INTERACTIONS BETWEEN A MAIN CRACK AND A MICRO CRACK IN A THERMOPIEZOELECTRIC PLATE

Qing-hua Qin and Yiu-wing Mai
Department of Mechanical Engineering, University of Sydney, Sydney, NSW 2006, Australia

Abstract: This paper treats the interactions between a main crack and a microcrack in an anisotropic piezoelectric solid. Based on the Green's function approach and the principle of superposition, a system of singular integral equations for the unknown temperature discontinuity defined on each crack face is developed and can be solved numerically. In the analysis, the residual heat flux, stress and electric displacement on microcrack location to be released are evaluated directly from the near-tip fields of main crack.

Key words: Crack, Fracture, Piezoelectric, Thermal stress, Singular integral equation

1. INTRODUCTION

The fracture mechanism of brittle material in the microscale is always concerned with micro defects such as cracks, holes and inclusions among which microcracking has received considerable attention in the past decades. Stress analysis of multiple crack problems in isotropic materials has been done by many researchers, such as Hoagland and Embury[1], Chen[2], Horri and Nemat-Nasser[3], Gong and Horii[4]. A history review of this topic was performed by Kachanov[5]. For anisotropic materials without considering the thermal effect, Hwu[6] obtained a solution for collinear cracks in an infinite plate. Chen and Hasebe[7] treated the elastic interaction between a main-crack and a parallel micro-crack in an orthotropic plate. Unlike in the case of anisotropic elasticity, relatively little work has been done for the analysis of multiple crack problems in piezoelectric materials. This work is a continuation of our previous studies[8,9]. In the paper, the Green's function approach and the principle of superposition are used to study thermoelectroelastic behaviour of a piezoelectric plate containing a main crack and a micro-crack. The geometry of the problem is shown in Fig. 1. After introducing the extended Stroh formalism and the thermoelectroelastic Green's function for temperature discontinuity, a system of singular integral equations for the unknown thermal analog of dislocation density defined on crack faces is derived by using the principle of superposition. The integral equations can be solved numerically and used to calculate stress and electric displacement(SED) intensity factors.

2. BASIC FORMULATIONS

Using the notation employed in[9,10], the general solution of a two dimensional thermopiezoelectric problem can be expressed as

\[ U = \text{Im}[\mathbf{A}f(z)q(\zeta)] \]

\[ \phi = \text{Im}[\mathbf{B}f(z)q(\zeta)] \]

\[ \Pi_1 = -\phi_2 \quad \Pi_2 = \phi_1 \]

\[ \theta = \text{Im}[g'(\zeta)] \]

\[ h_1 = \text{Im}[ik\tau g''(\zeta)] \]

\[ h_2 = \text{Im}[-ik\tau g''(\zeta)] \]

(1)

where overbar denotes complex conjugation, \( \text{Im} \) stands for the imaginary part of a complex number, \( q \) is a constant vector to be determined by the boundary conditions, \( \mathbf{U} = \{u_1 \ u_2 \ u_3 \ \varphi\}^T \), \( \Pi_j = \{\sigma_{ij} \ \sigma_{ji} \ \sigma_{ji} \ \sigma_{ji} \ \sigma_{ji} \ \sigma_{ji}\}^T \), \( j=1,2,3 \); \( i= \sqrt{-1} \), \( u \) and \( \varphi \) are the elastic displacement and electric potential, \( \theta \), \( h_1 \), \( \sigma_{ij} \) and \( D_i \) are temperature, heat flux, stress and electric displacement, respectively. \( f(z) = \text{diag}(f(z_1) f(z_2) f(z_3) f(z_4)) \), \( f(z) \) and \( g(z) \) are arbitrary functions with complex arguments \( z_1 \) and \( z_2 \), defined by \( z = x_1 + ip_1 \) and \( z_2 = x_2 + ip_2 \), \( k = \sqrt{k_{11}k_{22} - k_{12}^2} \), \( \tau \) and \( p \) are the heat and electroelastic eigenvalues of the materials whose imaginary parts are positive, respectively. \( \mathbf{A}, \mathbf{B}, \mathbf{c} \) and \( \mathbf{d} \) are well-defined in the literature (see [10], for example).

For an infinite piezoelectric plate subject to temperature discontinuity \( \theta_0 \) located at \( \zeta = \hat{x}_1 + \tau \hat{x}_2 \), the

\[ \theta = \theta_0 \quad \text{on} \quad \zeta = \hat{x}_1 + \tau \hat{x}_2 \]

Fig. 1 Geometry of the crack system
thermoelectroelastic Green's functions at point \( z(x_1, x_2) \) has been found by Qin and Mai\[8\]. For the sake of reference, the functions are listed below.

\[
\theta(z_c) = \theta_0 \text{Im}(\ln y_1) / 2\pi, \quad (2)
\]

\[
h_i = \frac{\theta_0}{2\pi} \text{Im} \left( \frac{k_{11} + p_i k_{21}}{y_1} \right), \quad (3)
\]

\[
U = \frac{\theta_0}{2\pi} \text{Im} \left\{ A f(z)q + c y_i (\ln y_i - 1) \right\}, \quad (4)
\]

\[
\phi = \frac{\theta_0}{2\pi} \text{Im} \left\{ B f(z)q + d y_i (\ln y_i - 1) \right\}, \quad (5)
\]

where

\[
f(z) = \text{diag} [f(y_1), f(y_2), f(y_3), f(y_4)],
\]

\[
y_1 = y - \xi, \quad y_1 = y, \quad (i = 1-4)
\]

\[
f(y) = y(\ln y - 1), \quad (6)
\]

\[
q = -\frac{1}{2} (\overline{B^{-1}B - A^{-1}A})^{-1}(A^{-1}(c + \overline{c})
\]

\[-\overline{B}^{-1}(d + \overline{d})) \]

3. SINGULAR INTEGRAL EQUATIONS

The geometrical configuration of the problem to be solved is depicted in Fig. 1, showing an infinite plate subjected to remote heat flow \( h_0 \) and with a main crack of length \( 2c \) and a microcrack of length \( 2c_0 \) located near the main-crack tip. The central point of the micro-crack is denoted as \( (x_{10}, x_{20}) \) and the orientation angle is denoted as \( \alpha \). The cracks are initially assumed to remain open and hence be free of tractions and charges as well as to prevent the transfer of heat between their faces. Moreover, the main crack is assumed much larger than the length of the micro-crack and the distance between the right tip of the main crack and the center of the micro crack, i.e. \( c >> c_0, \quad c >> d \). Under this assumption, the residual stresses to be released can be approximately calculated by the near-tip solutions of a single crack.

Using the Green's function described above and the superposition technique, the thermal problem shown in Fig. 1 can be formulated in the form

\[
\frac{1}{\pi} \int_{-c}^{c} \frac{\theta_0(\xi) d\xi}{\eta - \xi} + \int_{-c_0}^{c_0} K_{10}(\eta, \xi) \theta_1(\xi) d\xi = 0 \quad (7)
\]

\[
\frac{1}{\pi} \left\{ \int_{-c}^{c} \frac{\theta_0(\xi) d\xi}{\eta - \xi} + \int_{-c_0}^{c_0} K_{01}(\eta, \xi) \theta_0(\xi) d\xi \right\} = 2 h_{n1}^0(\xi) / k \quad (8)
\]

where \( \theta_0 \) and \( \theta_1 \) are the distributing Green's functions defined along the main crack and microcrack, respectively, subscripts 0 and 1 stand for main crack and microcrack, \( h_{n1}^0 \) is the residual heat flux to be released, which will be given in the Appendix, and

\[
K_{10}(\eta, \xi) = \text{Re} \left[ \frac{1}{\eta - z_{r1}^0 - \xi} \right],
\]

\[
K_{01}(\eta, \xi) = \text{Re} \left[ \frac{1}{\eta z_{r1}^0 + \eta z_{r1}^* - \xi} \right], \quad (9)
\]

where \( z_{r1}^0 = x_{10} + \tau x_0, \quad z_{r1}^* = \cos \alpha + \tau \sin \alpha \).

In addition to Eqs. (7) and (8), the single valuedness of the temperature around a closed contour surrounding a given crack requires that

\[
\int_{\Gamma_x} \theta_i(s) ds = 0 \quad (i=0,1; \quad x = c, c_0) \quad (10)
\]

The coupled singular integral equations for the temperature displacement density in Eqs. (7) and (8) combined with Eq. (10) can be solved numerically\[11\]. Since the solution for the functions, \( \theta_0(\xi) \), has a square root singular at the corresponding crack tip, it is more efficient for the numerical calculations by letting

\[
\theta_0(\xi) = \Theta_0(\xi) / \sqrt{x^2 - \xi^2} \quad (i=0,1; \quad x = c, c_0) \quad (11)
\]

where \( \Theta_0(\xi) \) is a regular function defined in a closed interval \( |\xi| \leq x \). Once the function \( \Theta_0(\xi) \) has been found, the corresponding SED can be given from Eqs. (1) and (5) in the form

\[
\Pi_1 = -\phi_{1z} = -\frac{1}{2\pi} \text{Im} \left\{ \int_{c_0}^{c} B P(\ln z_1) q + d \ln y_1 \theta_1(\xi) d\xi + \int_{c}^{c_0} B P(\ln z_0) q \right\} \quad (12)
\]

\[
+ \text{d} \ln y_1 \theta_0(\xi) d\xi \]

\[
\Pi_2 = \phi_{1z} = \frac{1}{2\pi} \text{Im} \left\{ \int_{c_0}^{c} B P(\ln z_1) q + d \ln y_1 \theta_1(\xi) d\xi + \int_{c}^{c_0} B P(\ln z_0) q \right\} \quad (13)
\]

where

\[
\left[ \ln z_0 \right] = \text{diag}[\ln y_{10}, \ln y_{20}, \ln y_{30}, \ln y_{40}],
\]

\[
\left[ \ln z_1 \right] = \text{diag}[\ln y_{11}, \ln y_{21}, \ln y_{31}, \ln y_{41}], \quad (14)
\]
Thus the traction-charge vectors on the crack faces are of the form

\[ t_{0i}(\eta) = -\Pi_{i}(\eta) \sin \alpha + \Pi_{2}(\eta) \cos \alpha \]

where

\[ \Pi_{i}(\eta) = -\int_{c}^{\infty} b_{i}(\xi) d\xi + \int_{c}^{\infty} (\ln z_{0}) q \]

Generally, \( t_{0i}(\eta) \) do not satisfy the given traction-charge boundary conditions on the crack faces. To satisfy those conditions, we must superpose a solution of the corresponding isothermal problem with a traction-charge vector induced by dislocation vector which, together with the results in Eqs.(16) and (17), will satisfy the given boundary conditions. The elastic solution for a singular dislocation of strength \( b_{0} \) obtained by Ting[12] is adopted.

The solution is now in the form

\[ \Pi_{i}(\eta) = -\frac{1}{2\pi} \text{Im} \left[ B^T \left( z_{j} - \xi \right)^{-1} B \right] b_{0} \]

where

\[ \left( \xi \right) = \text{diag} \left( \xi_{1}, \xi_{2}, \xi_{3}, \xi_{4} \right) \]

Similarly, based on Eqs.(18) and (19), the thermoelectroelastic problem shown in Fig. 1 can be formulated in the form

\[ y_{j} = z_{j} - \xi, \quad y_{j} = z_{j} - \hat{\xi}_{j} \]

\[ \hat{\xi}_{j} = z_{j}^{0} + \hat{\xi} z_{j}^{*}, \quad y_{j} = z_{j} - \hat{\xi}_{j} \quad (j = 1-4) \]

Here \( \Pi_{n}(\eta) \) is the residual SED to be released, which will be given in the Appendix.

Moreover, the single valued condition requires that

\[ \int_{c}^{\epsilon} b_{i}(s) ds = 0 \quad (i=0,1; \epsilon=c, c_{0}) \]

where

\[ L = -2iBB^{T} \]

As was done previously, let

\[ b_{i}(\xi) = \frac{\Theta(\xi)}{\sqrt{x^{2} - \xi^{2}}} \quad (i=0,1; \epsilon=c, c_{0}) \]

Once the function \( \Theta(\xi) \) has been found from Eqs.(21)–(23), the stresses and electric displacements near a given crack tip can be evaluated. For example, the SED, \( \Pi_{n}(\eta) \), in a coordinate local to the main crack line can be expressed in the form

\[ \Pi_{n}(\eta) = \frac{L}{2\pi} \int_{c}^{\epsilon} b_{0}(\xi) d\xi + \frac{1}{\pi} \text{Im} \int_{c}^{\epsilon} B \times \]

\[ \left( \frac{1}{\eta - \xi^{0} - \xi^{*}} \right) B^{T} b_{1}(\xi) d\xi \]

Using Eq.(26), we can evaluate the stress intensity factors \( K^{*}=(K_{n}, K_{m}, K_{l}, K_{d})^{T} \) at the tips, e.g., at the right tip (\( \xi=c \)) of the crack by following definition:

\[ K^{*} = \lim_{\xi \to c} \sqrt{2\pi(\xi - c)\Pi_{n}(\xi)} \]

Combined with the results of Eq.(26), one then leads to

\[ K^{*} = \sqrt{\frac{\pi}{4c}} L \Theta(c) \]
Thus the solution of the singular integral equation enables the direct determination of the stress intensity factors and study the effects of microcrack on the near-tip fields of a main crack.

REFERENCES

APPENDIX
For a crack of length 2c embedded in an infinite thermopiezoelectric plate subjected to remote uniform heat flux $h_o$, the near-tip fields can be found for the solution given by Qin et al.[13]:

\[h_a = -h_o \sqrt{\frac{c}{2r}} \text{Re}[(\cos \theta + \sin \theta)^{-1/2}]\]
\[h_\varphi = h_o \sqrt{\frac{c}{2r}} \text{Re}[(\cos \theta + \sin \theta)^{-1/2}]\]
\[\Pi_1 = -\frac{a^{3/2}h_0}{\sqrt{2r}} \text{Re} \left( \frac{p_k}{(\cos \theta + p_k \sin \theta)^{1/2}} \right) \times \left[ A^T (\text{Im}(d) + \text{Re}(-Lc + Hd)) + B^T (\text{Re}(Sc + Hd - L^{-1}(I + iS^T)d)) \right] \]
\[\Pi_2 = -\frac{a^{3/2}h_0}{\sqrt{2r}} \text{Re} \left( \frac{1}{(\cos \theta + p_k \sin \theta)^{1/2}} \right) \times \left[ A^T (\text{Im}(d) + \text{Re}(-Lc + Hd)) + B^T (\text{Re}(Sc + Hd - L^{-1}(I + iS^T)d)) \right] \]

where (see Fig. 1)
\[H = 2iAA^T, \quad S = i(2AB^T - I), \]
\[r^2 = (x_{10} - a + \xi \cos \alpha)^2 + (x_{20} + \xi \sin \alpha)^2, \]
\[\theta = \cos^{-1}[(x_{10} - a + \xi \cos \alpha)/r] \]

The residual fields, $h_{n1}^0$ and $\Pi_{n1}^0$ can be, then, calculated from the following relations
\[h_{n1}^0 = -h_a \sin \alpha + h_\varphi \cos \alpha \]
\[\Pi_{n1}^0 = -\Pi_1 \sin \alpha + \Pi_2 \cos \alpha \]