a layer we take the mean value of volume fraction of the layer as the layer's
volume fraction. As a consequence, the analysis described above for a thin and homogeneous bone can be applied here for the sub-layer in a straightforward manner. For instance, for the \( j \)th layer, Eq. (70) becomes

\[
\{ \mathbf{F}^{(i)}(h_j) \} = \{ \mathbf{D}^{(i)}(h_j) \} \{ \mathbf{F}^{(0)}(0) \} + \{ \mathbf{D}^{(i)}_L \} + \{ \mathbf{D}^{(i)}_T \}, \tag{73}
\]

where \( h_j \) denotes thickness of the \( j \)th sub-layer.

Consider the continuity of displacements and transverse stresses across the interfaces between these fictitious sub-layers, we have

\[
\{ \mathbf{F}^{(i)}(h_j) \} = \{ \mathbf{F}^{(i+1)}(0) \}. \tag{74}
\]

After establishing Eq. (73) for all sub-layers, the following equation can be obtained by using Eqs. (73) and (74) recursively:

\[
\begin{align*}
\{ \mathbf{F}(h_N) \} &= \{ \mathbf{D}^{(N)}(h_N) \} \{ \mathbf{F}(h_{N-1}) \} + [ \{ \mathbf{D}^{(N)}_L \} + \{ \mathbf{D}^{(N)}_T \} ] \\
&\quad + \{ \mathbf{D}^{(N-1)}_L \} + \{ \mathbf{D}^{(N-1)}_T \} ] \\
&\quad + \{ \mathbf{D}^{(N-2)}_L \} + \{ \mathbf{D}^{(N-2)}_T \} ] \\
&\quad + \cdots \\
&\quad + \{ \mathbf{D}^{(1)}_L \} + \{ \mathbf{D}^{(1)}_T \} ] \\
&\quad + \{ \mathbf{D}^{(0)}_L \} + \{ \mathbf{D}^{(0)}_T \} ] \\
&\quad + \{ \mathbf{D}^{(N)}(h_N) \} \{ \mathbf{D}^{(N-1)}(h_{N-1}) \} \\
&\quad \cdots \\
&\quad + \{ \mathbf{D}^{(N-j)}(h_{N-j}) \} \{ \mathbf{F}(h_{N-j-1}) \} \\
&\quad + \{ \mathbf{D}^{(N-j+1)}(h_{N-j+1}) \} \{ \mathbf{D}^{(N-j)}_L \} + \{ \mathbf{D}^{(N-j)}_T \} \\
&\quad + \{ \mathbf{D}^{(N-j+2)}(h_{N-j+2}) \} \{ \mathbf{D}^{(N-j+1)}_L \} + \{ \mathbf{D}^{(N-j+1)}_T \} \\
&\quad + \{ \mathbf{D}^{(N-j+3)}(h_{N-j+3}) \} \{ \mathbf{D}^{(N-j+2)}_L \} + \{ \mathbf{D}^{(N-j+2)}_T \} \\
&\quad + \cdots \\
&\quad + \{ \mathbf{D}^{(N-1)}(h_{N-1}) \} \{ \mathbf{D}^{(N-2)}_L \} + \{ \mathbf{D}^{(N-2)}_T \} \\
&\quad + \{ \mathbf{D}^{(N-1)}_L \} + \{ \mathbf{D}^{(N-1)}_T \} ] \\
&\quad + \{ \mathbf{D}^{(N)}_L \} + \{ \mathbf{D}^{(N)}_T \} ] \\
&\quad + \cdots \\
&\quad = \{ \mathbf{Y} \} \{ \mathbf{F}(0) \} + \{ \mathbf{\Omega} \}, \tag{75}
\end{align*}
\]

where

\[
\{ \mathbf{Y} \} = \prod_{j=N}^1 \{ \mathbf{D}^{(j)}(h_j) \},
\]

\[
\{ \mathbf{\Omega} \} = \sum_{j=2}^N \left( \prod_{j=N}^1 \{ \mathbf{D}^{(j)}(h_j) \} \right) \{ \{ \mathbf{D}^{(N-1)}_L \} + \{ \mathbf{D}^{(N-1)}_T \} \} \\
+ \{ \{ \mathbf{D}^{(N)}_L \} + \{ \mathbf{D}^{(N)}_T \} \}. \tag{76}
\]

It can be seen that Eq. (75) has the same structure and dimension as those of Eq. (70). After introducing the boundary condition imposed on the two transverse surfaces and considering Eq. (72), the surface displacements and/or stresses can be obtained. Introducing these solutions back to the equations at sub-layer level, the displacements, stresses and then strains within each sub-layers can be further calculated.

6. Surface bone remodeling induced by a medullar pin

Prosthetic devices often employ metallic pins fitted into the medulla of a long bone as a means of attachment. These medullar pins will cause the bone in the vicinity of the pin to change its internal structure and external shape. In this section we propose a model for external changes in bone shape. The theory is applied here to the problem of a pin force-fitted into the medulla to determine the changes in external bone shape that result. The diaphyseal region of a long bone is modeled here as a hollow circular cylinder, and external changes in shape are changes in the external and internal radii of the hollow circular cylinder.

The solution of this problem can be obtained by decomposing the title problem into two separate sub-problems: the problem of the remodeling of a hollow circular cylinder of adaptive bone material subjected to external loads, and the problem of an isotropic solid elastic cylinder subjected to an external pressure. These two problems are illustrated in Fig 1.

For an isotropic solid elastic cylinder subjected to an external pressure \( p(t) \), the displacement in the radial direction is given by

\[
u = \frac{-2(\mu + \lambda)p(t)r}{2\mu(3\lambda + 2\mu)}, \tag{77}
\]

where \( \lambda \) and \( \mu \) are Lamé’s constants for the isotropic solid elastic cylinder. The displacement of the endosteal surface of the bone is shown in Eq. (10), i.e.,

\[
u_r = \frac{r}{F_3} \left( c_{33} b_{11}^2 T_0 + p(t) + \frac{c_{33} C_{12}}{c_{11}} \right) + \frac{F_3 T_0 + P(t)}{\pi(b^2 - a^2)} c_{13} - F_3 T_0 c_{13} \right) \]

\[
+ \frac{a^2 b_{11}^2 T_0 + p(t)}{r(c_{11} - c_{12})} + \frac{\pi r[ln(r/a) - 1]}{c_{11}}. \tag{78}
\]

Fig. 1. The decomposition of the medullar pin problem into two separate sub-problems.
In this problem, we calculate the pressure of interaction \( p(t) \) which occurs when an isotropic solid cylinder of radius \( a_0 + \delta/2 \) is forced into a hollow adaptive bone cylinder of radius \( a_0 \).

Let \( a \) and \( b \) denote the inner and outer radii, respectively, of the hollow bone cylinder at the instant after the solid isotropic cylinder has been forced into the hollow cylinder. Although, the radii of the hollow cylinder will actually change during the adaptation process, the deviation of these quantities from \( a \) and \( b \) will be a small quantity negligible in the small strain theory.

At an arbitrary time instant after the two cylinders have been forced together the pressure of the interaction is \( p_1(t) \). The radial displacement of the solid cylinder at its surface is

\[
u_1 = \frac{-(2\mu + \lambda)p_1(t)a}{2\mu(3\lambda + 2\mu)}. \tag{79}\]

The radial displacement of the bone at its inner surface is

\[
u_2 = \frac{a}{F_3^2} \left( c_{33} \beta_1^*[\beta_2^* T_0 - p_1(t) + p(t)] + \frac{c_{33} c_{12}}{c_{11}} \right) \frac{F_2^* T_0 + p(t)}{\pi(b^2 - a^2)} \left( c_{13} - F_1^* T_0 c_{13} \right) + \frac{a \beta_1^*[\beta_2^* T_0 - p_1(t) + p(t)]}{(c_{11} - c_{12})} - \frac{\varepsilon \alpha}{c_{11}}. \tag{80}\]

Since it is assumed that the two surfaces have perfect contact, the two displacements have the following relationship:

\[
a_0 + \delta/2 + u_1 = a_0 + u_2. \tag{81}\]

Hence we find

\[
\delta = 2(u_2 - u_1). \tag{82}\]

Solving Eq. (82) for \( p_1(t) \) we obtain

\[
p_1(t) = -\frac{1}{H} \left[ \frac{\delta}{a} - \left( H_1 \frac{b^2}{b^2 - a^2} + H_2 \frac{1}{b^2 - a^2} \right) \right], \tag{83}\]

where

\[
H = \frac{2\mu + \lambda}{\mu(3\lambda + 2\mu)} - 2 \left( \frac{c_{33}}{F_3^2} + \frac{1}{c_{11} - c_{12}} \right) \frac{b^2}{b^2 - a^2}, \tag{84}\]

\[
H_1 = \frac{2}{F_3^2} \left[ c_{11} \left( \frac{c_{13}}{c_{11}} - \frac{1}{c_{11} - c_{12}} \right) \beta_1 - \beta_3 \right] T_0 \right] - 2 \left( \frac{c_{33}}{F_3^2} + \frac{1}{c_{11} - c_{12}} \right) \beta_2^* T_0 + p(t), \tag{85}\]

\[
H_2 = \frac{2}{F_3^2} \frac{c_{11} P(t)}{\pi}, \tag{86}\]

\[
H_3 = \frac{c_{33} c_{11} \beta_1 T_0}{F_3^2 c_{11}} - \frac{2}{F_3^2} \left( \frac{c_{13}}{c_{11}} \beta_1 - \beta_3 \right) c_{13} T_0 - \frac{\beta_1 T_0}{c_{11}}. \tag{87}\]

Introducing Eq. (83) into Eqs. (4)–(8) yields

\[
-H \frac{da}{dt} = N_1^e \frac{b^2}{b^2 - a^2} + N_2^e \frac{1}{\ln(b/a)} + N_3^e \frac{1}{b^2 - a^2} \left( M_1 p_1(t) \frac{b^2}{a^2 - b^2} - C_0^e \right), \tag{88}\]

where

\[
M_1 = \frac{(C_{rr}^e + C_{00}^e) c_{33} - 2 c_{11} C_{zz}^e}{c_{11} - c_{12}} - \frac{C_{rr}^e - C_{00}^e}{c_{11} - c_{12}}, \tag{89}\]

\[
M_2 = \frac{(C_{rr}^p + C_{00}^p) c_{33} - 2 c_{11} C_{zz}^p}{c_{11} - c_{12}}, \tag{90}\]

\[
M_3 = -\frac{C_{rr}^p - C_{00}^p}{c_{11} - c_{12}}. \tag{91}\]

It can be seen that Eq. (88) is similar to Eq. (20). It can also be simplified as

\[
\frac{d\varepsilon}{dt} = Y_1 \varepsilon + Y_2 \eta + Y_3, \tag{92}\]

where

\[
Y_1 = B_1 - M_1 \left( H_5 \left( \frac{\delta}{a_0} - H_1 L_0 - H_2 L_2 - H_3 L_1 \right) \right) + H_4 \left( 2L_2^2 H_1 a_0^2 + 2L_2^2 H_2 a_0^2 + H_2 L_1^2 + \frac{\delta}{a_0} \right), \tag{93}\]

\[
Y_2 = B_2 + M_1 \left( H_5 \left( \frac{\delta}{a_0} - H_1 L_0 - H_2 L_2 - H_3 L_1 \right) \right) + H_4 \left( 2L_2^2 H_1 a_0^2 + 2L_2^2 H_2 a_0^2 + H_2 L_1^2 \right), \tag{94}\]

\[
Y_3 = B_3 + M_1 H_4 \left( \frac{\delta}{a_0} - H_1 L_0 - H_2 L_2 - H_3 L_1 \right), \tag{95}\]
\( Y'_1 = B'_1 - M_2 \left[ H_5 \left( \frac{\delta}{a_0} - H_1 L_0 - H_2 L_2 - H_3 L_1 \right) \right. \]
\[ + H_4 \left( 2L_0^2 H_1 \frac{a_0^2}{b_0^2} + 2L_2^2 H_2 a_0^2 + H_3 L_2^2 + \frac{\delta}{a_0} \right) \]
\[ + M_3 \left[ H_7 \left( \frac{\delta}{a_0} - H_1 L_0 - H_2 L_2 - H_3 L_1 \right) \right. \]
\[ - H_6 \left( 2L_0^2 H_1 \frac{a_0^2}{b_0^2} + 2L_2^2 H_2 a_0^2 + H_3 L_2^2 + \frac{\delta}{a_0} \right) \].
\[ (96) \]

\( Y'_2 = B'_2 + M_2 \left[ H_5 \left( \frac{\delta}{a_0} - H_1 L_0 - H_2 L_2 - H_3 L_1 \right) \right. \]
\[ + H_4 \left( 2L_0^2 H_1 \frac{a_0^2}{b_0^2} + 2L_2^2 H_2 b_0^2 + H_3 L_2^2 \right) \]
\[ - M_3 \left[ H_7 \left( \frac{\delta}{a_0} - H_1 L_0 - H_2 L_2 - H_3 L_1 \right) \right. \]
\[ - H_6 \left( 2L_0^2 H_1 \frac{a_0^2}{b_0^2} + 2L_2^2 H_2 b_0^2 + H_3 L_2^2 \right) \].
\[ (97) \]

\( Y'_3 = B'_3 + (M_2 H_4 + M_3 H_6) \left( \frac{\delta}{a_0} - H_1 L_0 - H_2 L_2 \right. \)
\[ \left. - H_3 L_1 \right) \].
\[ (98) \]

\( H_4 = \frac{1}{2 \left( \frac{c_1}{c_1} + \frac{1}{c_{11} - c_{12}} \right) - \frac{2\mu + \nu}{\mu(3\mu + 2\nu)} \left( 1 - \frac{a_0^2}{b_0^2} \right) \}. \]
\[ (99) \]

\( H_5 = \left( \frac{4\mu + 2\kappa a_0^2}{\mu(3\mu + 2\nu) b_0^2} \right)^2 \frac{1}{2 \left( \frac{c_1}{c_1} + \frac{1}{c_{11} - c_{12}} \right) - \frac{2\mu + \nu}{\mu(3\mu + 2\nu)} \left( 1 - \frac{a_0^2}{b_0^2} \right) \}. \]
\[ (100) \]

\( H_6 = \left( \frac{2\mu + \nu}{\mu(3\mu + 2\nu)} + \left[ 2 \left( \frac{c_1}{c_1} + \frac{1}{c_{11} - c_{12}} \right) - \frac{2\mu + \nu}{\mu(3\mu + 2\nu)} \right] \frac{b_0^2}{a_0^2} \right)^{\frac{1}{2}} \}
\[ (101) \]

\( H_7 = \left( \frac{2\mu + \nu}{\mu(3\mu + 2\nu)} + \left[ 2 \left( \frac{c_1}{c_1} + \frac{1}{c_{11} - c_{12}} \right) - \frac{2\mu + \nu}{\mu(3\mu + 2\nu)} \right] \frac{b_0^2}{a_0^2} \right)^{\frac{1}{2}} \}
\[ (102) \]

Eq. (92) can be solved by following the solution process described in Section 4.

7. Numerical examples

7.1. A hollow, homogeneous circular cylindrical bone subjected to various external loads

As numerical illustration of the proposed analytical and semi-analytical solutions, we consider a femur with \( a = 25 \text{ mm} \) and \( b = 35 \text{ mm} \). The material properties assumed for the bone are:

\( c_{11} = 15 \text{ GPa}, \quad c_{12} = c_{13} = 6.6 \text{ GPa}, \)

\( c_{33} = 12 \text{ GPa}, \quad c_{44} = 4.4 \text{ GPa}, \)

\( \beta_1 = 0.621 \times 10^5 \text{ N K}^{-1} \text{ m}^{-2}, \)

\( \beta_3 = 0.551 \times 10^5 \text{ N K}^{-1} \text{ m}^{-2}, \)

\( p_3 = 0.0133 \text{ C K}^{-1} \text{ m}^{-2}, \)

\( e_{31} = -0.435 \text{ C/m}^2, \)

\( e_{33} = 1.75 \text{ C/m}^2, \quad e_{15} = 1.14 \text{ C/m}^2, \)

\( k_1 = 111.5 k_0, \quad k_3 = 126 k_0, \)

\( k_0 = 8.85 \times 10^{-12} \text{ C}^2/\text{N m}^2 = \text{permittivity of free space}. \)
\[ (103) \]

The surface remodeling rate coefficients are assumed to be:

\( C^e_{rr} = -9.6 \text{ m/day}, \quad C^e_{\theta\theta} = -7.2 \text{ m/day}, \)

\( C^e_{rr} = -5.4 \text{ m/day}, \quad C^e_{\theta\theta} = -8.4 \text{ m/day}, \)

\( C^p_{rr} = -12.6 \text{ m/day}, \quad C^p_{\theta\theta} = -10.8 \text{ m/day}, \)

\( C^p_{rr} = -9.6 \text{ m/day}, \quad C^p_{\theta\theta} = -12 \text{ m/day}, \)

and

\( C^e_0 = 0.0008373 \text{ m/day}, \quad C^p_0 = 0.00015843 \text{ m/day}. \)

The initial inner and outer radii are assumed to be

\( a_0 = 25 \text{ mm}, \quad b_0 = 35 \text{ mm} \)

and \( a_0 = 0, \quad \eta_0 = 0 \) are assumed.

We distinguish following five loading cases:

1. \( p(t) = n \times 2 \text{ MPa} \) (\( n = 0.8, 0.9, 1, 1.1 \) and 1.2), \( P(t) = 1500 \text{N} \), and no other types of loads are applied.

The extended results for this loading case are shown in Fig. 2 to study the effect of external pressure on the bone remodeling process. It displays that both the transverse and axial loads have the same effect on the bone. The inner radius of the bone decreases, while the outer one increases, as the external pressure increases. This results in an increase in the bone’s cross-sectional area and, consequently, a thicker and stronger bone. And when the external pressure decreases, the inner radius increases and the outer one decreases, which means that the bone becomes thinner and weaker. On the other hand, the larger pressure can increase the velocity of bone surface remodeling, which can accelerate the recovery of the injured bone.
Fig. 2 presents an interesting change of radius against time. The outer radius of the bone increases at first as the transverse pressure increases. It begins to decrease after a few days and finally converges to a stable value that is greater than its initial one. Similar result was presented by Cowin and Van Buskirk [19]. The earlier analysis was based on a model of surface remodeling and was due to the surrounding bone material becoming more or less stiff rather than being due to surface movement.

(2) \( T_0(t) = 29.5, 29.8, 30, 30.2, 30.5 \)˚C, \( \varphi_b - \varphi_a = 30 \) V, \( P(t) = 1 \) MPa, \( P(t) = 1500 \) N.

Fig. 3 shows the effects of temperature change on bone surface remodeling. In general, the radii of the bone decrease when the temperature increases and they increase when the temperature decreases. It can also be seen from Fig. 3 that \( \varepsilon \) and \( \eta \) are almost the same. Since \( a_0 < b_0 \), the change of the outer surface radius is normally greater than that of the inner one. The area of the bone cross-section decreases as the temperature increases. This also suggests that a lower temperature is likely to induce thicker bone structures, while a warmer environment may improve the remodeling process with a less thick bone structure. This result seems to coincide with the actual fact. Thicker and stronger bones maybe make a person living in Russia looks stronger than that living in Viet nam. It should be mentioned here that how this change may affect bone remodeling process is still an open question. As an initial investigation, the purpose of this study is to show how a bone may response to thermal loads and to provide information for possible use of imposed external temperature fields in medical treatment and controlling healing process of injured bones.

(3) \( \varphi_b - \varphi_a = -60, -30, 30, \) and \( 60 \) V, \( P(t) = 1 \) MPa, \( P(t) = 1500 \) N, and \( T_0 = 0 \).

Fig. 4 shows the variation of \( \varepsilon \) and \( \eta \) with time \( t \) for various values of electric potential difference. It can be seen that the effect of the electric potential is just opposite to that of temperature. A decrease of the intensity of electric field results in a decrease of the inner and outer surface radii of the bone by almost the same magnitude. Theoretically, the results suggest that the remodeling process may be improved by exposing a bone to an electric field. Apparently, further theoretical
and experimental studies are needed to investigate the implication of this in medical practice.

(4) $P = 1500 \, \text{N}$ and the internal pressure was produced by inserting a rigid pin whose radius $a^*$ is greater than $a$. The values of $\varepsilon$ and $\eta$ against $t$ for $\delta = a^* - a = 0.001 \, \text{mm}$, $0.003 \, \text{mm}$, and $0.005 \, \text{mm}$ are shown in Fig. 5. It can be seen that both the inner and the outer surface radii of the bone are reduced after the pin has been inserted, which will increase the tightness of fit. A tightly fitted pin can also increase the velocity of bone surface remodeling, which can accelerate the recovery of the injured bone. On the other hand, as the radius of the pin increases, the outer surface radius of the bone decreases more significantly than the inner surface one does. This results in a decrease of the bone's cross-sectional area and, hence, a thinner and weaker bone structure. Furthermore, if the fit is too tight, the pressure on the interface will cause damage to the bone structure. Thus, the radius of the pin should be restricted within a proper range.

7.2. A hollow, inhomogeneous circular cylindrical bone subjected to external loads

The geometrical and material parameters of this problem are the same as those used in the above cases except that all material constants in Eq. (103) is now modified by a multiplier $[1 - (1 - \xi)(b - r)/(b - a)]$, where $0 \leq \xi \leq 1$ and represents a percentage reduction of stiffness at the inner surface of the bone. It is worthwhile to mention that by using the semi-analytical approach, the form of stiffness variation in the radial direction can be arbitrary. Fig. 6 shows the results of $\varepsilon$ and $\eta$ at the outside surface of the bone for $\xi = 0.995$, 0.99 and 0.98. The external loads are $p = 1 \, \text{MPa}$, $P = 1500 \, \text{N}$, $T = 30^\circ \text{C}$, and $\phi_b - \phi_o = 30 \, \text{V}$. In general, the remodeling rate declines as the initial stiffness of inner bone surface decreases. The inhomogeneity of the bone has significant effect on the bone surface remodeling. A greater surface remodeling rate is always related to a higher level of inhomogeneity.
bone materials. Therefore further experimental validation is required before it can apply to the clinical practice.

Some observations have been made in regard to the effects of temperature change and inhomogeneity on a bone’s surface remodeling process, for which verifications from experimental investigations are needed.

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Appendix A. The solution of displacements $u_r$, $u_z$ and electric potential $\varphi$ [4]

Consider a hollow circular cylinder composed of linearly a thermopiezoelectric bone materials subjected to axisymmetric loading. The axial, circumferential and normal to the middle-surface coordinate length parameters are denoted with $z$, $\theta$ and $r$, respectively. The constitutive equations for the thermoelastolectric field in the cylindrical coordinate system are [21]

$$
\sigma_{rr} = c_{11}s_{rr} + c_{12}s_{r\theta} + c_{13}s_{rz} - e_{31}E_z - \beta_1 T ,
$$

$$
\sigma_{r\theta} = c_{12}s_{rr} + c_{11}s_{r\theta} + c_{13}s_{zz} - e_{31}E_z - \beta_1 T ,
$$

$$
\sigma_{zz} = c_{13}s_{rr} + c_{13}s_{r\theta} + c_{33}s_{zz} - e_{33}E_z - \beta_3 T ,
$$

$$
\sigma_{zz} = c_{33}s_{rr} - e_{13}E_r ,
$$

$$
\sigma_{zz} = c_{33}s_{r\theta} + \kappa_1 E_r ,
$$

$$
D_z = e_{31}(s_{rr} + s_{r\theta}) + e_{33}s_{zz} + \kappa_3 E_z - p_3 T ,
$$

$$
h_r = k_r H_r ,
$$

$$
h_z = k_z H_z ,
$$

where $\sigma_{ij}$, $D$, $s_{ij}$, $E_r$ denote components of stress, electric displacements, strains, and electric field intensities, respectively; $c_{ij}$ are elastic stiffness; $e_{ij}$ are piezoelectric constants; $\kappa_i$ are dielectric permittivities; $T$ denotes temperature change; $p_3$ is pyroelectric constant; $\beta_i$ are stress–temperature coefficients; $h_i$ are heat flow; $H_i$ are heat intensity; and $k_i$ are heat conduction coefficients. The associated strains, electric fields, and heat intensities are, respectively, related to displacements $u_i$, electric potential $\varphi$, and temperature change $T$ as

$$
S_{rr} = u_{r,r} ,
$$

$$
S_{r\theta} = u_{r,\theta} ,
$$

$$
S_{zz} = u_{z,z} ,
$$

$$
S_{rz} = u_{r,z} + u_{z,r} ,
$$

$$
E_r = -\varphi_r ,
$$

$$
E_z = -\varphi_z ,
$$

$$
H_r = -T_r ,
$$

$$
H_z = -T_z .
$$

(A.2)

For quasi-stationary behavior, in the absence of heat source, free electric charge and body forces, the set of equations for thermopiezoelectric theory of bones is completed by adding the following equations of equilibrium for heat flow, stress and electric

8. Conclusion

The problem of thermopiezoelectric bone surface remodeling has been addressed within the framework of adaptive elastic theory. Two thermoelastoelastic solutions for bone materials have been derived through the use of the adaptive elastic theory. By assuming a homogeneous bone material, the analytical solution can provide explicit solutions for analysing circular cylindrical bones. The semi-analytical solution is capable of modeling inhomogeneous circular cylindrical bones.

In the numerical analysis, various load conditions were considered, including loads induced by inserting a rigid pin to a bone as part of a prosthetic device.

The numerical results showed that apart from mechanical loads, both electric field and thermal load can affect bone remodeling process. This feature may be considered and utilized in controlling healing process of injured bones. It should be mentioned here that all the results are obtained on the basis of the numerical model which may have difference from those of individual...
displacements to Eqs. (A.1) and (A.2).
\[
\frac{\partial \sigma_{yy}}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} = 0, \quad \frac{\partial \sigma_{zz}}{\partial r} + \frac{\partial \sigma_{yy}}{\partial z} = 0,
\]
\[
\frac{\partial D_y}{\partial r} + \frac{\partial D_z}{\partial z} = 0, \quad \frac{\partial D_z}{\partial r} + \frac{\partial D_y}{\partial z} = 0, \quad \frac{\partial h_y}{\partial r} + \frac{h_z}{r} = 0, \quad \frac{\partial h_z}{\partial r} + \frac{h_y}{r} = 0,
\]
(A.3)

Using (A.1) and (A.2), differential Eq. (A.3) can be written as
\[
\left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) T = 0,
\]
\[
c_{11} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) u_r + \frac{c_{15}}{c_{11}} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) \varphi = 0,
\]
\[
c_{44} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) u_z + \frac{c_{15}}{c_{11}} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) \varphi = 0.
\]
(A.4)

The solution to the above equations satisfying boundary conditions (2)–(3) is given by
\[
u_r = \frac{r}{F_3^*} \left( c_{33} c_{12} [\beta_1^* T_0 + p(t)] + \frac{c_{33} c_{12}}{c_{11}} \right)
\]
\[
+ \frac{F_3^* T_0 + P(t)}{r (c_{11} - c_{12})} \left( a \beta_1^* [\beta_1^* T_0 + p(t)] + \frac{\varphi_r \ln(r/a) - 1}{c_{11}} \right),
\]
\[
u_z = \frac{z}{F_3^*} \left( F_3^* T_0 - F_3^* T_0 + P(t) \right) \frac{c_{11} + c_{12}}{\pi (b^2 - a^2)}
\]
\[
- \frac{2 c_{11} c_{12} \omega}{c_{11}} \left( c_{11} + c_{12} - 2 c_{15} \beta_1^* [\beta_1^* T_0 + p(t)] \right)
\]
\[
- \frac{c_{15}}{c_{11}} \left( \phi_h - \phi_a \right) \ln(r/a) \frac{c_{44}}{c_{11}} \ln(b/a),
\]
(A.5)

\[
\varphi = \frac{\ln(r/a)}{\ln(b/a)} (\phi_h - \phi_a) + \phi_a,
\]
(A.6)

\[
T = \frac{\ln(r/a)}{\ln(b/a)} T_0,
\]
(A.7)

where
\[
\omega = \frac{\beta_1^* T_0}{2 \ln(b/a)},
\]
(A.8)

\[
F_3^* = \frac{1}{\ln(b/a)} \left( \frac{c_{11} \beta_1^*}{c_{11}} - \frac{\beta_3^*}{2} \right),
\]
\[
F_2^* = \pi b^2 \left( \frac{c_{11}}{c_{11}} \beta_1^* - \beta_3^* \right),
\]
(A.9)

\[
F_3^* = c_{33}(c_{11} + c_{12}) - 2 c_{13}^2,
\]
\[
\beta_1^* = \frac{b^2}{(a^2 - b^2)}, \quad \beta_2^* = \frac{\beta_1^*}{2} \left( \frac{c_{12}}{c_{11}} - 1 \right).
\]
(A.10)

References