Trefftz finite element method and its special purpose elements

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Abstract In this report, the hybrid Trefftz finite element method based on nonsingular T-complete functions is first briefly reviewed for the purpose of establishing notation; then various special purpose elements are developed to illustrate the major advantage of Trefftz methods. In HT-FEM, elements containing local defects are treated by simply replacing the regular T-complete functions with appropriate special purpose “trial” functions. Elements containing such special “trial functions” is known as special purpose elements. It should be mentioned that a common characteristic of such special trial functions is that it is not only the governing partial differential equations which are satisfied exactly, but also some prescribed boundary conditions at a particular portion of the element boundary. This enables various singularities to be specifically taken into consideration without troublesome mesh refinement. Formulations of special purpose hole elements for both heat conduction and plane stress/strain problems are presented. Finally, a numerical example involving contact problems with an elliptic hole, are considered. Numerical results from the proposed models are presented and compared with those from ABAQUS.

Key words: Finite element method, Special purpose element, heat conduction, Trefftz function

INTRODUCTION

As a highly efficient and well established computational tool, hybrid Trefftz (HT) finite element method (FEM), initiated about three decades ago \cite{1}, has attracted considerable attention in practical engineering. This method preserves the advantages of conventional FEM and boundary element method (BEM) and avoids some of their drawbacks \cite{2,3}. The common characteristic is that the trial functions (so called Trefftz functions) exactly satisfy, \textit{a priori}, the governing differential equations, and for special purpose elements, they should also satisfy the boundary conditions on influential critical boundary portions (the hole surface here). Up to now, T-elements have been successfully applied to problems of elasticity \cite{2}, Kirchhoff plates \cite{4}, moderately thick Reissner-Mindlin plates \cite{5}, thick plates \cite{6}, general 3-D solid mechanics \cite{7}, potential problems \cite{8}, elastodynamic problems \cite{9}, transient heat conduction analysis \cite{10}, geometrically nonlinear plates \cite{11}, materially nonlinear elasticity \cite{12}, and Piezoelectric materials \cite{13}. This paper introduces three types of special purpose elements including hole elements for potential problems, circular hole elements for plane stress problems, elliptic hole elements for contact problems. Based on the developed special element models, Numerical examples involving holes are considered to assess the effectiveness and applicability of the special hole elements. Comparison of the present results with the predictions by the commercial FE software package ABAQUS has been made and a good agreement is observed, although the number of elements used here is much less than those used in the ABAQUS.

ASSUMED FIELDS IN TREFFTZ FEM

The main idea of the HT FE model is to establish a FE formulation whereby inter-element continuity is enforced on a non-conforming internal field chosen so as to \textit{a priori} satisfy the governing differential equation of the problem under consideration \cite{14,15}. With this method the solution domain $\Omega$ is subdivided into elements, and over each element “$e$”, the assumed intra-element fields are
where $c_{ij}$ stands for undetermined coefficients and $\mathbf{u}_e$ and $\mathbf{N}_e$ are, respectively, the particular and homogeneous solutions to the governing differential equations. The unknown coefficient $c_e$ may be calculated from the conditions on the external boundary and/or the continuity conditions on the inter-element boundary. Thus various Trefftz-element models can be obtained by using different approaches to enforce these conditions. In the majority of the approaches a hybrid technique is usually used, whereby the elements are linked through an auxiliary conforming displacement frame which has the same form as in conventional FE method. This means that, in the Trefftz FE approach, a conforming potential (or displacement in solid mechanics) field should be independently defined on the element boundary to enforce the potential continuity between elements and also to link the coefficient $c_e$, appearing in Eq (1), with nodal displacement $\mathbf{d}_e (= \{d_i\}_e)$. The frame is defined as

$$\mathbf{u}_e = \mathbf{N}_e \mathbf{d}_e \text{ on } \Gamma_e$$

being independently assumed along the element boundary in terms of nodal degrees of freedom $\mathbf{d}_e$, where

$$\Gamma_e = \Gamma_u \cup \Gamma_{ei} \cup \Gamma_{ed}, \text{ while } \Gamma_{ei} = \Gamma_e \cap \Gamma_{ei}, \text{ and } \Gamma_{ed} \text{ is the inter-element boundary, } \mathbf{N}_e \text{ are the shape functions (frame functions) defined in the customary way as in conventional FEM, typical examples of which are displayed in Fig. 1.}$$

The corresponding generalized stress field

$$\mathbf{\sigma}_e = \mathbf{\sigma}_e + \sum_{j=1}^{m} \mathbf{T}_{ij} c_{ij} = \mathbf{\sigma}_e + \mathbf{T}_e \mathbf{c}_e$$

as well as the boundary tractions

$$\mathbf{t}_e = \mathbf{t}_e + \sum_{j=1}^{m} \mathbf{Q}_{ij} c_{ij} = \mathbf{t}_e + \mathbf{Q}_e \mathbf{c}_e$$

can be readily deduced from

$$\mathbf{\sigma}_e = \{ \frac{\partial \mathbf{u}_e}{\partial x_1}, \frac{\partial \mathbf{u}_e}{\partial x_2} \}^T \text{ and } \mathbf{t}_e = \{ n_1, n_2 \} \mathbf{\sigma}_e$$

for potential problems, and

$$\mathbf{\sigma}_e = \mathbf{D} \mathbf{L}^T \mathbf{u}_e \text{ and } \mathbf{t}_e = \mathbf{A} \mathbf{\sigma}_e$$

for 2D elastic problems, where $\mathbf{L}$ is the differential operator matrix, $\mathbf{D}$ contains elastic constants and $\mathbf{A}$ contains components of a unit normal to the element boundary $\Gamma_e$ (see [2] for the detailed expressions of $\mathbf{L}$, $\mathbf{D}$, and $\mathbf{A}$).

**VARIATIONAL PRINCIPLE AND ELEMENT MATRIX EQUATION**

The HT FE formulation for 2D elastic problems (or potential problems) may be obtained by means of the following modified variational principle [15].
\[ \Psi_e = \frac{1}{2} \int_{\Omega_e} \varepsilon^T \sigma \, d\Omega - \int_{\Gamma_e} t^T \tilde{u} \, d\Gamma + \int_{\Gamma_e} \tilde{t}^T \tilde{u} \, d\Gamma \]  
(7)

for 2D elastic problems, and

\[ \Psi_e = \frac{1}{2} \int_{\Omega_e} \left( q_1^2 + q_2^2 \right) \, d\Omega - \int_{\Gamma_e} q \tilde{u} \, d\Gamma + \int_{\Gamma_e} \tilde{q} \tilde{u} \, d\Gamma \]  
(8)

for 2D potential problems, where the overhead bar is used to designate specified values.

Applying the stationary condition to functional (7) or (8) straightforwardly leads to the symmetric element stiffness equation

\[ K_e d_e = P_e \]  
(9)

where

\[ K_e = G_e^T H_e G_e \]  
(10)

\[ P_e = G_e^T H_e \mathbf{h}_e - \mathbf{g}_e \]  
(11)

Here the auxiliary matrices \( H_e, G_e, \mathbf{h}_e \) and \( \mathbf{g}_e \) are explicitly expressed as

\[ H_e = \int_{\Gamma_e} Q_e^T N_e \, d\Gamma = \int_{\Gamma_e} N_e^T Q_e \, d\Gamma \]  
(12)

\[ G_e = \int_{\Gamma_e} Q_e^T \tilde{N}_e \, d\Gamma \]  
(13)

\[ \mathbf{h}_e = \frac{1}{2} \int_{\Gamma_e} \left( N_e^T \mathbf{t} + Q_e^T \tilde{u}_e \right) \, d\Gamma + \frac{1}{2} \int_{\Omega_e} \tilde{N}_e^T \mathbf{b}_e \, d\Omega \]  
(14)

\[ \mathbf{g}_e = \int_{\Gamma_e} \tilde{N}_e^T \mathbf{t}_e \, d\Gamma - \int_{\Gamma_e} \tilde{\mathbf{N}}_e^T \tilde{\mathbf{t}}_e \, d\Gamma \]  
(15)

in which \( \mathbf{b}_e \) stand for the body forces.

**SPECIAL PURPOSE ELEMENTS**

It is well known that singularities induced by local defects such as angular corners, holes, and so on, can be accurately accounted for in the conventional FE model by way of appropriate local refinement of the element mesh. However, an important feature of the Trefftz element method is that such problems can be far more efficiently handled by the use of special purpose functions[2,14]. Elements containing local defects (see Fig. 2) are treated by simply replacing the standard regular functions \( N_e \) in Eq (1) by appropriate special purpose functions. One common characteristic of such trial functions is that it is not only the governing differential equations, which are satisfied exactly but also some prescribed boundary conditions at a particular portion \( \Gamma_{es} \) (see Fig. 2) of the element boundary. This enables various singularities to be specifically taken into account without troublesome mesh refinement. Since the whole element formulation remains unchanged (except that now the frame function \( \tilde{u}_e \) in Eq (2) is defined and the boundary integration is performed only at the portion \( \Gamma_e^* \) of the element boundary \( \Gamma_e = \Gamma_e^* + \Gamma_{es} \), see Fig. 2) [2], all that is needed to implement the elements containing such special trial functions is to provide the element subroutine of the standard, regular elements with a library of various optional sets of special purpose functions.

![Fig. 2 Special element containing a circular hole](image)
1) Special purpose circular hole elements for potential problems

In what follows we show how special purpose functions can be constructed to satisfy both the Laplace equation \( \nabla^2 u = 0 \) and the traction-free boundary conditions on the circular hole faces (Fig. 2). The derivation of such functions is based on the general solution of the two-dimensional Laplace equation:

\[
u_n(r, \theta) = a_0 + \sum_{n=1}^{\infty} \left( a_n r^{\beta_n} + b_n r^{-\beta_n} \right) \cos(\lambda_n \theta) + \sum_{n=1}^{\infty} (d_n r^{\beta_n} + e_n r^{-\beta_n}) \sin(\lambda_n \theta),
\]

where \( \beta_n , \lambda_n \) are the roots of the transcendental equation:

\[
\frac{\sinn(\beta_n a_0)}{\sinn(\beta_n a)} = \frac{\coss(\beta_n a_0)}{\coss(\beta_n a)}
\]

Appropriate trial functions for an element containing a circular hole are obtained by considering an infinite perforated domain (Fig. 2) for which the free boundary condition along the circular hole boundary is assumed as

\[
\frac{\partial u}{\partial r} = 0 \quad \text{along the hole boundary}
\]

After a series of mathematical derivation, we obtain

\[
u_n(r, \theta) = a_0 + \sum_{n=1}^{\infty} \left( r^n + b^n r^{-n} \right) \left[ a_n \cos(n\theta) + d_n \sin(n\theta) \right]
\]

For which the internal special Trefftz function defined in Eq (1) may be taken as

\[
N_{2k-1} = (r^k + b^k r^{-k}) \cos(k\theta)
\]

\[
N_{2k} = (r^k + b^k r^{-k}) \sin(k\theta)
\]

for \( k = 1, 2, 3 \ldots \)

2) Special purpose circular hole elements for elastic problems

Consider again the infinite perforated plane as shown in Fig. 2, the governing differential equations of linear elastic problems can be written in terms of displacements as \[15\]

\[
\begin{align*}
\nabla^2 u_{rr} + \left( \frac{1}{r} \frac{\partial}{\partial r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) u_{rr} + \frac{2}{r} \frac{\partial}{\partial r} u_{\theta \theta} + \frac{2}{r^2} \frac{\partial}{\partial r} u_{r \theta} - \frac{3}{r^2} \frac{\partial}{\partial \theta} u_{\theta 
\end{align*}
\]

Assume that the general solutions to Eq (19) have following form \[2\]

\[
\begin{align*}
\phi_n(r, \theta) = C_0 + C_1 r^2 + C_2 r^2 + C_3 + C_4 \ln r \\
g_1(r, \theta) = \frac{5 + \mu}{1 - 3\mu} C_0 r^2 + C_1 r^2 - C_3 + \frac{1 + \mu}{3 - \mu} \ln r C_4
\end{align*}
\]

and for the case \( k \geq 2 \), we obtain similarly

\[
\begin{align*}
f_k &= C_1^k r^{k+1} + C_2^k r^{-k+1} + C_3^k r^{k-1} + C_4^k r^{-(k-1)} \\
g_k &= \frac{k(1+\mu) + 4}{2(1-\mu) - k(1+\mu)} C_1^k r^{k+1} + C_2^k r^{-k+1} + \frac{k(1+\mu) - 4}{2(1-\mu) + k(1+\mu)} C_3^k r^{k-1} - C_4^k r^{-(k+1)}
\end{align*}
\]

It should be mentioned that the functions \( g_1, f_2, \) and \( g_2 \) given in page 1454 of Ref [16] are incorrect. Consequently, the corresponding functions derived from \( g_1, f_2, \) and \( g_2 \), which appeared in page 1470 of Ref [16], are also wrong. In the above equations, \( C_1^k, C_2^k, C_3^k, C_4^k \) are undetermined constants, and half of them can be eliminated by fulfilling (for any \( k \) separately) the homogeneous boundary conditions at the hole surface \( r = b \)

\[
\sigma_r \bigg|_{r=b} = \tau_{\theta \theta} \bigg|_{r=b} = 0
\]

The remaining constants become the undetermined coefficients \( c_j \) of the special purpose expansion set.

An alternative set of solutions can be found by assuming

\[
\begin{align*}
u_r &= f_j(r) \sin k\theta, \quad u_\theta = g_j(r) \cos k\theta \quad (k = 0, 1, 2, \cdots)
\end{align*}
\]
A similar procedure can be performed for determining the functions \( f_i \) and \( g_i \), which gives

\[
\begin{align*}
g_0 &= C_1^0 r + C_4^0 r^{-1} \\
f_0 &= C_1^1 r^2 + C_4^1 r^{-2} + C_1^1 + C_4^1 \ln r \\
g_1 &= \frac{5 + \mu}{1 - 3\mu} C_1^1 r^2 - C_4^1 r^{-2} + C_1^1 + \left(\frac{1 + \mu}{3 - \mu} + \ln r\right) C_4^1 \\
f_1 &= C_1^1 k^{r+1} + C_4^1 r^{-k-1} + C_1^1 r^{1-k} + C_4^1 r^{-(1-k)} \\
g_k &= \frac{k(1+\mu) + 4}{2(1-\mu)-k(1+\mu)} C_1^1 k^{r+1} - C_4^1 r^{-k-1} - \frac{k(1+\mu) - 4}{2(1-\mu)+k(1+\mu)} C_2^1 r^{1-k} + C_4^1 r^{1-k+k} 
\end{align*}
\]

(27)

The complete displacement homogeneous solutions \( \mathbf{N}_k = (N_{ijk}, N_{ijk})^T \) constructed from the two sets of displacement fields Eqs (21) and (25) can be written as

\[
\mathbf{N}_{10} = \begin{cases} 
1 + \mu b^2 r^{-1} + r \\
1 - \mu \\
0 
\end{cases}, \quad \mathbf{N}_{20} = \begin{bmatrix} 0 \\
r \end{bmatrix}
\]

(28)

for \( k=0 \), and

\[
\mathbf{N}_{11} = \begin{cases} 
F_1 \cos \theta \\
G_1 \sin \theta 
\end{cases}, \quad \mathbf{N}_{21} = \begin{cases} 
F_1 \sin \theta \\
-G_1 \cos \theta 
\end{cases}, \quad \mathbf{N}_{31} = \begin{cases} 
\cos \theta \\
-\sin \theta 
\end{cases}, \quad \mathbf{N}_{41} = \begin{cases} 
\sin \theta \\
\cos \theta 
\end{cases}
\]

(29)

for \( k=1 \), and

\[
\mathbf{N}_{1k} = \begin{cases} 
F_{2k} \cos k\theta \\
G_{2k} \sin k\theta 
\end{cases}, \quad \mathbf{N}_{2k} = \begin{cases} 
F_{2k} \sin k\theta \\
-G_{2k} \cos k\theta 
\end{cases}, \quad \mathbf{N}_{3k} = \begin{cases} 
F_{3k} \cos k\theta \\
G_{3k} \sin k\theta 
\end{cases}, \quad \mathbf{N}_{4k} = \begin{cases} 
F_{3k} \sin k\theta \\
-G_{3k} \cos k\theta 
\end{cases}
\]

(30)

for \( k \geq 2 \), where

\[
\begin{align*}
F_1 &= A_1 r^{-2} + r^2, \quad G_1 = A_2 r^{-2} + A_2 r^2, \quad F_{2k} = B_{2k} r^{1-k} + B_{4k} r^{k+1} + r^{k+1}, \\
G_{2k} &= B_{2k} r^{1-k} - B_{2k} r^{-k+1} + B_{4k} r^{k+1}, \quad F_{3k} = B_{2k} r^{1-k} + B_{4k} r^{k+1} + r^{-k+1}, \quad G_{3k} = B_{2k} r^{1-k} + B_{4k} r^{k+1} + r^{-k-1}
\end{align*}
\]

(31)

with

\[
\begin{align*}
A_1 &= \frac{1 + \mu}{1 - 3\mu} b^2, \quad A_2 = \frac{5 + \mu}{1 - 3\mu}, \quad B_{2k} &= \frac{2(1-\mu) + k(1+\mu)}{(k-1)[2(1-\mu) - k(1+\mu)]} b^{2k}, \\
B_{2k} &= \frac{k^2 (1+\mu)}{(k-1)[2(1-\mu) - k(1+\mu)]} b^{2k}, \quad B_{2k} &= \frac{2(1-\mu) + k(1+\mu)}{(k-1)[1+k(1+\mu)]} b^{-2k}, \\
B_{2k} &= \frac{k^2 (1+\mu)}{(k-1)[2(1-\mu) - k(1+\mu)]} b^{2k}, \quad B_{2k} &= \frac{2(1-\mu) + k(1+\mu)}{(k-1)[1+k(1+\mu)]} b^{-2k}, \quad B_{2k} &= \frac{2(1-\mu) + k(1+\mu)}{(k-1)[1+k(1+\mu)]} b^{-2k}
\end{align*}
\]

(32)

(33)

3) Special purpose elliptic hole elements

The derivation of special elliptic hole functions can be carried out by using following expressions of displacements and stresses [17]

\[
\begin{align*}
2G(u + iv) &= \kappa \phi(z) - z \bar{\phi}(z) - \bar{\psi}(z) \\
\sigma_{xx} + i \sigma_{xy} &= \phi(z) + \bar{\phi}(z) - z \bar{\phi}(z) - \bar{\psi}(z) \\
\sigma_{yy} - i \sigma_{xy} &= \phi(z) + \bar{\phi}(z) + z \bar{\phi}(z) + \bar{\psi}(z)
\end{align*}
\]

(34)

(35)

(36)
where \( z = x + iy \), \( i = \sqrt{-1} \), \( \phi(z) \) and \( \psi(z) \) are two analytical functions, \( G = E / 2(1+\mu) \), \( \kappa = (3-\mu)/(1+\mu) \), \( E \) and \( \mu \) are, respectively, Young’s modulus and Poisson’s ratio, \( (\cdot)^* \) denotes differentiation with respect to \( z \) and \( (\cdot)^\dagger \) represents complex conjugate.

Fig. 3 Conformal mapping for constructing special hole element

Due to the fact that Piltner’s hole element is based on the horizontal conformal transformation, it is tedious to treat structures with holes in arbitrary direction. To bypass this difficulty, a rotated mapping function

\[
\phi(\theta) = e^{i\theta}
\]

is introduced into the horizontal conformal transformation as

\[
z = f(\zeta) = \phi(\theta)c(\zeta + m\zeta^{-1}) = ce^{i\theta}(\zeta + m\zeta^{-1})
\]

where \( c = (a+b)/2 \), \( m = (a-b)/(a+b) \), \( a \) and \( b \) are, respectively, the semi-major axis and semi-minor axis, \( \theta \) is the angle between the semi-major axis and \( x \) axis (Fig. 3).

Substituting the inverse transformation

\[
z = f^{-1}(\zeta) = \frac{1}{2ce^{i\theta}}(z \pm \sqrt{z^2 + 4c^2me^{2i\theta}})
\]

into Eqs (34)-(36) produces the displacements and stresses in the \( \xi \)-plane as:

\[
2G(u_x + iu_y) = \kappa\phi - f\frac{\phi}{f} - \psi
\]

\[
\sigma_{xx} - i\tau_{xy} = \frac{\phi}{f} + \frac{\bar{\phi}}{f} - \frac{\bar{f}(j\phi - j\bar{\phi})}{f^3} - \frac{\bar{\psi}}{f}
\]

\[
\sigma_{yy} - i\tau_{yx} = \frac{\phi}{f} + \frac{\bar{\phi}}{f} + \frac{\bar{f}(j\phi - j\bar{\phi})}{f^3} + \frac{\bar{\psi}}{f}
\]

Here, the sign in Eq (39) is chosen in a similar way to that in Ref [17].

In general, it is impossible to find a formulation for \( \phi(\xi) \) and \( \psi(\xi) \) in closed form for arbitrary geometry and boundary conditions. Following the way of Piltner [17], we can expanded the two holomorphic functions in the general expressions of elasticity solutions into two complex Laurent series respectively as follows

\[
\phi(\xi) = \sum_{j=-N}^{M} c_j \xi^j
\]

\[
\psi(\xi) = -\sum_{j=-N}^{M} \left[ \bar{c}_j \xi^{-j} + c_j \bar{\xi}^{-j} (\zeta^{-2} + m\zeta^{-2}) \right]
\]

where \( c_j = a_j + ib_j \), are complex coefficients, \( M \) and \( N \) are the upper and lower limits of the Laurent series and \( M \) is generally set to be \( N \) for symmetry, Eq (44) is obtained according to the traction-free condition along the hole boundary. Therefore, the displacement and stress fields are given in the following form

\[
2G(u_x + iu_y) = \sum_{j=-N}^{M} \left[ (\xi_1 - \xi_2) a_j + i(\xi_1 + \xi_2) b_j \right]
\]
\[\sigma_{xx} - i\tau_{xy} = \sum_{j=-N}^{M} \left[ X_1 + X_2 - X_3 - X_4 + X_5 \right] a_j + \left[ X_1 - X_2 + X_3 + X_4 - X_5 \right] b_j \quad (46)\]

\[\sigma_{yy} + i\tau_{xy} = \sum_{j=-N}^{M} \left[ X_1 + X_2 + X_3 + X_4 - X_5 \right] a_j + \left[ X_1 - X_2 + X_3 - X_4 + X_5 \right] b_j \quad (47)\]

where

\[\xi_1 = \kappa \zeta^j + \bar{\zeta}^{-j} \quad (48)\]

\[\xi_2 = \frac{je^{i2\theta} \zeta^{-j-1} \left[ \xi - \bar{\zeta}^{-1} + m(\zeta^{-1} - \bar{\zeta}) \right]}{1 - m}\]

\[X_1 = \frac{j\zeta^{-1}}{c e^{i\theta} (1 - m\zeta^{-2})} \quad (50)\]

\[X_2 = \frac{j \zeta^{-j-1} e^{i\theta} \zeta^{-1}}{c \left( 1 - m\zeta^{-2} \right)} \quad (51)\]

\[X_3 = \frac{j \zeta^{j-2} (j-1) \zeta^{-j-2} + (j+1) \zeta^{-j-4}}{c e^{i\theta} (1 - m\zeta^{-2})^3} \quad (52)\]

\[X_4 = \frac{j \zeta^{j-2} e^{i\theta} \zeta^{-1}}{c \left( 1 - m\zeta^{-2} \right)} \quad (53)\]

\[X_5 = \frac{\sqrt{((j-2) - m^2 (j+2)) \zeta^{j-2} - mj \zeta^{-j-2} + mj \zeta^{-j-4}}}{c e^{i\theta} (1 - m\zeta^{-2})^3} \quad (54)\]

From Eqs (40)-(42) the special Trefftz functions \(N_e\) and \(T_e\) may be written as follows

\[N_e = \frac{1}{2G} \begin{bmatrix} \text{Re} U_{-N} & \cdots & \text{Re} U_M & \text{Re} U_{M+N+1} & \cdots & \text{Re} U_{2(M+N)} \\ \text{Im} U_{-N} & \cdots & \text{Im} U_M & \text{Im} U_{M+N+1} & \cdots & \text{Im} U_{2(M+N)} \end{bmatrix} \quad (55)\]

\[T_e = \begin{bmatrix} \text{Re} S_{1,-N} & \cdots & \text{Re} S_{1,M} & \text{Re} S_{1,M+N+1} & \cdots & \text{Re} S_{1,2(M+N)} \\ \text{Re} S_{2,-N} & \cdots & \text{Re} S_{2,M} & \text{Re} S_{2,M+N+1} & \cdots & \text{Re} S_{2,2(M+N)} \\ \text{Im} S_{3,-N} & \cdots & \text{Im} S_{3,M} & \text{Im} S_{3,M+N+1} & \cdots & \text{Im} S_{3,2(M+N)} \end{bmatrix} \quad (56)\]

where

\[U_j = \xi_1 - \xi_2 \quad (57)\]

\[U_{M+N+j} = i(\xi_1 + \xi_2) \quad (58)\]

\[S_{h,j} = X_1 + X_2 - X_3 - X_4 + X_5 \quad (59)\]

\[S_{1,M+N+j} = X_1 + X_2 - X_3 + X_4 + X_5 \quad (60)\]

\[S_{2,j} = X_1 + X_2 + X_3 + X_4 - X_5 \quad (61)\]

\[S_{2,M+N+j} = X_1 - X_2 + X_3 - X_4 + X_5 \quad (62)\]

\[S_{3,j} = X_1 + X_4 - X_5 \quad (63)\]

\[S_{3,M+N+j} = X_3 - X_4 - X_5 \quad (64)\]

In numerical implementation, it is necessary to eliminating the rigid body movement in the above expressions. For a circular hole element, one has \(\text{Re} S_{1,M+N+1} = \text{Re} S_{2,M+N+1} = \text{Im} S_{3,M+N+1} = 0\), the coefficient \(b_j\) has to be zero, thus the number of Trefftz terms \(m\) is \(2(M + N) - 1\), whereas for elliptical hole element, \(m\) is chosen to be \(2(M + N)\) because all terms make contributions to stresses. Besides, \(m\) should be approximately equal to the degrees of freedom of the element. It is clear that Piltner’s recommendation on the choice of the number of special Trefftz functions can accurately represents an element with a circular hole only.
**NUMERICAL ASSESSMENT**

As shown in Fig. 4, an elastic rectangular punch pressed on an elastic foundation with a hole or holes is considered to assess the performance of the proposed special purpose element model. The upper surface of the punch is subjected to a uniform pressure \( q = 1.2 \text{ MPa} \) in the vertical direction. The same material properties characterized by Young’s modulus of 4000MPa and Poisson’s ratio of 0.35 are assumed for both bodies. In our contact analysis, the plane stain state is assumed and the direct constraint-Trefftz FEM is used [18]. Both 4-node regular Trefftz plane element (C2D4T) [18] and RHOL16 or RHOL32 element (Fig. 5) are employed to discrete the bodies in contact. C2D4T is used to model regular element (elements without hole); while RHOL16 or RHOL32 is used to model hole element.

![Fig. 4 Configuration of the contact problem](image)

In the following effect of radius of a circular hole on the contact stress is analyzed. Here a range of radii \( r = 4, 6, 8, 10 \text{mm} \) is considered. The foundation with one individual circular hole, whose center locates at (0, -20mm), is partitioned into 264 C2D4T elements and one 20mm×20mm RHOL16 element (Fig. 6a). The same boundary value problem is analyzed by ABAQUS and corresponding mesh for four kinds of radii is shown in Fig. 6(b-e), in which 783, 769, 760 and 687 (ABAQUS Element CPE4H) elements are respectively used. It can be seen that use of special hole element leads to significant reduction in the number of elements as well as the total degrees of freedom. As shown in Fig. 7, the same contact pressure distribution for a range of radii of circular holes, caused by a uniform load, are obtained using the coarse HT FE mesh without sacrificing accuracy. This clearly demonstrates the efficiency of the proposed special purpose element model. Furthermore, as the radius becomes larger, the contact pressure shows a marked decrease within \( |X| < 13.8 \text{mm} \) while opposite trend occurs within \( 13.8 \text{mm} < |X| < 45 \text{mm} \). However, the contact pressure at \( |X| = 13.8 \text{mm}, 50 \text{mm} \) is hardly affected. For the case of \( r = 10 \text{mm} \), the contact pressure dropped approximately by 43.21% compared to the model without holes.

**CONCLUSIONS**

A set of special purpose hole elements is developed for analysing heat conduction and elastic problems with holes. In particular, the special purpose elliptic element is constructed by introducing a rotated mapping function and using different terms of special Trefftz functions. The element can treat the structure with an elliptical hole in arbitrary direction without any difficulty. Using the proposed elliptic hole element and the direct constraint technique[18], the contact pressure distributions along the interface between an elastic rectangular punch and an elastic foundation with holes are investigated. Compared with the conventional FEM (ABAQUS), HT FEM can accurately simulate the mechanical behavior around a hole without troublesome mesh refinement locally. This demonstrates the efficiency of the special purpose hole elements discussed in this paper. Also, comparisons of the results with ABAQUS models and HT FE analyses have generally shown good agreement, although inaccuracies in some numerical simulations have been observed.
The study presents a parametric investigation of the contact behavior of a punch and a foundation with holes. We note that the larger the radius of a circular hole embedded in the foundation and the closer the distance from the center of a hole to the contact interface, the more remarkable effect of holes on the contact pressure distribution.

Fig. 6 Element meshes

Fig. 7 Effect of radius on the contact pressure distribution
REFERENCES