# TREFFTZ PLANE ELEMENT OF PIEZOELECTRIC PLATE WITH *p*-EXTENSION CAPABILITIES

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## Abstract

This paper is concerned with the development of Trefftz *p*-element for two-dimentional piezoelectric materials. Solutions in Stroh formalism for transversely isotropic piezoelectric materials are used for the intra-element displacement and electric potential (DEP) fields together with an independent DEP frame function field along element boundaries. The formulation is based on a modified variational functional in which electric field and strain are taken as basic variables. The final unknowns are the parameters of the frame function field consisting of the usual degrees of freedom (DOF) at corner nodes and an optional number of hierarchic DOF associated with the mid-side nodes. Some numerical examples are considered to show the application of the proposed formulation.

## 1. Introduction

During the past decades the hybrid-Trefftz finite element (FE) model, originating in 1977 [1], has been considerably improved and has now become a highly efficient computational tool for the solution of complex boundary value problems. Up to now, T-elements have been successfully applied to problems of elasticity, Kirchhoff plates, moderately thick Reissner-Mindlin plates, thick plates, general 3-D solid mechanics, axisymmetric solid mechanics, potential problems, shells, elastodynamic problems, transient heat conduction analysis, geometrically nonlinear plates, materially nonlinear elasticity, and piezoelectric materials. Most of these developments can be found in [2,3]. Furthermore, the concept of special purpose functions has been found to be of great efficiency in dealing with various geometry or load-dependent singularities and local effects (e.g., obtuse or reentrant corners, cracks, circular or elliptic holes, and concentrated loads) [2].

This work aims at developing a Trefftz finite element, *p*-element, for modeling two-dimensional piezoelectric material. Solutions in Stroh formalism for transversely isotropic piezoelectric materials are used for the intra-element DEP. The modified variational functional used was based on a free energy density with strain and

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electric fields as independent variables. The stationary conditions of the variational functional and the theorem on the existence of extremum are discussed. Numerical results are presented to show the applicability of the proposed formulation.

## 2. Governing Equations and Their Trefftz Functions

### 2.1 BASIC FUNCTIONS

In this section we recall briefly the governing equations of linear piezoelectricity. The summation convention is invoked over repeated indices. For convenience, matrices are represented by bold face letters and a comma after a variable implies differentiation with respect to Cartesian coordinates. Then, for a linear piezoelectric material, the differential governing equations of linear piezoelectric material in the Cartesian coordinates $x_i$ ($i$=1, 2, 3) are given by

$$\sigma_{ij,j} + \bar{b}_i = 0, \quad D_{i,i} + \bar{q}_b = 0 \qquad \text{in } \Omega \tag{1}$$

where $\sigma_{ij}$ is the stress tensor, $D_i$ is the electric displacement vector, $\bar{b}_i$ is the body force vector, $\bar{q}_b$ is the electric charge density, $\Omega$ is the solution domain. For an anisotropic piezoelectric material, the constitutive relation is [3]

$$\sigma_{ij} = \frac{\partial H(\boldsymbol{\varepsilon}, \mathbf{E})}{\partial \varepsilon_{ij}} = c^E_{ijkl}\varepsilon_{kl} - e_{kij}E_k, \quad D_i = -\frac{\partial H(\boldsymbol{\varepsilon}, \mathbf{E})}{\partial E_i} = e_{ikl}\varepsilon_{kl} + \kappa^{\varepsilon}_{ik}E_k \tag{2}$$

where $2H(\boldsymbol{\varepsilon}, \mathbf{E}) = c^E_{ijkl}\varepsilon_{ij}\varepsilon_{kl} - \kappa^{\varepsilon}_{ij}E_iE_j - 2e_{kij}\varepsilon_{ij}E_k$, $c^E_{ijkl}$ is the stiffness coefficient tensor for **E**=0, $\kappa^{\varepsilon}_{ij}$ the permittivity constant matrix for **ε**=0, $\varepsilon_{ij}$ and $E_i$ are, respectively, the elastic strain tensor and the electric field intensity vector, $e_{kij}$ is piezoelectric stress constants.

The relation between the strain tensor and the displacement, $u_i$, is given by

$$2\varepsilon_{ij} = (u_{i,j} + u_{j,i}) \tag{3}$$

and the electric field components are related to the electric potential $\phi$ by

$$E_i = -\phi_{,i}\,. \tag{4}$$

The boundary conditions of the boundary value problem (1)-(4) are given by:

$$u_i = \bar{u}_i \text{ on } \Gamma_u,\ t_i = \sigma_{ij}n_j = \bar{t}_i \text{ on } \Gamma_t,\ \phi = \bar{\phi} \text{ on } \Gamma_\phi,\ D_n = D_in_i = -q_n = \bar{D}_n \text{ on } \Gamma_D \tag{5}$$

where $\bar{u}_i, \bar{t}_i, \bar{q}_n$ and $\bar{\phi}$ are, respectively, prescribed boundary displacement, traction vector, surface charge and electric potential. An overhead bar denotes prescribed value, $\Gamma=\Gamma_u+\Gamma_t=\Gamma_D+\Gamma_\phi$ is the boundary of the solution domain $\Omega$.

Moreover, in the Trefftz finite element formulation, equations (1)-(5) should be completed by the following inter-element continuity requirements:

$$u_{ie} = u_{if}, \quad \phi_e = \phi_f, \quad (\text{on } \Gamma_e \cap \Gamma_f, \text{ conformity}), \tag{6}$$

$$t_{ie} + t_{if} = 0, \quad D_{ne} + D_{nf} = 0 \quad (\text{on } \Gamma_e \cap \Gamma_f, \text{ reciprocity}) \tag{7}$$

where '*e*' and '*f*' stand for any two neighboring elements. Equations (1)-(7) are taken as the basis to establish the modified variational principle for Trefftz finite element analysis of piezoelectric materials.

### 2.2 TREFFTZ FUNCTIONS

The Trefftz function plays an important role in the derivation of Trefftz finite element formulation. A complete system of homogeneous solutions to equation (1) can be generated in a systematic way from the Stroh formalism technique [4]

$$\mathbf{u} = 2\,\mathrm{Re}\{\mathbf{A}\langle f(z_\alpha)\rangle\mathbf{c}\} \tag{8}$$

where 'Re' stands for the real part of a complex number, $\mathbf{A}$ is the material eigenvector matrix which was well defined in the literature [4], $\langle f(z_\alpha)\rangle = \mathrm{diag}[f(z_1)\, f(z_2)\, f(z_3)\, f(z_4)]$ is a diagonal 4×4 matrix, while $f(z_i)$ is an arbitrary function with argument $z_i = x_1 + p_i x_2$. $p_i$ ($i$=1-4) are the material eigenvalues. Of particular interest is a complete set of polynomial solutions which may be generated by setting in equation (8) in turn

$$f(z_\alpha) = z_\alpha^k, \quad f(z_\alpha) = iz_\alpha^k, \quad (k = 1,2,\cdots) \tag{9}$$

where $i = \sqrt{-1}$. Thus the Trefftz functions of equation (1) can be given from the following

$$\mathbf{u} = \sum_{j=1}^{\infty} [\mathrm{Re}\{\mathbf{A}\langle z_\alpha^j\rangle\}\mathbf{a}_j + \mathrm{Re}\{\mathbf{A}\langle iz_\alpha^j\rangle\}\mathbf{b}_j]. \tag{10}$$

## 3. Elemental Stiffness Formulation

### 3.1 ASSUMED FIELDS

The Trefftz finite element model is based on assuming two distinct DEP fields: the internal field $\mathbf{u}$ and the frame function $\tilde{\mathbf{u}}$ [2]. The field $\mathbf{u}$ fulfills identically the governing differential equations (1) and is assumed as

$$\mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \\ \phi \end{Bmatrix} = \begin{Bmatrix} \breve{u}_1 \\ \breve{u}_2 \\ \breve{\phi} \end{Bmatrix} + \begin{Bmatrix} \mathbf{N}_1 \\ \mathbf{N}_2 \\ \mathbf{N}_3 \end{Bmatrix}\mathbf{c} = \breve{\mathbf{u}} + \sum_{j=1} \mathbf{N}_j \mathbf{c}_j = \breve{\mathbf{u}} + \mathbf{N}\mathbf{c} \tag{11}$$

for a two-dimensional problem, where $\mathbf{c}_i$ stands for undetermined coefficient, and $\breve{\mathbf{u}}$ (= $\{\breve{u}_1, \breve{u}_2, \breve{\phi}\}^T$ ) and $\mathbf{N}_j$ are known functions. If the governing differential equation (1) is rewritten in a general form

$$\Re\mathbf{u}(\mathbf{x}) + \bar{\mathbf{b}}(\mathbf{x}) = 0, \qquad (\mathbf{x} \in \Omega_e) \tag{12}$$

where $\Re$ stands for the differential operator matrix for equation (1), $\mathbf{x}$ for the position vector, $\bar{\mathbf{b}}$ (={ $\{\bar{b}_1, \bar{b}_2, \bar{q}_b\}^T$ for the known right-hand side term, the overhead bar indicates the imposed quantities and $\Omega_e$ stands for the *e*th element sub-domain, then $\breve{\mathbf{u}} = \breve{\mathbf{u}}(\mathbf{x})$ and $\mathbf{N} = \mathbf{N}(\mathbf{x})$ in equation (11) have to be chosen so that

$$\Re\breve{\mathbf{u}} + \bar{\mathbf{b}} = 0 \quad \text{and} \quad \Re\mathbf{N} = 0 \tag{13}$$

everywhere in $\Omega_e$. Thus $\mathbf{N}_j$ in (11) can be formed by a suitably truncated complete system of (10). For example, we may set

$$\mathbf{N}_{2j} = 2\,\mathrm{Re}\{\mathbf{A}\langle z_\alpha^j\rangle\}, \; \mathbf{N}_{2j+1} = 2\,\mathrm{Re}\{\mathbf{A}\langle iz_\alpha^j\rangle\}. \tag{14}$$

The unknown coefficient **c** may be calculated from the conditions on the external boundary and/or the continuity conditions on the inter-element boundary. Thus various Trefftz element models can be obtained by using different approaches to enforce these conditions. In the majority of cases a hybrid technique is used, whereby the elements are linked through an auxiliary conforming displacement frame which has the same form as in the conventional finite element method. This means that, in the Trefftz finite element approach, a conforming DEP field should be independently defined on the element boundary to enforce the field continuity between elements and also to link the coefficient **c**, appearing in equation (11), with the nodal DEP **d** (={*d*}). The frame is defined as

$$\tilde{\mathbf{u}}(\mathbf{x}) = \begin{Bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{\phi} \end{Bmatrix} = \begin{Bmatrix} \tilde{\mathbf{N}}_1 \\ \tilde{\mathbf{N}}_2 \\ \tilde{\mathbf{N}}_3 \end{Bmatrix} \mathbf{d} = \tilde{\mathbf{N}}\mathbf{d}, \qquad (\mathbf{x} \in \Gamma_e) \tag{15}$$

for a two-dimensional problem, where the symbol "~" is used to specify that the field is defined on the element boundary only, **d** = **d**(**c**) stands for the vector of the nodal displacements which are the final unknowns of the problem, $\Gamma_e$ represents the boundary of element *e*, and $\tilde{\mathbf{N}}$ is a matrix of the corresponding shape functions which are similar to those in the conventional finite element formulation.

In the development of the present *p*-element, the following assumptions are adopted. First of all, the problem is assumed to be plane strain of transversely isotropic piezoelectric solid referred to a Cartesian system ($x_1$, $x_2$) [5]. Secondly, the element may be of a general quadrilateral shape or a triangle shape with three degrees of freedom (DOF) $(u_1, u_2, \phi)$ at each corner node (see Figure 1). Thirdly, to achieve higher order variations, an optional number of extra hierarchic modes is introduced along with the hierarchic DOF, $a_i$ for $\tilde{u}_1, b_i$ for $\tilde{u}_2, p_i$ for $\tilde{\phi}$, which are conveniently associated with the mid-side node *C* (see Figure 1). Thus, along the side *A-C-B* of a particular element (see Figure 1), a simple interpolation of the frame DEP field can be given in the form

$$\begin{Bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{\phi} \end{Bmatrix} = \begin{bmatrix} \tilde{N}_1 & 0 & 0 & \tilde{N}_2 & 0 & 0 \\ 0 & \tilde{N}_1 & 0 & 0 & \tilde{N}_2 & 0 \\ 0 & 0 & \tilde{N}_1 & 0 & 0 & \tilde{N}_2 \end{bmatrix} \mathbf{d}_{AB} + \sum_{i=1}^{M} \gamma^{i-1} R_i \begin{Bmatrix} a_{Ci} \\ b_{Ci} \\ p_{Ci} \end{Bmatrix} \tag{16}$$

where $M$ is the order of the hierarchical DOF, and $\mathbf{d}_{AB} = \{u_{1A}, u_{2A}, \phi_A, u_{1B}, u_{2B}, \phi_B\}^T$. The shape functions are

$$\tilde{N}_1 = \frac{1-\xi}{2}, \quad \tilde{N}_2 = \frac{1+\xi}{2}, \quad R_i = \xi^{i-1}(1-\xi^2) \tag{17}$$

where $\xi$ is defined in Figure 1.

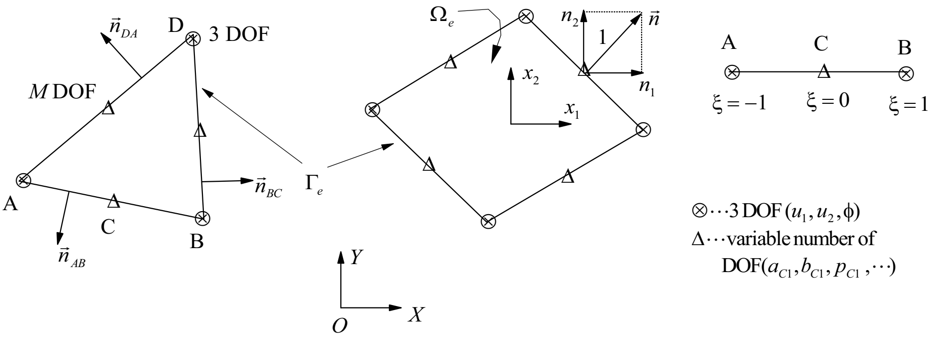

*Figure 1.* The Trefftz *p*-element

The coefficient $\gamma$ is equal to +1 or -1 according to the orientation of the side *A-C-B* (see Figure 1) in the global coordinate system (X,Y):

$$\gamma = \begin{cases} +1 & \text{if } X_B - X_A \le Y_B - Y_A \\ -1 & \text{if } X_B - X_A > Y_B - Y_A \end{cases}. \tag{18}$$

The purpose of using the coefficient $\gamma$ is to ensure a univocal definition of the frame functions $\tilde{\mathbf{u}}$ in terms of parameters $a_i$, $b_i$ and $p_i$, common to two elements sharing the mid-side node *Z*.

Using the definitions in (2), $(5)_2$ and $(5)_4$ the generalized boundary forces and electric displacements can be given as

$$\mathbf{T} = \begin{Bmatrix} t_1 \\ t_2 \\ D_n \end{Bmatrix} = \begin{Bmatrix} \sigma_{1j} n_j \\ \sigma_{2j} n_j \\ D_j n_j \end{Bmatrix} = \begin{Bmatrix} \breve{t}_1 \\ \breve{t}_2 \\ \breve{D}_n \end{Bmatrix} + \begin{Bmatrix} \mathbf{Q}_1 \\ \mathbf{Q}_2 \\ \mathbf{Q}_3 \end{Bmatrix} \mathbf{c} = \breve{\mathbf{T}} + \mathbf{Qc},$$

$$\tilde{\mathbf{T}} = \begin{Bmatrix} \tilde{t}_1 \\ \tilde{t}_2 \\ \tilde{D}_n \end{Bmatrix} = \begin{Bmatrix} \tilde{\mathbf{Q}}_1 \\ \tilde{\mathbf{Q}}_2 \\ \tilde{\mathbf{Q}}_3 \end{Bmatrix} \mathbf{c} = \tilde{\mathbf{Q}}\mathbf{c} \tag{19}$$

where $\breve{t}_i$ and $\breve{D}_n$ are derived from $\breve{\mathbf{u}}$.

### 3.2 PARTICULAR SOLUTIONS

The particular solution of $\breve{\mathbf{u}}$ can be obtained by means of their Green's functions. The Green's functions of (1) are as follows [5]

$$u_{ij}^{*}(p,q) = \frac{1}{4\pi}\left\{ a_{ij}^{(0)} \ln\left|r_{pq}\right| + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}[a_{ij}^{(k)} \cos(2k\theta) + b_{ij}^{(k)} \sin(2k\theta)\right\} \tag{20}$$

with

$$r_{pq} = \sqrt{(x_q - x_p)^2 + (y_q - y_p)^2} \tag{21}$$

where $u_{ij}^{*}(p,q)$ designates the Green's function of the in-plane displacement (for $I$ = 1,2) or electric potential (for $i$ = 3) at field point $p$ of an infinite plane when a unit point force is applied at the source point $q$ in the $j$-direction. $a_{ij}^{(k)}$ and $b_{ij}^{(k)}$ are calculated as: $a_{ij}^{(k)} = 0$ if $i \neq j; k = 0,1,2,\ldots$, $b_{ij}^{(k)} = 0$ if $i = j; k = 1,2,\ldots$, and the non-zero values of $a_{ij}^{(k)}$ and $b_{ij}^{(k)}$ can be found in Table 1 of Ref. [5].

Thus the particular solutions of (1) can be expressed as

$$\breve{\mathbf{u}} = \{\breve{u}_1, \breve{u}_2, \breve{\phi}\}^T = \iint_{\Omega} \bar{b}_j \{u_{1j}^{*}, u_{2j}^{*}, u_{3j}^{*}\}^{\mathrm{T}} \, d\Omega \tag{22}$$

where $\bar{b}_3 = \bar{q}_b$. The area integration in (22) is performed by numerical quadrature using the Gauss-Legendre rule.

### 3.3 MODIFIED VARIATIONAL PRINCIPLE

The Trefftz finite element equation for piezoelectric materials can be established by the variational approach [2]. Since the stationary conditions of the traditional potential and the complementary variational functional cannot satisfy the inter-element continuity condition which is required in the Trefftz finite element analysis, some new variational functionals need to be developed. For this purpose, we present the following modified variational functional suitable for Trefftz finite element analysis:

$$\Pi_m = \sum_e \Pi_{me} = \sum_e \{\Pi_e + \int_{\Gamma_{\phi e}} (\bar{\phi} - \phi)\tilde{D}_n ds + \int_{\Gamma_{ue}} (\bar{u}_i - u_i)\tilde{t}_i ds \\ - \int_{\Gamma_{Ie}} (\tilde{\phi} D_n + \tilde{u}_i t_i ds\} \tag{23}$$

where

$$\Pi_e = \iint_{\Omega_e} [H(\boldsymbol{\varepsilon}, \mathbf{E}) - \bar{b}_i u_i - \bar{q}_b \phi] d\Omega - \int_{\Gamma_{te}} \bar{t}_i u_i ds - \int_{\Gamma_{De}} \bar{D}_n \phi ds \tag{24}$$

and equation (1) is assumed to be satisfied, *a priori*. The terminology “modified principle” refers here, to the use of a conventional functional ( $\Pi_e$ here) and some modified terms for the construction of a special variational principle to account for additional requirements such as the condition defined in equations (6) and (7).

The boundary $\Gamma_e$ of a particular element consists of the following parts:

$$\Gamma_e = \Gamma_{ue} \cup \Gamma_{te} \cup \Gamma_{Ie} = \Gamma_{\phi e} \cup \Gamma_{De} \cup \Gamma_{Ie} \tag{25}$$

where

$$\Gamma_{ue} = \Gamma_u \cap \Gamma_e, \;\; \Gamma_{te} = \Gamma_t \cap \Gamma_e, \;\; \Gamma_{\phi e} = \Gamma_\phi \cap \Gamma_e, \;\; \Gamma_{De} = \Gamma_D \cap \Gamma_e, \tag{26}$$

and $\Gamma_{Ie}$ is the inter-element boundary of the element ‘*e*’. We now show that the stationary condition of the functional (23) leads to equations (5)-(7), $(t_i = \tilde{t}_i \text{ on } \Gamma_u)$, $(D_n = \tilde{D}_n \text{ on } \Gamma_\phi)$, and present the theorem on the existence of extremum of the functional, which ensures that an approximate solution can converge to the exact one. For the functional (23), we have the following two statements:

(a) *Modified complementary principle*

$$\delta\Pi_m^{\varepsilon E} = 0 \;\; \Rightarrow \;\; (5)-(7), \; (t_i = \tilde{t}_i \text{ on } \Gamma_u) \text{ and } (D_n = \tilde{D}_n \text{ on } \Gamma_\phi) \tag{27}$$

where δ stands for the variation symbol.

(b) *Theorem on the existence of extremum*

If the expression

$$\iint_\Omega \delta^2 H(\boldsymbol{\varepsilon}, \mathbf{E}) d\Omega - \int_{\Gamma_u} \delta\tilde{t}_i \delta u_i ds - \int_{\Gamma_\phi} \delta\tilde{D}_n \delta\phi ds \\ - \sum_e \int_{\Gamma_{Ie}} (\delta\tilde{\phi} \delta D_n + \delta\tilde{u}_i \delta t_i) ds \tag{28}$$

is uniformly positive (or negative) in the neighborhood of $\mathbf{u}_0$, where $\mathbf{u}_0$ is such a value that $\Pi_m(\mathbf{u}_0) = (\Pi_m)_0$, and where $(\Pi_m)_0$ stands for the stationary value of $\Pi_m$, we have

$$\Pi_m \geq (\Pi_m)_0 \; [\text{or } \Pi_m \leq (\Pi_m)_0] \tag{29}$$

in which the relation that $\tilde{\mathbf{u}}_e = \tilde{\mathbf{u}}_f$ is identical on $\Gamma_e \cap \Gamma_f$ has been used.

*PROOF*: First, we derive the stationary conditions of functional (23). To this end, performing a variation of $\Pi_m$ and noting that eqn (1) holds true *a priori* by the previous assumption, we obtain

$$\begin{aligned}\delta\Pi_m = &\int_{\Gamma_u}[(\bar{u}_i-u_i)\delta\tilde{t}_i+(t_i-\tilde{t}_i)\delta u_i]ds+\int_{\Gamma_\phi}[(\bar{\phi}-\phi)\delta\tilde{D}_n+(D_n-\tilde{D}_n)\delta\phi]ds\\ &-\int_{\Gamma_t}(\bar{t}_i-t_i)\delta u_i ds-\int_{\Gamma_D}(\bar{D}_n-D_n)\delta\phi ds\\ &-\sum_e\int_{\Gamma_{Ie}}[t_i\delta(\tilde{u}_i-u_i)+D_n\delta(\tilde{\phi}-\phi)+\tilde{u}_i\delta t_i+\tilde{\phi}\delta D_n]ds\end{aligned} \quad . \tag{30}$$

Therefore, the Euler equations for expression (30) are equations (5)-(7), $(t_i=\tilde{t}_i \text{ on } \Gamma_u)$ , and $(D_n=\tilde{D}_n \text{ on } \Gamma_\phi)$ , since the quantities $\delta t_i,\ \delta u_i, \delta\phi, \delta D_n, \delta\tilde{u}_i, \delta\tilde{t}_\mathrm{i}, \delta\tilde{D}_n \text{ and } \delta\tilde{\phi}$ may be arbitrary. The principle (27) has thus been proved. This indicates that the stationary condition of the functional satisfies both the required boundary and inter-element continuity equations and can thus be used for deriving Trefftz finite element formulation.

As for the proof of the theorem on the existence of extremum, we may complete it by way of the so-called "second variational approach" [6]. In doing this, performing variation of $\delta\Pi_m$ and using the constrained conditions (1), we find

$$\begin{aligned}\delta^2\Pi_m = &\iint_\Omega \delta^2 H(\boldsymbol{\varepsilon},\mathbf{E})d\Omega-\int_{\Gamma_t}\delta\tilde{t}_i\delta u_i ds-\int_{\Gamma_D}\delta\tilde{D}_n\delta\phi ds\\ &-\sum_e\int_{\Gamma_{Ie}}(\delta\tilde{\phi}\delta D_n+\delta\tilde{u}_i\delta t_i)ds=\text{expression (28)}\end{aligned} \quad . \tag{31}$$

Therefore the theorem has been proved from the sufficient condition of the existence of a local extreme of a functional [6]. This completes the proof.

## 3.4 ELEMENT MATRIX EQUATION

The element matrix equation can be generated by setting $\delta\Pi_{me}=0$. To simplify the derivation, we first transform all domain integrals in (23) into boundary ones. In fact, by reason of the solution properties of the intra-element trial functions, the functional $\Pi_{me}$ can be simplified to

$$\begin{aligned}\Pi_{me}=&\frac{1}{2}\int_{\Gamma_e}(t_iu_i+D_n\phi)ds-\frac{1}{2}\int_\Omega(\bar{b}_iu_i+\bar{q}_b\phi)d\Omega+\int_{\Gamma_{\phi e}}(\bar{\phi}-\phi)\tilde{D}_n ds\\ &+\int_{\Gamma_{ue}}(\bar{u}_i-u_i)\tilde{t}_i ds-\int_{\Gamma_{Ie}}(D_n\tilde{\phi}+t_i\tilde{u}_i)ds-\int_{\Gamma_{te}}\bar{t}_iu_i ds-\int_{\Gamma_{De}}\bar{D}_n\phi ds\end{aligned} \quad . \tag{32}$$

Substituting the expressions given in eqns (11), (16), and (19) into (32) produces

$$\Pi_{me}=\frac{1}{2}\mathbf{c}^T\mathbf{H}\mathbf{c}+\mathbf{c}^T\mathbf{S}\mathbf{d}+\mathbf{c}^T\mathbf{r}_1+\mathbf{d}^T\mathbf{r}_2+\text{terms without }\mathbf{c}\text{ or }\mathbf{d} \tag{33}$$

in which the matrices **H**, **S** and the vectors $\mathbf{r}_1$, $\mathbf{r}_2$ are defined by

$$\mathbf{H}=\int_{\Gamma_e}\mathbf{Q}^T\mathbf{N}ds \tag{34}$$

$$\mathbf{S} = -\int_{\Gamma_{\phi e}} \mathbf{N}_3^T \tilde{\mathbf{Q}}_3 ds - \int_{\Gamma_{ue}} \begin{bmatrix} \mathbf{N}_1 \\ \mathbf{N}_2 \end{bmatrix}^T \begin{bmatrix} \tilde{\mathbf{Q}}_1 \\ \tilde{\mathbf{Q}}_2 \end{bmatrix} ds - \int_{\Gamma_{Ie}} \mathbf{Q}^T \tilde{\mathbf{N}} ds \tag{35}$$

$$\mathbf{r}_1 = \frac{1}{2}\int_{\Gamma_e} (\mathbf{N}^T \breve{\mathbf{T}} + \mathbf{Q}^T \breve{\mathbf{u}}) ds - \frac{1}{2}\int_{\Omega} \mathbf{N}^T \overline{\mathbf{b}} d\Omega - \int_{\Gamma_{De}} \mathbf{N}_3^T \overline{D}_n ds - \int_{\Gamma_{te}} \begin{bmatrix} \mathbf{N}_1 \\ \mathbf{N}_2 \end{bmatrix}^T \begin{Bmatrix} \bar{t}_1 \\ \bar{t}_2 \end{Bmatrix} ds \tag{36}$$

$$\mathbf{r}_2 = \int_{\Gamma_{\phi e}} \tilde{\mathbf{Q}}_3^T (\bar{\phi} - \breve{\phi}) ds + \int_{\Gamma_{\phi e}} \begin{bmatrix} \tilde{\mathbf{Q}}_1 \\ \tilde{\mathbf{Q}}_2 \end{bmatrix}^T \begin{Bmatrix} \bar{u}_1 - \breve{u}_1 \\ \bar{u}_2 - \breve{u}_2 \end{Bmatrix} ds - \int_{\Gamma_{Ie}} \begin{bmatrix} \tilde{\mathbf{N}}_1 \\ \tilde{\mathbf{N}}_2 \\ \tilde{\mathbf{N}}_3 \end{bmatrix}^T \begin{Bmatrix} \breve{t}_1 \\ \breve{t}_2 \\ \breve{D}_n \end{Bmatrix} ds\,. \tag{37}$$

After a series of mathematical derivations and variational calculations, equation (33) leads to

$$\mathbf{Kd} = \mathbf{P} \tag{38}$$

where $\mathbf{K}=\mathbf{S}^T\mathbf{H}^{-1}\mathbf{S}$ and $\mathbf{P}=\mathbf{S}^T\mathbf{g}\text{-}\mathbf{r}_2$ are, respectively, the element stiffness matrix and the equivalent nodal flow vector. The expression (38) is the elemental stiffness-matrix equation for Trefftz finite element analysis.

## 4. Numerical Examples

Since the main purpose of this paper is to outline the basic principles of the Trefftz finite element method in piezoelectric materials, the assessment will be limited to two simple examples. In order to allow for comparisons with other solutions reported in reference [7], the obtained results are limited to a piezoelectric prism subjected to simple tension.

Table 1 $u_1$ of Trefftz BEM results at A for several values of $M$

| $M$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $u_1 \times 10^{10}$ (m) | -0.9670 | -0.9671 | -0.9671 | -0.9671 |

Consider a PZT-4 ceramic prism subjected to a simple tension P=10Nm$^{-2}$ in the $y$-direction (see Figure 2 in ref. [3]). The properties of the material are the same as those in [7]. The boundary conditions of the prism are

$$\sigma_{yy} = P,\ \sigma_{xy} = D_y = 0 \quad \text{on edges } y = \pm b$$

$$\sigma_{xx} = \sigma_{xy} = D_x = 0 \qquad \text{on edges } x = \pm a$$

where $a$=3m, $b$=10m. Owing to the symmetry about load, boundary conditions and geometry, only one quadrant of the prism is modeled by 25 ($x$-direction) and 40 ($y$-direction) elements in the TFEM analysis. Table 1 shows the displacement at point A with several values of $M$. It is evident that the same result is obtained when $M \leq 2$.

## 5. Conclusion

A Trefftz finite element model with *p*-capabilities has been presented for analysis of plane piezoelectric plate. It includes a modified variational functional which are based on a free energy density with (**ε**, **E**) as the basic independent variables. The proof of the stationary conditions of the variational functional and the theorem on the existence of extremum are provided in this paper. The stationary conditions are displacement and electric potential conditions on the boundary, surface traction and surface charge condition, and inter-element continuity condition. Based on the assumed intra-element and frame fields as well as the variational functional, an element stiffness matrix equation is obtained which is implemented into the computer programs for numerical analysis. The numerical results obtained here are in excellent agreement with the analytical ones and show that they can converge to the exact solution quickly along with an increase in the order *M* of the *p*-element.

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