Fracture Analysis of Piezoelectric Materials by Boundary and Trefftz Finite Element Methods

Qing-Hua Qin

Department of Engineering, FEIT, Australian National University, Acton ACT 0200, Australia
e-mail: qinghua.qin@anu.edu.au

Abstract This paper contains two main parts. First, applications of Trefftz finite element methods to mode III fracture problems are presented in this paper. Trial functions for electroelastic fields are presented for both regular element and cracked element. These functions are used in deriving Trefftz finite element formulation together with modified variational principle. The performance of the proposed element model is assessed by an example and comparison is made with those obtained by conventional finite element model or by other approaches. Second, boundary element formulation for analyzing fracture behaviour of thermopiezoelectric materials is presented. Based on the Green’s function of piezoelectric materials with various defects and total potential variational principle, a boundary element model for thermopiezoelectric solid with multiple cracks has been developed and can be used to calculate the stress intensity factors of the multiple crack problem.

Key words: Fracture, finite element method, boundary element method, Trefftz method

INTRODUCTION

Fracture analysis of piezoelectric solid is of considerable importance in promoting applications of piezoelectric materials in engineering structures, as piezoelectric materials often contain many internal microcracks before in use and have strong inference in their mechanical properties. Analytical analysis of piezoelectric solid with defects is, however, difficult due to its complex geometry and mathematical formulation. Thus efficient numerical techniques such as trefftz method or boundary element method is required to develop. In 1994, Lee and Jiang [1] derived a boundary integral equation of piezoelectric media by the method of weighted residuals and also obtained the fundamental solution for plane piezoelectricity by using the double Fourier transform technology. Ding et al. [2] developed a boundary integral formulation which is efficient for analyzing crack problems in piezoelectric material. Xu and Rajapakse [3] presented several versions of boundary element formulations for analyzing piezoelectric solids with various defects (cavities, inclusions, cracks, etc.). Qin [4] and Qin and Lu [5] proposed a boundary element formulation for fracture analysis of thermopiezoelectricity based on the dislocation method and the potential variational principle. More recently, Qin[6] presented a Trefftz finite element model for solving anti-plane fracture problem of piezoelectric materials. In this paper, applications of Trefftz finite element and boundary element methods to fracture problems are presented. At first, Mode III fracture problems are analyzed using Trefftz finite element method. And then fracture problems of thermopiezoelectric materials are studied using boundary element method. The boundary element formulation is derived based on the Green’s function of piezoelectric materials with various defects and total potential variational principle.

TREFFTZ METHOD FOR ANTI-PLANE PROBLEM

1. Basic equations In the case of anti-plane shear deformation involving only out-of-plane displacement $uz$ and in-plane electric fields, we have
where $\phi$ is electrical potential. The differential governing equation can be written as

$$c_{44} \nabla^2 u_z + e_{15} \nabla^2 \phi = 0, \quad e_{15} \nabla^2 u_z - \kappa_{11} \nabla^2 \phi = 0 \quad \text{in } \Omega$$

with the constitutive equations

$$
\begin{bmatrix}
\sigma_{zz} \\
\sigma_{zy} \\
D_x \\
D_y
\end{bmatrix} =
\begin{bmatrix}
c_{44} & 0 & -e_{15} & 0 \\
0 & c_{44} & 0 & -e_{15} \\
e_{15} & 0 & \kappa_{11} & 0 \\
0 & e_{15} & 0 & \kappa_{11}
\end{bmatrix}
\begin{bmatrix}
\gamma_{zz} \\
\gamma_{zy} \\
E_x \\
E_y
\end{bmatrix}
$$

where the symbols appeared in this section have been defined in [6] and we omit those details for saving space.

The boundary conditions of the boundary value problem (1)-(3) can be given by:

$$u_z = u \quad \text{on } \Gamma_u$$

$$n \cdot \sigma = \sigma \quad \text{on } \Gamma_t$$

$$\mathbf{D} n \cdot \mathbf{n} = -\mathbf{D} \quad \text{on } \Gamma_D$$

$$\phi = \phi \quad \text{on } \Gamma_D$$

2. Trefftz finite element formulation

For the boundary value problem described by eqns (1)-(7), the corresponding variational functional is constructed in the form

$$\Pi_m^D = \sum_e \Pi_m^{D,e} = \sum_e \{ \Pi_m^{D,e} \} - \int_{\Gamma_w} (\mathbf{D}_a - \mathbf{D}_n) \partial \hat{u}_z ds - \int_{\Gamma_w} (\mathbf{t} - \mathbf{t}) \hat{u}_z ds$$

$$+ \int_{\Gamma_w} (\mathbf{D}_n \partial + \mathbf{t} \hat{u}_z) ds$$

where

$$\Pi_m^{D,e} = \int_{\Omega_e} H(\sigma_{y}, D_k) d\Omega + \int_{\Gamma_w} \mathbf{t} \hat{u}_z ds + \int_{\Gamma_w} \mathbf{D}_n \partial ds$$

$$H(\sigma_{y}, D_k) = -\frac{1}{2} s_{44} (\sigma_{zz}^2 + \sigma_{zy}^2) - g_{15} \sigma_{zz} D_x - g_{15} \sigma_{zy} D_y + \frac{1}{2} \lambda_{11} (D_x^2 + D_y^2)$$

The non-conforming intra-element field \{u_z\, \phi\}T is expressed by

$$\mathbf{u} = \{ u_z \, \phi \} = \sum_{j=1}^{N} \begin{bmatrix} N_{1j} \\ 0 \end{bmatrix} \begin{bmatrix} c_{uj} \\ c_{0j} \end{bmatrix} = \begin{bmatrix} N_1 \\ 0 \end{bmatrix} \mathbf{c} = \mathbf{N} \mathbf{c}$$

where $\mathbf{c}$ is a vector of undetermined coefficient, $\mathbf{N}$, are taken from the component of the series:

$$u_z(r, \theta) = \sum_{m=0}^{\infty} r^m (a_m \cos m\theta + b_m \sin m\theta)$$

$$\phi(r, \theta) = \sum_{m=0}^{\infty} r^m (c_m \cos m\theta + d_m \sin m\theta)$$

for a regular element, and

$$u_z(r, \theta) = a_0 + \sum_{n=1}^{\infty} a_n r^n \cos(n\theta) + \sum_{n=1,3,5} b_n r^n \sin(n\theta)$$

$$\phi(r, \theta) = c_0 + \sum_{n=1}^{\infty} c_n r^n \cos(n\theta) + \sum_{n=1,3,5} d_n r^n \sin(n\theta)$$
for an element containing an edge crack, where $m$ is its number of components. The choice of $m$ has
been discussed in Section 2.6 of Ref. [7].

The auxiliary conforming field $\{\tilde{u}_z, \tilde{\phi}\}$ in eq. (8) is given by

$$\tilde{\mathbf{u}} = \{\tilde{u}_z, \tilde{\phi}\} = \begin{bmatrix} \tilde{N}_1 & 0 \\ 0 & \tilde{N}_2 \end{bmatrix} \begin{bmatrix} d_u \\ d_w \end{bmatrix} + \begin{bmatrix} \tilde{N}_{1c} & 0 \\ 0 & \tilde{N}_{2c} \end{bmatrix} \begin{bmatrix} d_{uc} \\ d_{wc} \end{bmatrix} = \tilde{N}d + \tilde{N}_c d_c,$$

where the definitions of $\tilde{N}_i$ and $\tilde{N}_{ic}$ can be found in [6].

**THERMOPIEZOELECTRIC MATERIALS WITH BOUNDARY ELEMENTS**

Consider a two-dimensional thermoelectroelastic solid inside of which there are a number of cracks. The
numerical approach to such a thermoelectroelastic problem will involve two steps: (1) solve a heat
transfer problem first to obtain the steady-state $T$ field; (2) calculate the stress and electric displacement
(SED) caused by the $T$ field, then derive an isothermal solution to satisfy the corresponding mechanical
and electric boundary conditions, and finally, solve the modified problem for the elastic
displacement-electric potential and SED fields. The details are as follows.

1. **Boundary element for temperature discontinuity problem** Consider a thermal finite region $\Omega_1$
with a number of cracks bounded by $\Gamma$. The heat transfer problem to be considered may be stated as

$$k \frac{\partial T}{\partial t} = 0, \quad \text{in} \quad \Omega_1,$$

$$h_n = h_n \frac{\partial T}{\partial n}, \quad \text{on} \quad \Gamma_h,$$

$$T = T_0, \quad \text{on} \quad \Gamma_T,$$

$$h_n n_\tau = 0, \quad \text{on} \quad L,$$

The total generalized potential energy for the thermal problem defined above can be given as

$$P(\tilde{T}) = -\frac{1}{2} \int_L \tilde{g}(\tilde{T}) \tilde{T}_s ds + \int_\Gamma h_n \tilde{T} ds.$$

where $\tilde{T}$ is the temperature discontinuity. As in the conventional boundary element method, the
boundaries $\Gamma$ and $L$ are divided into $M_\Gamma$ and $M_L$ linear boundary elements, for which the temperature
discontinuity may be approximated by the sum of elemental temperature discontinuities:

$$\tilde{T}(s) = \sum_{m=1}^{M} \tilde{T}_m F_m(s)$$

With the use of Green’s functions presented in [8], the temperature and heat-flow function at point $z_i$ (or
$\zeta_i(z_i)$) can be given as

$$T(\zeta_i) = \sum_{m=1}^{M} \Im[a_m(\zeta_i)] \tilde{T}_m$$

$$g(\zeta_i) = -k \sum_{m=1}^{M} \Re[a_m(\zeta_i)] \tilde{T}_m$$

where $a_m(\zeta_i)$ has different form for different problem (see [8] for details).

In particular the temperature at node $j$ can be written as

$$T_j[f(d_j)] = \sum_{m=1}^{M} \Im[a_m[f(d_j)]] \tilde{T}_m$$

where $d_j = x_{ij} + p_i x_{2j}$, $(x_{ij}, x_{2j})$ are the coordinates at node $j$.

For the total potential energy (21), through the substitution of eqns (23) and (24) into it, one obtains
\[ P(\hat{T}) \approx \sum_{j=1}^{M} \left[ \sum_{m=1}^{M} \left( -\frac{1}{2} K_{mj} \hat{T}_m \hat{T}_j \right) + G_j \hat{T}_j \right] \] (26)

where \( K_{mj} \) is the so-called stiffness matrix and \( G_j \) the equivalent nodal heat flow vector, with the form

\[ K_{mj} = -\frac{k}{l_{j-1}} \int_{l_{j-1}} \text{Re}[a_m(\zeta_{j-1}^{-\infty})] ds + \frac{k}{l_j} \int_{l_j} \text{Re}[a_m(\zeta_{j-0}^{+\infty})] ds, \] (27)

\[ G_j = \int_{l_j} h_{\text{v}}(s) F_j(s) ds, \] (28)

where \( h_{\text{v}} = h_0 \) when \( s \in \Gamma_h \), and \( h_{\text{v}} = 0 \) for other cases. The minimization of \( P(\hat{T}) \) yields

\[ \sum_{j=1}^{M} K_{mj} \hat{T}_j = G_m. \] (29)

The final form of linear equation to be solved is obtained by selecting the appropriate equation from eqns (25) and (29).

2. Boundary element for displacement and potential discontinuity problems

Consider again the region \( \Omega \) with a number of cracks bounded by \( \Gamma \), in which the governing equation and its boundary conditions are described as follows[8]:

\[ \Pi_{ij} = 0, \quad \text{in } \Omega, \] (30)

\[ t_{ni} = \Pi_{ij} n_j = t_j^0 - (t_n^0), \quad \text{on } \Gamma_e, \] (31)

\[ u_j = u_j^0 -(U^0), \quad \text{on } \Gamma_u, \] (32)

\[ t_m \big|_{L'} = -t_m \big|_{L'} = -(t_n^0), \quad b_j = u_j \big|_{L'} - u_j \big|_{L'} = (U^0) \big|_{L'} - (U^0) \big|_{L'}, \quad \text{on } L \] (33)

where \( \Gamma_e \) and \( \Gamma_u \) are the boundaries on which the prescribed values of stress \( t_j^0 \) and displacement \( u_j^0 \) are imposed. The related potential energy for the electroelastic problem can be given as

\[ P(\mathbf{b}) = \frac{1}{2} \int_{L'} [\mathbf{\varphi}(\mathbf{b}) \cdot \mathbf{b}, + 2t_n^0 \cdot \mathbf{b}] ds - \int_{\Gamma} (t_n^0 - t_n^0) \cdot \mathbf{b} ds \] (34)

where the electroelastic solutions of functions \( \varphi(\mathbf{b}) \) and \( \mathbf{U}(\mathbf{b}) \) appearing later have been defined in Chapter 4 of Ref.[8].

As treated before, the boundaries \( L \) and \( \Gamma \) are divided into a series of boundary elements, for which the EDEP discontinuity may be approximated through linear interpolation as

\[ \mathbf{b}(s) = \sum_{m=1}^{M} \mathbf{b}_m F_m (s). \] (35)

With the approximation (35) and the use of Green’s functions presented in [8], the elastic displacement-electric potential and SED functions can be expressed in the form

\[ \mathbf{U}(\zeta) = \sum_{m=1}^{M} \text{Im}[\mathbf{AD}_m (\zeta)] \mathbf{b}_m, \quad \varphi(\zeta) = \sum_{m=1}^{M} \text{Im}[\mathbf{BD}_m (\zeta)] \mathbf{b}_m \] (36)

where \( \mathbf{D}_m \) has different forms for different problems and can be find in [8].

In particular the displacement at node \( j \) is given by

\[ \mathbf{U}[(f(d_{aj}))] = \sum_{m=1}^{M} \text{Im}\{\mathbf{AD}_m \{[(f(d_{aj}))]\}) \mathbf{b}_m. \] (37)

Substituting eq. (36) into eq. (34), we have
\[ P(b) = \sum_{i=1}^{M} \left( b_i^T \left( \sum_{j=1}^{M} k_{ij} b_j \right) / 2 - g_i \right) \] (38)

where

\[ k_{ij} = \frac{1}{l_{j-1}} \int_{l_{j-1}} \text{Im}[D_i^T (\xi_{j-1})B^T] \text{d}s - \frac{1}{l_j} \int_{l_j} \text{Im}[D_i^T (\xi_{j+1})B^T] \text{d}s \] (39)

and \[ g_j = \int_{l_{j-1}}^{l_j} G_j F_j(s) \text{d}s \] (40)

and \[ G_j = -t_{n}^0 \] when node \( j \) is located at the boundary \( L \), \[ G_j = t_{n}^0 - t_{n}^0 \] for the other nodes. The minimization of eqn (38) leads to a set of linear equations:

\[ \sum_{j=1}^{M} k_{ij} b_j = g_i. \] (41)

the final form of the linear equations to be solved is obtained by selecting the appropriate equation, from eqns (37) and (41). Equation (37) will be chosen for those nodes at which the elastic displacement-electric potential is prescribed, and eqn (41) will be chosen for the remaining nodes.

\[ \sum_{j=1}^{M} k_{ij} b_j = g_i. \] (41)

NUMERICAL ILLUSTRATION

As a numerical illustration of the proposed formulation we consider an anti-plane crack of length \( 2c \) embedded in an infinite PZT-5H medium which is subjected to a uniform shear traction, \( \sigma_{xy} = \tau_{xy} \), and a uniform electric displacement, \( D_y = D_y \) at infinity (see Fig. 1). The material properties of PZT-5H are as given by [8]: \( c_{44} = 3.53 \times 10^{10} \text{ N/m}^2 \), \( e_{15} = 17.0 \text{ C/m}^2 \), \( \kappa_{11} = 1.51 \times 10^{-8} \text{ Cl/(Vm)} \), \( J_{cr} = 5.0 \text{ N/m} \), where \( J_{cr} \) is the critical energy release rate. Since the trial functions for the crack element satisfy the crack face condition and represent the singularity at crack tip, it is unnecessary to increase the mesh density near the crack tip. The energy release rate for PZT-5H material with a crack of length \( 2c=0.02 \text{ m} \) and \( acl=15 \) is plotted in Fig. 2 as a function of electrical load with the mechanical load fixed such that \( J=J_{cr} \) at zero electric load. The results are compared with those from Ref. [8]. It is found from Fig. 2 that the energy release rate can be negative which means the crack growth may be arrested.

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Fig. 1 Configuration of the cracked infinite piezoelectric medium
CONCLUSION

This paper discussed the fracture problem by Trefftz finite element method and boundary element method. In the Trefftz finite element method, the element containing an edge crack can be accurately modelled by using special Trefftz functions. While for the boundary element method, the Green’s function for is shown to be useful in establishing boundary element formulation for cracked thermopiezoelectric solid. It is interesting to note that the energy release rate can be negative which means the crack growth may be arrested.

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REFERENCES