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A new RBF-Trefftz meshless method for partial differential equations

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Abstract. Based on the radial basis functions (RBF) and T-Trefftz solution, this paper presents a new meshless method for numerically solving various partial differential equation systems. First, the analog equation method (AEM) is used to convert the original partial differential equation to an equivalent Poisson’s equation. Then, the radial basis functions (RBF) are employed to approximate the inhomogeneous term, while the homogeneous solution is obtained by linear combination of a set of T-Trefftz solutions. The present scheme, named RBF-Trefftz has the advantage over the fundamental solution (MFS) method due to the use of nonsingular T-Trefftz solution rather than singular fundamental solutions, so it does not require the artificial boundary. The application and efficiency of the proposed method are validated through several examples which include different type of differential equations, such as Laplace equation, Helmholtz equation, convection-diffusion equation and time-dependent equation.

1. Introduction

In the last decade, meshless methods have been emerging as important schemes for numerical solution of partial differential equations (PDEs), because the meshless methods can avoid the complicated meshing problem in finite element methods (FEM) and boundary element method (BEM) effectively. The methods based on the radial basis function (RBF) are inherently meshfree due to the independent of dimensionality and complexity of geometry, so the research on the RBF for PDEs has become very active.

Among the existing RBF schemes, the so-called Kansa’s method is a domain-type collocation technique, while the BBF-MFS technique which is evolved form the dual reciprocity boundary element method (DRBEM) [1] is a typical boundary-type methodology. In the RBF-MFS, the method of fundamental solutions (MFS) [2] has been used instead of boundary element method (BEM) in the
process of the DRBEM. Despite the advantages, the main drawback of the RBF-MFS is the use of the fictitious boundaries outside the physical domain to avoid the singularities of the fundamental solution. The creation of the fictitious boundaries is the troublesome issue related to the stability and accuracy of the methodology and there are no effective rules to follow.

In this study, we introduce a new RBF-Trefftz meshless method based on applying the radial basis function and T-Trefftz solution. In contrast to the RBF-MFS which is based on MFS, the RBF-Trefftz is based on Trefftz method. Trefftz method is formulated using the family of T-complete functions which are homogeneous solutions for the governing equation. The Trefftz method was initiated in 1926 [3]. Since then, it has been studied by many researchers (Cheung et al. [4, 5], Zielinski [6], Qin [7, 8], Kita [9]). Unlike in the method of fundamental solution which needs source points to be placed outside the domain in order to avoid singularity, T-Trefftz functions are non-singular inside and on the boundary of the given region. Also, the analog equation method (AEM) is developed to convert the original partial differential equation to an equivalent Poisson’s equation which has a simpler T-Trefftz solution than that of the original problem required.

2. Numerical method and algorithms

Consider a general differential equation

\[ Lu(X) = f(X), \quad X \in \Omega \]  

(1)

with boundary conditions

\[ u(X) = g_1(X), \quad X \subset \Gamma_u \]  

(2)

\[ \frac{\partial u(X)}{\partial n} = g_2(X), \quad X \subset \Gamma_q \]  

(3)

where \( L \) is a differential operator, \( f(X) \) is a known forcing function, \( n \) is the unit outward normal to the boundary \( \Gamma_q \) and \( \frac{\partial u(X)}{\partial n} \) is the directional derivative in direction \( n \). \( X \in \mathbb{R}^d, d \) is the dimension of geometry domain.

We start by converting Equation (1) into a simple Poisson equation through analog equation method, then introduce the RBF-MFS method to solve the equivalent Poisson equation.

2.1. The analog equation method (AEM) [10]

It is assumed here that the operator \( L \) includes the Laplace operator, namely,

\[ L = \nabla^2 + L_q \]  

(4)

It should be pointed out that this assumption is not necessary.

Equation (1) can be restated as

\[ \nabla^2 u(X) = f(X) - L_q u(X) \]  

(5)

that is

\[ \nabla^2 u(X) = b(X) \]  

(6)

where \( b(X) = f(X) - L_q u(X) \)  

(7)

The solution of the above Equation (6) can be expressed as a summation of a particular solution \( u_p \) and a homogeneous solution \( u_h \), that is:
\[ u = u_p + u_h \]  

(8)

where \( u_p \) satisfies the inhomogenous equation

\[ \nabla^2 u_p(X) = b(X) \quad X \in \Omega \]  

(9)

but does not necessarily satisfy the boundary conditions (2)-(3), and \( u_h \) satisfies:

\[ \nabla^2 u_h(X) = 0 \quad X \in \Omega \]  

(10)

\[
\begin{align*}
    u_h(X) &= g(X) - u_p(X) & X \in \Gamma_u \\
    \frac{\partial u_h(X)}{\partial n} &= g(X) - \frac{\partial u_p(X)}{\partial n} & X \in \Gamma_q
\end{align*}
\]  

(11)

2.2. RBF-Trefftz scheme

2.2.1. RBF approximation for particular solution

The particular solution \( u_p \) can be evaluated by RBF. To do this, the right-hand side term of Equation (9) is approximated by RBF [11], yielding

\[ b(X) = \sum_{i=1}^{N_f} \alpha_i \phi_i(X) \quad X \in \Omega \]  

(12)

where \( N_f \) is the number of interpolation points in the domain under consideration. Here, \( \phi_i(X) = \phi(r) = \phi([X - X_i]) \) denotes radial basis functions with the reference point \( X_i \) and \( \alpha_i \) are interpolating coefficients to be determined.

Simultaneously, the particular solution \( u_p \) is similarly expressed as

\[ u_p(X) = \sum_{i=1}^{N_f} \alpha_i \psi_i(X) \]  

(13)

where \( \psi_i \) represent corresponding approximated particular solutions which satisfy the following differential equations:

\[ \nabla^2 \psi_i = \phi_i \]  

(14)

noted the relation between the particular solution \( u_p \) and function \( f(X) \) in Equation (9).

The choice of the RBFs \( \phi \) is very important because it affects the effectiveness and accuracy of the interpolation. There are several type of RBF, like \textit{ad hoc} basis function \((1 + r)\), the polyharmonic splines, Thin Plate Spline (TPS) and mutiquadrics (MQ). In the mathematical and statistical literature TPS and MQs seem to be the most widely used RBFs [12]. In this study, we choose \( \phi \) as a TPS:

\[ \phi = r^2 \ln r \]  

(15)

An additional polynomial term \( P \) is required to assure nonsingularity of the interpolation matrix if the RBF is conditionally positive definite such as TPS [13, 14]. And also, to achieve higher convergence rates for \( f(X) \), the higher order splines are considered [15]. For example,
\[ \varphi = r^{2n} \ln r \quad n \geq 1, \text{ in } \mathbb{R}^2 \] (16)

then

\[ f(X) = \sum_{i=1}^{N_i} \alpha_i \varphi^{[n]}(X) + P_n \] (17)

where \([n]\) is the order of the TPS, \(P_n\) is a polynomial of total degree \(n\) and let \(\{b^l\}_{l=1}^{l_n}\) be a basis for \(P_n\) \((l_n = \binom{n+d}{d})\) is the dimension of \(P_n\), and \(d=2\) for a 2 dimension problem). The corresponding boundary conditions are given by

\[ \sum_{i=1}^{N_i} \alpha_i b_l(P_l) = 0, \quad 1 \leq l \leq l_n \] (18)

However, from Equation (12), it is obvious that the unknown coefficients \(\alpha_i\) can not be straightly determined since the inhomogeneous term \(b(X)\) is an unknown function depending on the unknown function \(u(X)\). The problem can be dealt with in the way described below.

### 2.2.2. Trefftz function for homogeneous solution

Introducing polar coordinates \((r, \theta)\) with \(r = 0\) at the centroid of \(\Omega\), it is known that the set

\[ N = \{r^n \cos n\theta\}_{n=0}^{\infty} \cup \{r^n \sin n\theta\}_{n=1}^{\infty} \] (19)

are T-Trefftz solutions of the Laplace equation.

Hence, the homogeneous solution to Equation (10) is approximated as

\[ u_h(X) = \sum_{j=1}^{m} c_j N_j(X) \] (20)

where \(c_j\) are the coefficients to be determined and \(m\) is its number of components. The terms \(N_j(X) = N(r) = N(\|X - X_j\|)\) are the T-Trefftz solutions of the Laplace operator \(\nabla\), and \(\{X_j\}_{j=1}^{N_j}\) are collocation points placed on the physical boundary of the solution domain. As an illustration, the internal function \(N_j\) in Equation (20) can be given in the form

\[ N_1 = 1, N_2 = r \cos \theta, N_3 = r \sin \theta, \ldots \] (21)

So, Equation (20) can be written as

\[ u_h(X) = \sum_{n=0}^{k} r^n \cos n\theta + \sum_{n=1}^{k} r^n \sin n\theta \] (22)

where \(m = 2k + 1\). Noted that \(u_h\) in Equation (20) and (22) automatically satisfies the given differential equation (10), all we need to do is to enforce \(u_h\) to satisfy the modified boundary conditions (11). To do this, collocation points \(\{X_j\}_{j=1}^{N_j}\) are placed on the physical boundary to fit the boundary condition (11).
2.3. The construction of the solution system

Based on the equations derived above, the solutions $u(X)$ to differential equation (1) and (2)-(3) can be written as

$$u(X) = \sum_{i=1}^{N_i} \alpha_i \psi_i(X) + \sum_{j=1}^{m} c_j N_j(X) \quad X \in \Omega \quad (23)$$

and

$$\frac{\partial u(X)}{\partial n} = \sum_{i=1}^{N_i} \frac{\partial \psi_i(X)}{\partial n} + \sum_{j=1}^{m} c_j \frac{\partial N_j(X)}{\partial n} \quad X \in \Omega \quad (24)$$

differentiating Equation (23) with respect to $x$ or $y$ yields

$$u_x = \sum_{i=1}^{N_i} \alpha_i (\psi_i)_x + \sum_{j=1}^{m} c_j (N_j)_x \quad (25)$$

$$u_y = \sum_{i=1}^{N_i} \alpha_i (\psi_i)_y + \sum_{j=1}^{m} c_j (N_j)_y \quad (25-1)$$

$$u_{xx} = \sum_{i=1}^{N_i} \alpha_i (\psi_i)_{xx} + \sum_{j=1}^{m} c_j (N_j)_{xx} \quad (25-2)$$

$$u_{yy} = \sum_{i=1}^{N_i} \alpha_i (\psi_i)_{yy} + \sum_{j=1}^{m} c_j (N_j)_{yy} \quad (25-3)$$

It is convenient for computer programming to utilize the vector form. Therefore, Equation (23) can be written as

$$u(X) = \{U(X)\} \{\beta\} \quad (26)$$

where

$$\{U(X)\} = \{\psi_1, \psi_2, \ldots, \psi_{N_i}, N_1, N_2, \ldots, N_m\}_{N_i+m}$$

$$\{\beta\} = \{\alpha_1, \alpha_2, \ldots, \alpha_{N_i}, c_1, c_2, \ldots, c_m\}_{N_i+m} \quad (27)$$

Using Equation (23)-(25), satisfaction of the governing equation (9) at $N_i$ interpolation points inside $\Omega$ and the boundary condition (11) at $N_s$ collocation points on the physical boundary provides $N_i + N_s$ equations to determine unknowns $\alpha_i$ and $c_j$

$$\begin{align*}
\sum_{i=1}^{N_i} \alpha_i (\nabla^2 + \mathbf{L}_1) \psi_i + \sum_{j=1}^{m} c_j (\nabla^2 + \mathbf{L}_1) N_j &= f(X) \\
\sum_{i=1}^{N_i} \alpha_i \psi_i + \sum_{j=1}^{m} c_j N_j &= g_1(X) \\
\sum_{i=1}^{N_i} \alpha_i \frac{\partial \psi_i(X)}{\partial n} + \sum_{j=1}^{m} c_j \frac{\partial N_j(X)}{\partial n} &= g_2(X)
\end{align*} \quad (28)$$
It leads to a system of linear algebraic equations in matrix form:

\[
\begin{bmatrix} 
M_{(N_x+N_y) \times (N_x+N_y)} 
\end{bmatrix} 
\begin{bmatrix} 
\beta_1 
\end{bmatrix} 
_{(N_x+N_y) \times 1} = 
\begin{bmatrix} 
\gamma_1 
\end{bmatrix} 
_{(N_x+N_y) \times 1} 
\tag{29}
\]

with

\[
\{ \beta \} = \{ \alpha_1, \alpha_2, \ldots, \alpha_{N_x}, c_1, c_2, \ldots, c_m \}, \quad \{ \gamma \} = \{ f_1, \ldots, f_{N_x}, g_1, \ldots, g_N \} 
\tag{30}
\]

If the number of components equals to the number of collocation points on the physical boundary (\( m = N_x \)), this leads to properly determined equations. Alternatively, in case the number of components is smaller than the number of collocation points (\( m < N_x \)), this results in over-determined equations. The least square method can be used to solve the over-determined equations. Once \( \{ \beta \} \) is obtained, \( u \) can be computed using Equation (23).

3. Numerical implementation

In order to evaluate the performance of the proposed approach, here we consider typical benchmark problems which are taken from the References [1, 16] for solving differential equations. The geometry of the test problems is an ellipse featured with major axis of length 3 and minor axis of length 1 unless otherwise specified. For comparison, the same nodes as Reference [1] are employed to discretize the domain. The results are compared with other numerical methods and the analytical solution.

3.1. Laplace Equation

The 2D Laplace equation is given by

\[
\nabla^2 u = 0 
\tag{31}
\]

the boundary condition is applied by the particular function

\[
u = x + y
\tag{32}\]

The numerical results are presented in Table 1 together with those by the BEM and boundary knot method (BKM) for comparison. It can be seen clearly from Table 1 that the results obtained by proposed RBF-Trefftz methods agree well with the exact solution and appear to be more accurate than the results obtained from BEM.

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>exact</th>
<th>BEM</th>
<th>BKM</th>
<th>RBF-Trefftz</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>0.0</td>
<td>1.500</td>
<td>1.507</td>
<td>1.500</td>
<td>1.500</td>
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<td>1.2</td>
<td>-0.35</td>
<td>0.850</td>
<td>0.857</td>
<td>0.850</td>
<td>0.850</td>
</tr>
<tr>
<td>0.6</td>
<td>-0.45</td>
<td>0.150</td>
<td>0.154</td>
<td>0.150</td>
<td>0.150</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>-0.450</td>
<td>-0.451</td>
<td>-0.450</td>
<td>-0.450</td>
</tr>
<tr>
<td>0.9</td>
<td>0.0</td>
<td>0.900</td>
<td>0.913</td>
<td>0.900</td>
<td>0.900</td>
</tr>
<tr>
<td>0.3</td>
<td>0.0</td>
<td>0.300</td>
<td>0.304</td>
<td>0.300</td>
<td>0.300</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>
3.2. *Helmholtz equation*

The 2D Helmholtz equation is given by

\[ \nabla^2 u + u = 0 \]  \hspace{1cm} (33)

with inhomogeneous boundary condition

\[ u = \sin x \]  \hspace{1cm} (34)

The results are displayed in Table 2. It can be seen that the RBF-Trefftz method can achieve higher accuracy than other methods.

**Table 2.** Results for a Helmholtz equation.

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>exact</th>
<th>DRBEM</th>
<th>BKM</th>
<th>RBF-Trefftz</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>0.0</td>
<td>0.998</td>
<td>0.994</td>
<td>0.997</td>
<td>0.998</td>
</tr>
<tr>
<td>1.2</td>
<td>-0.35</td>
<td>0.932</td>
<td>0.928</td>
<td>0.932</td>
<td>0.932</td>
</tr>
<tr>
<td>0.6</td>
<td>-0.45</td>
<td>0.565</td>
<td>0.562</td>
<td>0.565</td>
<td>0.565</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>0.9</td>
<td>0.0</td>
<td>0.783</td>
<td>0.780</td>
<td>0.783</td>
<td>0.783</td>
</tr>
<tr>
<td>0.3</td>
<td>0.0</td>
<td>0.296</td>
<td>0.294</td>
<td>0.296</td>
<td>0.296</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

3.3. *Varying parameter Helmholtz problem*

Consider the varying-parameter Helmholtz equation

\[ \nabla^2 u - \frac{2}{x^2} u = 0 \]  \hspace{1cm} (35)

with the particular solution applied as boundary condition

\[ u = -\frac{2}{x} \]  \hspace{1cm} (36)

The origin of the Cartesian coordinates system is dislocated to the node (3.0) to circumvent singularity at \( x = 0 \). From Table 3 we can see that the accuracy and efficiency of the RBF-Trefftz scheme are very encouraging.

3.4. *Convection-Diffusion problems*

Consider the equation

\[ \nabla^2 u = -\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} \]  \hspace{1cm} (37)

with boundary conditions

\[ u = e^{-x} + e^{-y} \]  \hspace{1cm} (38)

which is also a particular solution of Equation (37). It can be seen from Table 4 that the results obtained by proposed meshless methods agree well with the exact solution and appear to be more accurate than the results obtained from other methods.
Table 3. Absolute errors for varying parameter Helmholtz problem.

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>DRBEM</th>
<th>BKM</th>
<th>RBF-Trefitz</th>
</tr>
</thead>
<tbody>
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<td>4.5</td>
<td>0.0</td>
<td>2.3e-3</td>
<td>2.6e-3</td>
<td>1.0e-4</td>
</tr>
<tr>
<td>4.2</td>
<td>-0.35</td>
<td>2.1e-1</td>
<td>3.3e-3</td>
<td>1.5e-4</td>
</tr>
<tr>
<td>3.6</td>
<td>-0.45</td>
<td>5.4e-3</td>
<td>4.7e-3</td>
<td>3.0e-4</td>
</tr>
<tr>
<td>3.0</td>
<td>-0.45</td>
<td>4.5e-3</td>
<td>4.4e-3</td>
<td>5.4e-4</td>
</tr>
<tr>
<td>2.4</td>
<td>-0.45</td>
<td>1.2e-3</td>
<td>9.1e-4</td>
<td>8.7e-4</td>
</tr>
<tr>
<td>1.8</td>
<td>-0.35</td>
<td>9.0e-4</td>
<td>1.7e-2</td>
<td>1.4e-3</td>
</tr>
<tr>
<td>1.5</td>
<td>0.0</td>
<td>1.0e-4</td>
<td>3.4e-2</td>
<td>2.0e-3</td>
</tr>
<tr>
<td>3.9</td>
<td>0.0</td>
<td>3.9e-3</td>
<td>5.3e-3</td>
<td>2.7e-4</td>
</tr>
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<td>0.0</td>
<td>3.3e-3</td>
<td>6.3e-3</td>
<td>5.2e-4</td>
</tr>
<tr>
<td>3.0</td>
<td>0.0</td>
<td>4.5e-3</td>
<td>5.6e-3</td>
<td>6.8e-4</td>
</tr>
<tr>
<td>2.7</td>
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<td>2.7e-3</td>
<td>3.4e-3</td>
<td>8.7e-4</td>
</tr>
<tr>
<td>2.1</td>
<td>0.0</td>
<td>3.2e-3</td>
<td>8.8e-3</td>
<td>1.4e-3</td>
</tr>
</tbody>
</table>

Table 4. Results for a Convection-Diffusion equation.

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>exact</th>
<th>DRBEM</th>
<th>BKM</th>
<th>RBF-Trefitz</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>0.0</td>
<td>1.223</td>
<td>1.231</td>
<td>1.224</td>
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<td>1.720</td>
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<td>2.602</td>
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<td>0.0</td>
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<td>2.335</td>
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<tr>
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<td>0.0</td>
<td>2.000</td>
<td>1.989</td>
<td>0.993</td>
<td>2.068</td>
</tr>
</tbody>
</table>

3.5. *Modified inhomogeneous equation*

Consider the equation

\[ \nabla^2 u - ku = 4 - k(x^2 + y^2) \quad (39) \]

with boundary conditions

\[ u = x^2 + y^2 \quad (40) \]

which is also a particular solution of Equation (39). In our calculation, the solution domain is a square with side length 3 and \( k = 9 \). The distribution of the numerical solution is shown in figure.1 and the corresponding isolines are shown in figure.2. It can be seen that the numerical solution matches very well with the analytical solution.
3.6. Time-dependent partial differential equation

Let us consider a time-dependent partial differential equation

$$\nabla^2 u(x, y, t) - \frac{\partial u(x, y, t)}{\partial t} = 0$$

(41)

with boundary conditions and initial condition

$$u(x, y, t) = \frac{1}{4\pi(t + 0.1)} \exp\left(-\frac{(x^2 + y^2)}{4(t + 0.1)}\right)$$

(42)

The solution domain is an ellipse with major axis 3 and minor axis 1.

In the literature there are different approaches to handle time variable, two of which are: (1) Laplace transform; (2) time-stepping method. Since numerical inversion of the Laplace transform is often ill-posed, here we apply time-stepping method scheme to handle the time variable. For a typical time interval $[t^n, t^{n+1}] \subset [0, T]$, $u(X, t)$, its derivative with respect to time variable $t$

$$u(X, t) = \theta u^{n+1}(X) + (1 - \theta) u^n(X)$$

$$\frac{\partial u(X, t)}{\partial t} = \frac{u^{n+1}(X) - u^n(X)}{\tau}$$

(43)

It is proved that the Partial differential Equation (PDE) can be solved accurately using the implicit scheme ($\theta = 1$) [17]. Hence, we use $\theta = 1$ and $\tau = 0.01$ in our analysis. Table 5 shows the solution at $(0, 0)$

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Table 5. Maximum absolute errors at various time steps at point (0,0)
4. Conclusion
A new RBF-Trefftz meshless approach is developed for solving partial differential equations. After transferring the partial differential equation into equivalent Poisson’s equation by the analog equation method, the inhomogeneous term is approximated by linear combination of radial basis functions (RBF) and the homogeneous term by Trefftz bases. RBF-Trefftz is easier to implement since it is non-singular, so it is unnecessary to place source points outside the domain for avoiding singularity which does occur in RBF-MFS. The numerical examples show clearly that the method presented is very effective.

References

[16] Chen W and Tanaka M 2002 A meshless, integration-free, and boundary-only RBF technique, *Computers and Mathematics with Applications*. 43. 379-391,