Interaction between crack and hole in thermopiezoelectric materials by BEM

Qing-Hua Qin and Yiu-Wing Mai

Centre for Advanced Materials Technology, Department of Mechanical and Mechatronics Engineering, University of Sydney, Sydney, NSW 2006, Australia

Abstract

The boundary element formulation for analysing the interaction between crack and hole in piezoelectric materials is presented in this paper. Using the Green's function for holes and the potential variational principle, a boundary element model (BEM) for 2-D thermo-piezoelectric material solid with cracks and holes has been developed and used to calculate the stress intensity factors of the crack-hole problem. The method is available for multiple crack problems in both finite and infinite solids. Numerical results for SED intensity factors at a particular crack tip in a crack-hole system of piezoelectric materials are presented to illustrate the application of the proposed formulation.

1. Introduction

The thermoelectroelastic analysis of crack-hole system in a piezoelectric solid is of considerable importance in the field of fracture mechanics, as the piezoelectric materials often contains many internal microcracks before in use. Stress analysis of crack problems in isotropic materials has been reported by many researchers[1,2]. For anisotropic materials, a solution for collinear cracks in an infinite plate has been obtained in[2]. Most of the developments in the field can also be found in[3]. Relatively little work has been, however, found in the literature for the thermal analysis of multiple crack problems in piezoelectric solid. In this paper, a boundary element model is presented to study fracture behaviour of crack-hole system in a piezoelectric material. Based on the thermoelectroelastic Green's function for a piezoelectric solid with a hole, a system of boundary element formulation for temperature discontinuity as well as dislocation of elastic displacement and electric potential(EDEP) is presented and used to calculate stress and electric displacement(SED) intensity factors. One numerical example is considered to illustrate the application of the proposed formulation.
2. Basic formulations

Consider a linear piezoelectric solid in which all fields are assumed to depend only on in-plane coordinates \( x_1 \) and \( x_2 \). Boldfaced symbols stand for either column vectors or matrices, depending on whether lower case or upper case is used. The SED vector \( \Pi_1 \), The EDEP vector \( \mathbf{u} \), temperature \( T \) and heat flux \( h_1 \) in the solid subjected to loading can be expressed in terms of complex analytic functions as follows[2]

\[
T = 2 \text{Re}[g'(z_1)], \quad \Phi = 2 \text{Im}[kg'(z_1)], \quad h_1 = -\Phi_2, \quad h_2 = \Phi_1, \quad \mathbf{u} = 2 \text{Re}[\mathbf{A}(z)\mathbf{q} + c\mathbf{g}(z_1)],
\]

\[
\phi = 2 \text{Re}[\mathbf{B}(z)\mathbf{q} + d\mathbf{g}(z_1)], \quad \Pi_1 = -\phi_2, \quad \Pi_2 = \phi_1,
\]

with

\[
\mathbf{F}(z) = \text{diag}[f(z_1)f(z_2)f(z_3)f(z_4)], \quad z_i = x_1 + \alpha x_2, \quad z_i = x_1 + p_i x_2
\]

where overbar denotes the complex conjugate, a prime represents the differentiation, \( \mathbf{q} \) is a constant vector to be determined by the boundary conditions, \( \mathbf{u} = \{u_1 u_2 u_3 \}^T, \quad \Pi_1 = \{\sigma_1 \sigma_2 \sigma_3 \}^T, \quad j, k, l = 1, 2 \), \( \sigma_i, \tau_i \) are the coefficients of heat conduction, \( u_i \) and \( \phi \) are the elastic displacement and electric potential, \( T, \quad h_i \), \( \sigma_j \) and \( D_i \) are temperature, heat flux, stress and electric displacement, \( \Phi \) and \( \phi \) are known as heat-flux function and SED function[2], \( \tau \) and \( p_i \) are the heat and electroelastic eigenvalues of the materials whose imaginary parts are positive, \( f(z_i) \) and \( g(z_i) \) are arbitrary functions with complex arguments \( z_i \) and \( z_o \), respectively, \( \mathbf{A}, \mathbf{B}, \mathbf{c} \) and \( \mathbf{d} \) are well-defined in the literature[2].

Green's function used in this paper is described as follows. For a material plate with an elliptic hole, the basic solutions due to a discrete temperature discontinuity of magnitude \( \hat{T} \) at a point \( z_o = x_{o1} + \alpha x_{o2} \) (see Fig. 1), are given by[4]

\[
T = \frac{\hat{T}}{2\pi} \text{Im}[\ln(\zeta_i - \zeta_{o1}) - \ln(\zeta_i^{-1} - \zeta_{o0})], \quad \Phi = \frac{\hat{T}}{2\pi} \text{Re}[\ln(\zeta_i - \zeta_{o1}) - \ln(\zeta_i^{-1} - \zeta_{o0})]
\]

\[
\mathbf{u} = \frac{\hat{T}}{2\pi} \text{Im}\{\mathbf{A}[f_1(z) + f_2(z)P^{-1}\tau]\mathbf{B}^{-1}\mathbf{d} - c\mathbf{g}(z_1)\}
\]

\[
\phi = \frac{\hat{T}}{2\pi} \text{Im}\{\mathbf{B}[f_1(z) + f_2(z)P^{-1}\tau]\mathbf{B}^{-1}\mathbf{d} - d\mathbf{g}(z_1)\}
\]

where \( \zeta_i \) and \( \zeta_{o0} \) are related to \( z_i \) and \( z_{o0} \) by the relations[4]

\[
\zeta_i = \frac{z_i + \sqrt{z_i^2 - a^2 - \tau^2 b^2}}{a - itb}, \quad \zeta_{o0} = \frac{z_{o0} + \sqrt{z_{o0}^2 - a^2 - \tau^2 b^2}}{a - itb}
\]

in which \( a \) and \( b \) are the length of the semi-axes of the ellipse. The functions \( f_1(z), f_2(z) \) and \( g(z_1) \) are well-documented in [4] and we omit them here for the limitation of space.

3. BEM for thermopiezoelectric problem

Consider a two-dimensional thermopiezoelectric solid inside of which there are a hole and a crack with arbitrary orientation and size. The numerical approach to such a problem usually
involves two steps: (i) solve a heat transfer problem first to obtain the steady-state $T$ field; (ii) calculate the electroelastic field caused by the $T$ field, then plus an isothermal solution to satisfy the corresponding mechanical boundary conditions, and finally, solve the modified problem for electroelastic fields. In what follows, we begin with deriving the variation principle for temperature discontinuity and then extended it to the case of thermoelectroelasticity.

3.1 BEM for temperature discontinuity problem

Let us consider a finite region $\Omega_1$ bounded by $\Gamma = \Gamma_\text{h} + \Gamma_\text{T}$, as shown in Fig. 2(a). The heat transfer problem to be considered is stated as

$$
k_y T_{xy} = 0 \quad \text{in } \Omega_1$$

$$
h_n = h_n = h_0 \quad \text{on } \Gamma_\text{h}$$

$$
T = T_0 \quad \text{on } \Gamma_\text{T}$$

$$
h_n = 0, \quad \hat{T} = T|_L - T|_L - \text{ on } L$$

where $n_i$ is the normal to the boundary $\Gamma = \Gamma_\text{h} + \Gamma_\text{T}$, $h_0$ and $T_0$ are the prescribed values of heat flow and temperature, which act on the boundaries $\Gamma_\text{h}$ and $\Gamma_\text{T}$, respectively, $\hat{T}$ is the temperature discontinuity, $L$ is the union of all cracks, $L^+$ and $L^-$ are defined in Fig. 2(b). It should be pointed out that the boundary condition along the hole is automatically satisfied due to the use of the Green function given in Eq.(3)-(5). Naturally, the hole boundary condition is not involved in the below analysis.

Further, if we let $\Omega_2$ be the complementary region of $\Omega_1$ (i.e., the union of $\Omega_1$ and $\Omega_2$ forms the infinite region $\Omega$) and $\hat{T} = T|_{\Gamma_+} - T|_{\Gamma_-} = T_0$, the problem shown in Fig 2(a) can be extended to the infinite one (see Fig. 2b). In this case $\Gamma = \Gamma^+ + \Gamma^-$, where $\Gamma^+$ and $\Gamma^-$ stands for the boundaries of $\Omega_1$ and $\Omega_2$ respectively [see Fig. 2(b)]. In a way similar to that in [5], the total generalised potential energy for the thermal problem defined above can be given as

$$
P(T, \hat{T}) = \frac{1}{2} \int_{\Omega} k_y T_{xy} T_{xy} d\Omega + \int_{\Gamma} h_n \hat{T} dL$$

By transforming the area integral in Eq.(11) into boundary one, we have

$$
P(T, \hat{T}) = -\frac{1}{2} \int_{\Omega} k_y(T, \hat{T}) T_{xy} d\Omega + \int_{\Gamma} h_n \hat{T} ds$$

in which the relation

$$
h_n = -k_y T_{xy} \quad \text{and} \quad \int_{\Gamma} h_n \hat{T} ds = [((\hat{T})_+ - \hat{T})_-] ds$$

end.
and the temperature discontinuity being assumed to be continuous over $L$ and being zero at the ends of $L$ have been used. Moreover, temperature $T$ in Eq.(12) can be expressed in terms of $\hat{T}$ through use of Eq.(3). Therefore, the potential energy can be further written as

$$P(\hat{T}) = -\frac{1}{2} \int_{\Gamma} \Theta(\hat{T}) \hat{T}^2 \, ds + \int_{L} h_n \hat{T} \, ds$$  \hspace{1cm} (14)$$

The analytical results for the minimum of potential (14) is not, in general, possible, and therefore a numerical procedure must be used to solve the problem. As in the conventional BEM, the boundaries $\Gamma$ and $L$ are divided into a series of linear boundary elements for which the temperature discontinuity may be approximated by a linear function. To illustrate this, take a particular element $m$, which is a line connected by nodes $m$ and $m+1$, as an example (see Fig. 3)

$$\hat{T}(s) = \hat{T}_m F_m(s) + \hat{T}_{m+1} F_{m+1}(s)$$  \hspace{1cm} (15)$$

where $\hat{T}_m$ is the temperature discontinuity at node $m$, and functions $F_m(s)$, $F_{m+1}(s)$ are shown in Figure 3.

On the use of Eqs.(3) and (15), the temperature and heat-flux function at point $z_i$ are

$$T(z_i) = \sum_{m=1}^{M} \text{Im}[a_m(z_i)] \hat{T}_m$$  \hspace{1cm} (16)$$

$$\Theta(z_i) = -k \sum_{m=1}^{M} \text{Re}[a_m(z_i)] \hat{T}_m$$  \hspace{1cm} (17)$$

where

$$a_m(\zeta_j) = \frac{1}{2\pi} \left[ \ln(\zeta_j - \zeta_{m-1}) + \ln(\zeta_j - \zeta_{m-1}) \right] \frac{l_{m-1} - s}{l_{m-1}} ds$$  \hspace{1cm} (18)$$

In particular the temperature at node $j$ can be written as

$$T_j(\zeta_j) = \sum_{m=1}^{M} \text{Im}[a_m(\zeta_j)] \hat{T}_m$$  \hspace{1cm} (19)$$

The substitution of Eq.(15) into (14), one sees

$$P(\hat{T}) \approx \sum_{j=1}^{M} \left[ \sum_{m=1}^{M} \left( -\frac{k}{2} K_{mj} \hat{T}_m \hat{T}_j + G_j \right) \right]$$  \hspace{1cm} (20)$$

where $[K_{mj}]$ is called as stiffness matrix and $G_j$ the equivalent nodal heat flux vector, which have the form

$$K_{mj} = -\frac{k}{l_{j-1}} \int_{j_{j-1}} \text{Re}[a_m(\zeta_{j-1})] ds + \frac{k}{l_{j}} \int_{j_{j-1}} \text{Re}[a_m(\zeta_{j-1})] ds , \quad G_j = \int_{j_{j-1}} h_n F_j(s) ds$$  \hspace{1cm} (21)$$

The minimization of $P(\hat{T})$ yields

$$\sum_{j=1}^{M} K_{mj} \hat{T}_j = G_m$$  \hspace{1cm} (22)$$
The final form of linear equations to be solved is obtained by selecting the appropriate ones, from among Eqs. (19) and (22). Eq. (19) will be chosen for those nodes at which the temperature is prescribed, and Eq. (23) will be chosen for the remaining nodes. After the nodal temperature discontinuities have been calculated, the EDEP and SED at any point in the region can be evaluated through use of Eqs. (1), (4) and (5). They are

$$u = \sum_{j=1}^{M} w_j \hat{T}_j, \quad \Pi_1 = \sum_{j=1}^{M} x_j \hat{T}_j, \quad \Pi_2 = \sum_{j=1}^{M} y_j \hat{T}_j \quad (23)$$

where

$$w_j = -\frac{1}{2\pi} \text{Im}\left\{ \int_{j-1}^{j} \left[ A(f(z) + f(z)P^{-1}\tau)B^{-1}\delta - \text{cg}(z) \right] \frac{l_{j-1} - s}{l_j} ds \right\}$$

$$-\frac{1}{2\pi} \text{Im}\left\{ \int_{j-1}^{j} \left[ A(f(z) + f(z)P^{-1}\tau)B^{-1}\delta - \text{cg}(z) \right] \frac{s}{l_j} ds \right\} \quad (24)$$

$$x_j = \frac{1}{2\pi} \text{Im}\left\{ \int_{j-1}^{j} \left[ B(f(z)P + f(z)P^{-1}\tau)B^{-1}\delta - \text{cg}'(z) \right] \frac{l_{j-1} - s}{l_j} ds \right\}$$

$$+ \frac{1}{2\pi} \text{Im}\left\{ \int_{j-1}^{j} \left[ B(f(z)P + f(z)P^{-1}\tau)B^{-1}\delta - \text{cg}'(z) \right] \frac{s}{l_j} ds \right\} \quad (25)$$

$$y_j = -\frac{1}{2\pi} \text{Im}\left\{ \int_{j-1}^{j} \left[ B(f(z)P + f(z)P^{-1}\tau)B^{-1}\delta - \text{cg}'(z) \right] \frac{l_{j-1} - s}{l_j} ds \right\}$$

$$-\frac{1}{2\pi} \text{Im}\left\{ \int_{j-1}^{j} \left[ B(f(z)P + f(z)P^{-1}\tau)B^{-1}\delta - \text{cg}'(z) \right] \frac{s}{l_j} ds \right\} \quad (26)$$

Thus, the surface traction-charge and EDEP induced by the temperature discontinuity are of the form

$$t_n^0(s) = \Pi_i n_i = \sum_{j=1}^{M} (x_j, n_j + y_j n_j) \hat{T}_j, \quad u^0(s) = \sum_{j=1}^{M} w_j(s) \hat{T}_j \quad (27)$$

In general, $t_n^0(s) \neq 0$ over $\Gamma_i$ (the boundary on which SED is prescribed) and $u^0(s) \neq 0$ over $\Gamma_u$ (the boundary on which EDEP is prescribed). To satisfy the SED (or EDEP) on the corresponding boundaries, we must superpose a solution of the corresponding isothermal problem with a SED (or a EDEP) equal and opposite to those of Eq. (27). The details will be given in the following subsection.

### 3.2 BEM for EDEP discontinuity problem

Consider again the domain $\Omega_i$, the governing equation and its boundary conditions are described as follows

$$\Pi_{y,j} = 0 \quad \text{in } \Omega_i \quad (28)$$

$$t_n = \Pi_y n_j = t_n^0 - \hat{t}_n \quad \text{on } \Gamma_i \quad (29)$$

$$u_i = u_i^0 - (u_i^0) \quad \text{on } \Gamma_u \quad (30)$$

$$t_n \big|_L = -t_n \big|_L = -t_n^0 \quad \text{on } L \quad (31)$$

where

$$\Pi(\hat{u})$$

and

$$I_1 = \frac{1}{2\pi} \text{Im}\left\{ \int_{j-1}^{j} \left[ A(f(z) + f(z)P^{-1}\tau)B^{-1}\delta - \text{cg}(z) \right] \frac{l_{j-1} - s}{l_j} ds \right\}$$

With

$$\phi(\hat{u})$$

and

$$\Pi(\hat{u})$$

and

$$I_1 = \frac{1}{2\pi} \text{Im}\left\{ \int_{j-1}^{j} \left[ A(f(z) + f(z)P^{-1}\tau)B^{-1}\delta - \text{cg}(z) \right] \frac{s}{l_j} ds \right\}$$

Thus, the surface traction-charge and EDEP induced by the temperature discontinuity are of the form

$$t_n^0(s) = \Pi_i n_i = \sum_{j=1}^{M} (x_j, n_j + y_j n_j) \hat{T}_j, \quad u^0(s) = \sum_{j=1}^{M} w_j(s) \hat{T}_j \quad (27)$$

In general, $t_n^0(s) \neq 0$ over $\Gamma_i$ (the boundary on which SED is prescribed) and $u^0(s) \neq 0$ over $\Gamma_u$ (the boundary on which EDEP is prescribed). To satisfy the SED (or EDEP) on the corresponding boundaries, we must superpose a solution of the corresponding isothermal problem with a SED (or a EDEP) equal and opposite to those of Eq. (27). The details will be given in the following subsection.

### 3.2 BEM for EDEP discontinuity problem

Consider again the domain $\Omega_i$, the governing equation and its boundary conditions are described as follows

$$\Pi_{y,j} = 0 \quad \text{in } \Omega_i \quad (28)$$

$$t_n = \Pi_y n_j = t_n^0 - (t_n^0) \quad \text{on } \Gamma_i \quad (29)$$

$$u_i = u_i^0 - (u_i^0) \quad \text{on } \Gamma_u \quad (30)$$

$$t_n \big|_L = -t_n \big|_L = -(t_n^0) \quad \text{on } L \quad (31)$$

where

$$\Pi(\hat{u})$$

and

$$I_1 = \frac{1}{2\pi} \text{Im}\left\{ \int_{j-1}^{j} \left[ A(f(z) + f(z)P^{-1}\tau)B^{-1}\delta - \text{cg}(z) \right] \frac{l_{j-1} - s}{l_j} ds \right\}$$

With

$$\phi(\hat{u})$$

and

$$I_1 = \frac{1}{2\pi} \text{Im}\left\{ \int_{j-1}^{j} \left[ A(f(z) + f(z)P^{-1}\tau)B^{-1}\delta - \text{cg}(z) \right] \frac{s}{l_j} ds \right\}$$

Thus, the surface traction-charge and EDEP induced by the temperature discontinuity are of the form

$$t_n^0(s) = \Pi_i n_i = \sum_{j=1}^{M} (x_j, n_j + y_j n_j) \hat{T}_j, \quad u^0(s) = \sum_{j=1}^{M} w_j(s) \hat{T}_j \quad (27)$$

In general, $t_n^0(s) \neq 0$ over $\Gamma_i$ (the boundary on which SED is prescribed) and $u^0(s) \neq 0$ over $\Gamma_u$ (the boundary on which EDEP is prescribed). To satisfy the SED (or EDEP) on the corresponding boundaries, we must superpose a solution of the corresponding isothermal problem with a SED (or a EDEP) equal and opposite to those of Eq. (27). The details will be given in the following subsection.
where $\Gamma_r$ and $\Gamma_u$ are the boundaries on which the prescribed values of SED $\tau^0_r$ and EDEP $U^0_u$ are imposed. Similarly, the total potential energy for the electroelastic problem can be given as

$$
\Pi(\tilde{u}) = \frac{1}{2} \int \left[ \phi(\tilde{u}) \cdot \tilde{u} - 2 \tau^0_r \cdot \tilde{u} + 2 \tau^0_u \cdot \tilde{u} \right] ds - \int \left( \tau^0 - \tau^0_u \right) \cdot \tilde{u} ds
$$

where the elastic solution of functions $\phi(\tilde{u})$ and $u(\tilde{u})$ appeared later has been given in [4]. These solutions are as follows

$$u(\tilde{u}) = \frac{1}{\pi} \int \{\ln(\zeta - \zeta_{0u})\} B^T \tilde{u} + \sum_{\beta=1}^{4} \text{Im}[A(\ln(\zeta_{\beta} - \zeta_{0})B^{-1}B\tilde{u})]$$

$$\phi(\tilde{u}) = \frac{1}{\pi} \int \{\ln(\zeta - \zeta_{0u})\} B^T \tilde{u} + \sum_{\beta=1}^{4} \text{Im}[B(\ln(\zeta_{\beta} - \zeta_{0})B^{-1}B\tilde{u})]$$

and

$$I_1 = \text{diag}[1,0,0,0], \quad I_2 = \text{diag}[0,1,0,0], \quad I_3 = \text{diag}[0,0,1,0], \quad I_4 = \text{diag}[0,0,0,1]$$

As treated before the boundaries $L$ and $\Gamma$ are divided into a series of boundary elements, for which the EDEP discontinuity may be approximated through linear interpolation as

$$\tilde{u}(s) = \tilde{u}_m F_m(s) + \tilde{u}_{m+1} F_{m+1}(s)$$

With the approximation (36), the EDEP and SED function given in Eqs.(33) and (34) can now be expressed in the form

$$u(z) = \sum_{m=1}^{M} \text{Im}[AD_m(z)]\tilde{u}_m, \quad \phi(z) = \sum_{m=1}^{M} \text{Im}[BD_m(z)]\tilde{u}_m$$

where

$$D_m(\zeta) = \frac{1}{\pi} \int_{\zeta_{m-1}}^{\zeta_{m}} \{(\ln(\zeta - \zeta_{0u})\} B^T + \sum_{\beta=1}^{4} \{\ln(\zeta_{\beta} - \zeta_{0})B^{-1}B\} I_{m-1} - \frac{s}{I_{m-1}} ds$$

In particular the displacement at node $j$ is given by

$$u(\zeta_j) = \sum_{m=1}^{M} \text{Im}[AD_m(\zeta_j)]\tilde{u}_m$$

The substitution of Eq.(36) into Eq.(32), we have

$$\Pi(\tilde{u}) = \sum_{m=1}^{M} \left[ \tilde{u}_m^T \cdot (\sum_{i=1}^{M} k_{ij} \tilde{u}_j) / 2 - g_j \right]$$

where

$$k_{ij} = \frac{1}{l_{j-i}} \int_{l_{j-i}} \text{Im}[D^T(\zeta_{j-i})B^T] ds - \frac{1}{l_{j-i}} \int_{j-i} \text{Im}[D^T(\zeta_j)B^T] ds$$

and $G_j = -t^0_j$ when node $j$ located at the boundary $L$, $G_j = t^0 - t^0_j$ for other nodes. The minimization of Eq.(40) leads to a set of linear equations

$$\sum_{j=1}^{M} K_{ij} \tilde{u}_j = g_i$$
Similarly, the final form of the linear equations to be solved is obtained by selecting the appropriate ones, from among Eqs.(39) and (42). Eq.(39) will be chosen for those nodes at which the EDEP is prescribed, and Eq.(42) will be chosen for the remaining nodes. Once the EDEP discontinuity \( \hat{u} \) has been found, the SED at any point can be expressed as

\[
\Pi_1 = -\sum_{m=1}^{M} \text{Im}[\text{BD}_m'(z)]\hat{u}_m, \quad \Pi_2 = \sum_{m=1}^{M} \text{Im}[\text{BD}'_m(z)]\hat{u}_m
\]  

(43)

Therefore the SED, \( \Pi_n \), in a coordinate system local to the crack line can be expressed in the form

\[
\Pi_n = \Phi(\alpha)\{-\Pi_1 \sin \alpha + \Pi_2 \cos \alpha\}^T
\]

(44)

where \( \Phi(\alpha) \) has been defined in[4].

Using Eq.(44) we can evaluate the SED intensity factors by the following definition

\[
K(c) = (K_{II}, K_{III}, K_D)^T = \lim_{r \to 0} \sqrt{2\pi r} \Pi_n (r)
\]

(45)

In practical calculation, one can evaluate the SED intensity factors in several ways such as extrapolation formulae, traction formulae and J-integral formulae [6]. In our analysis, the first method is used to calculate the SED intensity factors in BEM. In the method, \( \Pi_n \) at points A and B ahead of a crack-tip (see Fig. 4) is first derived and then the substitution of them into Eq.(45), we have

\[
K^A = \sigma^A \sqrt{2\pi r_A}, \quad K^B = \sigma^B \sqrt{2\pi r_B}
\]

(46)

where \( r_A \) (or \( r_B \)) are the distance from crack-tip to point A (or B). Finally, the SED intensity factors \( K \) can be obtained by the linear extrapolation of \( K^A \) and \( K^B \) to the crack tip, that is

\[
K = K^A \left(1 - \frac{r_B}{r_A}\right) - K^B
\]

(47)

4. Numerical results

Since the main purpose of this paper is to outline the basic principles of the proposed method, the numerical assessment is limited to an infinite thermapiezolectric material containing an elliptic hole and a crack as shown in Fig. 5, in which \( x_{10} = 0, \ x_{20} = 2b \) and \( c=0.5b \). The uniform heat flow \( h_0 \) is applied on each crack face only. The material is assumed to be BaTiO\(_3\), whose constants can be found in[4].

In our analysis, the plane strain deformation is assumed and the crack lines are assumed to be in the \( x_1-x_2 \) plane, i.e., \( D_3 = u_3 = 0 \). Therefore the stress intensity factor vector \( K^* \) now has only three components \( (K_I, K_{III}, K_D) \). In all calculations, \( r_A = l/7 \) and \( r_B = l/5 \) have been used, where \( l \) is the length of the related element. In the calculations, four meshes \( (N=10, 20, 40 \text{ and } 80 \text{ elements}) \) for the crack have been used to see the convergence of the BEM results, in which \( N \) represents the element number of the crack. In Fig. 6 the coefficients of SED intensity factors \( \beta_i \), at point B (see Fig. 5) are presented as a function of crack orientation angle \( \alpha \) for \( N=80 \), where \( \beta_i \) are defined by[4]

\[
K_I(A) = h_0 c y_1 \beta_1(\alpha) \sqrt{\pi c} / k, \quad K_{III}(A) = h_0 c y_1 \beta_2(\alpha) \sqrt{\pi c} / k, \quad K_D(A) = h_0 c y_3 \beta_3(\alpha) \sqrt{\pi c} / k
\]

(48)
Table 1 shows the results of SED intensity factors at point B versus mesh refinement for $\alpha=0$. It is found from Fig. 6 that the SED intensity factors are sensitive to the crack orientation in this example, respectively. It is also found from Table 1 that the BEM results can converge to a particular value along with the mesh refinement.

Table 1: SED intensity factors versus mesh refinement

<table>
<thead>
<tr>
<th>Mesh</th>
<th>$\beta_1$</th>
<th>$\beta_2$</th>
<th>$\beta_D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N=10$</td>
<td>0.6969</td>
<td>0.4697</td>
<td>0.3188</td>
</tr>
<tr>
<td>20</td>
<td>0.6990</td>
<td>0.4722</td>
<td>0.3205</td>
</tr>
<tr>
<td>40</td>
<td>0.7011</td>
<td>0.4743</td>
<td>0.3223</td>
</tr>
<tr>
<td>80</td>
<td>0.7021</td>
<td>0.4752</td>
<td>0.3231</td>
</tr>
</tbody>
</table>

5. Conclusion

This investigation presented a boundary element formulation for crack-hole problem of a thermopiezoelectric material plate. A system of boundary element equations is developed with the aid of Green's function approach and the potential variational principle. Solutions for the thermal, electric and elastic fields are then obtained for the crack-hole system in an piezoelectric material plate subjected to external heat flux disturbances. For an infinite material plate with a hole and a crack, the numerical results show that the crack angle $\alpha$ has a strong effect on the SED intensity factors.

Acknowledgments

The work was supported by the Australian Research Council. QHQ was supported through a Queen Elizabeth II fellowship.

References


Fig. 5. Configuration of the crack-hole system

Fig. 6 SED intensity factors vs crack angle