Chapter 2  Solution Methods

In this chapter, the solution methods commonly used in analyzing the mechanical behavior of piezoelectric material are reviewed. The chapter begins with a summary of the potential function method in piezoelectricity, followed by a discussion of other methods including Lekhnitskii formalism, techniques of Fourier transformation, the Trefftz finite element method, the Fredholm integral equation and Abel equation, the shear-lag model, the symplectic method, and the state space approach.

2.1  Potential function method

Potential function formulation is well known for solving the system of equations in both the classical theory of elasticity and piezoelectricity. In this section, the potential function method for boundary value problems of three-dimensional (3D) piezoelectricity is briefly summarized [1,2]. For a 3D piezoelectric problem of hexagonal solids of class 6\(mm\), the four unknowns \(u_1, u_2, u_3, \phi\) are to be expressed in terms of four potential functions \(\zeta(x_1, x_2, x_3), \chi(x_1, x_2, x_3), \omega(x_1, x_2, x_3)\) and \(\Theta(x_1, x_2, x_3)\) in such a way that [1]

\[
\begin{align*}
    u_1 &= \zeta_1 + \chi_2, \\
    u_2 &= \zeta_2 - \chi_1, \\
    u_3 &= k \zeta_3 + \omega_3, \\
    \phi &= \Theta_3
\end{align*}
\]

where \(k\) is an unknown coefficient. Then, consider the problem of the piezoelectricity of a hexagonal body of class 6\(mm\) subjected to electroelastic loadings. The constitutive equations for the electroelastic field are expressed as

\[
\sigma = c_\varepsilon \varepsilon - e^T E, \quad D = e \varepsilon + \kappa E
\]

where the superscript represents the transpose of a matrix, and

\[
\begin{align*}
    \sigma &= \begin{bmatrix} 
        \sigma_{11} & \sigma_{12} & \sigma_{13} & 0 & 0 & 0 \\
        \sigma_{22} & \sigma_{23} & \sigma_{33} & 0 & 0 & 0 \\
        \sigma_{33} & \sigma_{23} & \sigma_{33} & 0 & 0 & 0 \\
        \sigma_{23} & \sigma_{33} & \sigma_{23} & 0 & 0 & 0 \\
        \sigma_{31} & \sigma_{31} & \sigma_{33} & 0 & 0 & 0 \\
        \sigma_{12} & \sigma_{12} & \sigma_{12} & 0 & 0 & 0 \\
    \end{bmatrix}, \\
    \varepsilon &= \begin{bmatrix} 
        \varepsilon_{11} \\
        \varepsilon_{22} \\
        \varepsilon_{33} \\
        \varepsilon_{23} \\
        \varepsilon_{31} \\
        \varepsilon_{12} \\
    \end{bmatrix}, \\
    D &= \begin{bmatrix} 
        d_{11} \\
        d_{22} \\
        d_{33} \\
        d_{23} \\
        d_{31} \\
        d_{12} \\
    \end{bmatrix}, \\
    E &= \begin{bmatrix} 
        e_{11} \\
        e_{22} \\
        e_{33} \\
        e_{12} \\
        e_{31} \\
        e_{13} \\
    \end{bmatrix}, \\
    \kappa &= \begin{bmatrix} 
        \kappa_{11} & 0 & 0 \\
        0 & \kappa_{11} & 0 \\
        0 & 0 & \kappa_{33} \\
    \end{bmatrix}, \\
    \varepsilon &= \begin{bmatrix} 
        \varepsilon_{11} \\
        \varepsilon_{22} \\
        \varepsilon_{33} \\
        \varepsilon_{23} \\
        \varepsilon_{31} \\
        \varepsilon_{12} \\
    \end{bmatrix}, \\
    \sigma &= \begin{bmatrix} 
        \sigma_{11} \\
        \sigma_{22} \\
        \sigma_{33} \\
        \sigma_{23} \\
        \sigma_{31} \\
        \sigma_{12} \\
    \end{bmatrix}, \\
    \kappa &= \begin{bmatrix} 
        \kappa_{11} & 0 & 0 \\
        0 & \kappa_{11} & 0 \\
        0 & 0 & \kappa_{33} \\
    \end{bmatrix}
\end{align*}
\]
Making use of Eqs. (1.2) and (2.2) and then substituting Eq. (2.1) into the governing differential equations (1.10), in which all body forces and free charges are assumed to be zero, the following four equations result:

\[\begin{bmatrix}
    c_{11}V + \alpha_1 \frac{\partial^2}{\partial z^2} \\
    (c_{13} + \alpha_2)V + kc_{33} \frac{\partial^2}{\partial z^2} \\
    (e_{31} + \alpha_3)V + ke_{33} \frac{\partial^2}{\partial z^2}
\end{bmatrix}
= \begin{bmatrix}
    \zeta \\
    \sigma \Theta \\
    \Theta
\end{bmatrix} = 0 \quad (2.3)

(\Delta \chi + \xi \frac{\partial^2 \chi}{\partial z^2} = 0) \quad (2.4)

where

\[ V = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad \alpha_1 = k c_{13} + (1+k)c_{44}, \quad \alpha_2 = (1+k)c_{44}, \]

\[ \alpha_3 = (1+k)e_{13}, \quad \xi = \frac{2c_{44}}{c_{11} - c_{12}} \]

This reduces to the formulation in \cite{2} when \(k = 0\). In the following, we review briefly the results presented in \cite{2}. To obtain the solution to Eq. (2.3), Wang and Agrawal \cite{2} assumed that the solution of \(\zeta, \sigma, \Theta\) had the following form:

\[\begin{bmatrix}
    \zeta \\
    \sigma \\
    \Theta
\end{bmatrix} = \int_0^\infty \int_0^\infty \begin{bmatrix}
    A \\
    B \\
    C
\end{bmatrix} \cos(\alpha x) \cos(\beta y) e^{mz} \, dx \, dy \quad (2.5)\]

Substituting Eq. (2.5) into Eq. (2.3), we obtain

\[\begin{bmatrix}
    -c_{11}r^2 + \alpha m^2 \\
    -(c_{13} + \alpha_2)r^2 + kc_{33}m^2 \\
    -(e_{31} + \alpha_3)r^2 + ke_{33}m^2
\end{bmatrix}
= \begin{bmatrix}
    (c_{13} + c_{44})m^2 \\
    -(c_{44}r^2 + c_{33}m^2) \\
    -(e_{55}r^2 + e_{33}m^2)
\end{bmatrix} \begin{bmatrix}
    A \\
    B \\
    C
\end{bmatrix} = 0 \quad (2.6)

where \(r^2 = \alpha^2 + \beta^2\). For simplicity, define

\[ m^2 = \frac{\gamma^2}{\mu} \quad (2.7)\]
Substituting Eq. (2.7) into Eq. (2.6) and setting the determinant of the matrix to zero, we obtain

\[ R_3 \mu^3 + R_2 \mu^2 + R_1 \mu + R_0 = 0 \]  \hspace{1cm} (2.8)

where

\[ \begin{align*}
R_0 & = -c_{44} (e_{13}^2 + c_{33} k_{33}), \\
R_1 & = -2e_{14} e_{33} c_{13} + c_{33} c_{11} k_{33} - 2e_{33} e_{31} (c_{13} + c_{44}) \\
& \quad + c_3 e_{12}^2 + c_{11} e_{32}^2 - 2c_{13} c_{44} k_{33} + 2c_{33} e_{31} e_{31} \\
& \quad + c_{33} e_{44} k_{11} - c_{13} k_{33} + e_{33}, \\
R_2 & = -2e_{14} e_{33} c_{13} + c_{33} c_{11} k_{11} + 2c_{11} e_{11}^2 - c_{33} c_{11} k_{11} \\
& \quad + 2e_{31} e_{31} c_{13} - c_{44} e_{12}^2 - c_{11} c_{44} k_{33} + 2c_{33} c_{44} k_{11}, \\
R_3 & = c_{13} c_{44} k_{11} + c_{11} e_{33}^2
\end{align*} \]  \hspace{1cm} (2.9)

The three roots of Eq. (2.8) are denoted by \( \mu_j \) \((j=1, 2, 3)\). Corresponding to the three roots, the roots of Eq. (2.6) can be written as

\[ m = \pm \frac{\gamma}{\sqrt{\mu_1}}, \quad \pm \frac{\gamma}{\sqrt{\mu_2}}, \quad \pm \frac{\gamma}{\sqrt{\mu_3}} \]  \hspace{1cm} (2.10)

It is obvious that the solution to Eq. (2.6) is not unique. To solve this equation, Wang and Agrawal took \( A=1 \) and solved the resulting equation. After a series of mathematical operations the solution of Eq. (2.5) is obtained as

\[ \zeta = \int_0^{\infty} \int_0^{\infty} \cos(\alpha x) \cos(\beta y) \sum_{i=1}^{3} \left[ G_i \cosh \left( \frac{\gamma z}{\sqrt{\mu_i}} \right) \right] d\alpha d\beta \]  \hspace{1cm} (2.11)

\[ \sigma = \int_0^{\infty} \int_0^{\infty} \cos(\alpha x) \cos(\beta y) \sum_{i=1}^{3} h_i \left[ G_i \cosh \left( \frac{\gamma z}{\sqrt{\mu_i}} \right) \right] d\alpha d\beta \]  \hspace{1cm} (2.12)

\[ \Theta = \int_0^{\infty} \int_0^{\infty} \cos(\alpha x) \cos(\beta y) \sum_{i=1}^{3} c_i \left[ G_i \cosh \left( \frac{\gamma z}{\sqrt{\mu_i}} \right) \right] d\alpha d\beta \]  \hspace{1cm} (2.13)

where \((b_1,c_1), (b_2,c_2), \) and \((b_3,c_3)\) are the solutions of \((B,C)\) of Eq. (2.6) corresponding to \( m = \pm \gamma / \sqrt{\mu_1}, \pm \gamma / \sqrt{\mu_2} \) and \( \pm \gamma / \sqrt{\mu_3} \) respectively, and \( G_i \) and \( H_i \) are arbitrary constants which are determined using the boundary conditions.

### 2.2 Solution with Lekhnitskii formalism

The mathematical method known as the Lekhnitskii formalism was developed...
originally to solve two-dimensional problems in elastic anisotropic materials [3]. The evolution of the method and a number of extensions to electroelastic problems were described in Refs. [4–7]. In this section the Lekhnitskii formalism of generalized plane piezoelectricity presented in [7] is briefly summarized. For a complete derivation and discussion, the reader is referred to Refs. [3–6].

Consider a generalized plane problem of piezoelectric materials, in which all physical quantities, such as stresses, strains, displacements, electric fields, electric displacements and the electric potential, are functions of \( x \) and \( y \) only. The generalized plane strain constitutive equations are governed by Eq. (2.2) or equations located in the second column and fourth row of Table 1.1 as follows:

\[
\begin{align*}
\sigma_{11} & = f_{11} \varepsilon_{11} + f_{12} \varepsilon_{22} + f_{14} \varepsilon_{14} + f_{15} \varepsilon_{23} + f_{16} \varepsilon_{16} + g_{11} \varepsilon_{11} + g_{21} \varepsilon_{21} \\
\sigma_{22} & = f_{22} \varepsilon_{22} + f_{24} \varepsilon_{14} + f_{25} \varepsilon_{23} + f_{26} \varepsilon_{16} + g_{12} \varepsilon_{12} + g_{22} \varepsilon_{22} \\
\sigma_{23} & = f_{14} \varepsilon_{14} + f_{44} \varepsilon_{44} + f_{45} \varepsilon_{23} + f_{46} \varepsilon_{16} + g_{14} \varepsilon_{14} + g_{24} \varepsilon_{24} \\
\sigma_{12} & = f_{12} \varepsilon_{22} + f_{25} \varepsilon_{25} + f_{55} \varepsilon_{55} + f_{56} \varepsilon_{56} + g_{16} \varepsilon_{16} + g_{26} \varepsilon_{26} \\
\sigma_{13} & = f_{15} \varepsilon_{23} + f_{55} \varepsilon_{23} + f_{56} \varepsilon_{23} + f_{56} \varepsilon_{23} + g_{15} \varepsilon_{15} + g_{25} \varepsilon_{25} \\
\sigma_{11} & = g_{31} \varepsilon_{31} + g_{22} \varepsilon_{22} + g_{44} \varepsilon_{44} + g_{45} \varepsilon_{23} + g_{46} \varepsilon_{16} + \beta_{11} \varepsilon_{11} + \beta_{12} \varepsilon_{12} \\
\sigma_{22} & = g_{32} \varepsilon_{32} + g_{22} \varepsilon_{22} + g_{44} \varepsilon_{44} + g_{45} \varepsilon_{23} + g_{46} \varepsilon_{16} + \beta_{12} \varepsilon_{12} + \beta_{22} \varepsilon_{22}
\end{align*}
\]

(2.14)

where the materials \( f_{ij}, g_{ij} \), and \( \beta_{ij} \) are defined in Eq. (1.8) and Table 1.1. The derivation of these constants can be found in Ref. [7].

Eq. (2.14) constitutes a system of seven equations in 14 unknowns. Additional equations are provided by elastic equilibrium and Gauss’ law:

\[
\begin{align*}
\sigma_{11,1} + \sigma_{12,2} = 0, & \quad \sigma_{12,1} + \sigma_{22,2} = 0, & \quad \sigma_{13,1} + \sigma_{23,2} = 0, & \quad D_{1,1} + D_{2,2} = 0
\end{align*}
\]

(2.15)

in which the absence of body forces and free electric volume charge has been assumed, and by two elastic conditions and one electric compatibility condition

\[
\begin{align*}
\varepsilon_{11,22} + \varepsilon_{22,11} - 2\varepsilon_{12,12} = 0, & \quad \varepsilon_{13,2} - \varepsilon_{23,1} = 0, & \quad E_{1,2} - E_{2,1} = 0
\end{align*}
\]

(2.16)

Having formulated the generalized plane problem, we seek a solution to Eqs. (2.14)–(2.16) subjected to a given loading and boundary condition. To this end, the well-known Lekhnitskii stress functions \( F \), \( \Psi \) and induction function \( V \) satisfying the foregoing equilibrium equations are introduced as follows [7]:

\[
\begin{align*}
\sigma_{11} & = F_{22}, & \quad \sigma_{22} = F_{11}, & \quad \sigma_{12} = -F_{12}, & \quad \sigma_{13} = \Psi_{22}, & \quad \sigma_{23} = -\Psi_{11}, & \quad D_{1} = V_{2}, & \quad D_{2} = -V_{1}
\end{align*}
\]

(2.17)

Inserting Eq. (2.17) into Eq. (2.14), and later into Eq. (2.16) leads to

\[
\begin{align*}
\begin{bmatrix} L_4 & L_3 & L_3^* \\ L_3 & L_2 & L_2^* \\ L_3^* & L_2^* & L_2^* \\
\end{bmatrix}
\begin{bmatrix} F \\ \Psi \\ V \end{bmatrix} = 0
\end{align*}
\]

(2.18)
where

\[ L_4 = f_{22} \frac{\partial^4}{\partial x_1^4} - 2f_{36} \frac{\partial^4}{\partial x_1^3 \partial x_2} + (2f_{12} + f_{66}) \frac{\partial^4}{\partial x_1^2 \partial x_2^2} - 2f_{16} \frac{\partial^4}{\partial x_1 \partial x_2^3} + f_{11} \frac{\partial^4}{\partial x_2^4}, \]

\[ L_3 = -f_{24} \frac{\partial^3}{\partial x_1^3} + (f_{25} + f_{46}) \frac{\partial^3}{\partial x_1^2 \partial x_2} - (f_{14} + f_{66}) \frac{\partial^3}{\partial x_1 \partial x_2^2} + f_{15} \frac{\partial^3}{\partial x_2^3}, \]

\[ L_2 = -g_{22} \frac{\partial^3}{\partial x_1^2} + (g_{12} + g_{26}) \frac{\partial^3}{\partial x_1 \partial x_2} - (g_{21} + g_{16}) \frac{\partial^3}{\partial x_2^2} + g_{11} \frac{\partial^3}{\partial x_2^3}, \] (2.19)

Eliminating \( \Psi \) and \( V \) from Eq. (2.18) yields

\[ (L_4L_2L_2^* + 2L_3L_2L_2 - L_4L_2^*L_2 - L_4L_2^* - L_3L_2^*) F = 0 \] (2.20)

As discussed in [4] within the framework of anisotropic elasticity, Eq. (2.20) can be solved by assuming a solution of \( F(z) \) such that

\[ F(z) = F(x_1 + i\mu x_2), \quad \mu = \alpha + i\beta \] (2.21)

where \( \alpha \) and \( \beta \) are real numbers. By introducing Eq. (2.21) into Eq. (2.20), and using the chain rule of differentiation, an expression of the form \( \{F(z)^{\alpha}\} = 0 \) is obtained. A nontrivial solution follows by setting the characteristic equation equal to zero:

\[ (L_4L_2L_2^* + 2L_3L_2L_2 - L_4L_2^*L_2 - L_4L_2^* - L_3L_2^*) F = 0 \] (2.22)

Owing to the particular material symmetry of the piezoelectricity under investigation, the polynomial is expressed in terms of even powers of \( \mu \). This allows us to solve Eq. (2.22) analytically, rendering

\[ \mu_k = \alpha_k + i\beta_k, \quad k = 1, 2, 3, 4 \] (2.23)

where \( i = \sqrt{-1} \). Once the roots \( \mu_k, k=1, 2, 3, 4 \) are known, the solution for the functions \( F, \Psi, \) and \( V \) is written as

\[ F(x_1, x_2) = 2 \text{Re} \sum_{j=1}^{4} F_j(z_j) \] (2.24)
\[ \Psi(x_1, x_2) = 2 \text{Re} \sum_{j=1}^{4} \Psi_j(z_j) \]  
\[ V(x_1, x_2) = 2 \text{Re} \sum_{j=1}^{4} V_j(z_j) \]

where \( z_j = x + p_j y \). By eliminating \( \Psi \) or \( V \) from Eq. (2.18), we can express the functions \( \Psi \) and \( V \) in terms of the function \( F \) as

\[ \Psi_k = \begin{cases} A_k F'_k & \text{for } k=1,2,4 \\ F'/A_k & \text{for } k=3 \end{cases} \]

\[ V_k = \begin{cases} \Omega_k F'_k & \text{for } k=1,2,3 \\ F'/\Omega_k & \text{for } k=4 \end{cases} \]

where \( F'_k = dF_k / dz_k \), and

\[ A_k = \begin{cases} \frac{l_1(\mu_k)l^*_2(\mu_k) - l^*_1(\mu_k)l_2(\mu_k)}{l_2(\mu_k)l^*_2(\mu_k) - l^*_2(\mu_k)l_2(\mu_k)} & \text{for } k=1,2 \\ \frac{l_4(\mu_k)l^*_2(\mu_k) - l^*_4(\mu_k)l_2(\mu_k)}{l_2(\mu_k)l^*_2(\mu_k) - l^*_2(\mu_k)l_2(\mu_k)} & \text{for } k=3 \\ \frac{l_4(\mu_k)l^*_2(\mu_k) - l^*_4(\mu_k)l_2(\mu_k)}{l_2(\mu_k)l^*_2(\mu_k) - l^*_2(\mu_k)l_2(\mu_k)} & \text{for } k=4 \end{cases} \]

\[ \Omega_k = \begin{cases} \frac{l_2(\mu_k)l^*_2(\mu_k) - l^*_2(\mu_k)l_2(\mu_k)}{l_2(\mu_k)l^*_2(\mu_k) - l^*_2(\mu_k)l_2(\mu_k)} & \text{for } k=1,2 \\ \frac{l_4(\mu_k)l^*_2(\mu_k) - l^*_4(\mu_k)l_2(\mu_k)}{l_2(\mu_k)l^*_2(\mu_k) - l^*_2(\mu_k)l_2(\mu_k)} & \text{for } k=3 \\ \frac{l_4(\mu_k)l^*_2(\mu_k) - l^*_4(\mu_k)l_2(\mu_k)}{l_2(\mu_k)l^*_2(\mu_k) - l^*_2(\mu_k)l_2(\mu_k)} & \text{for } k=4 \end{cases} \]

with

\[ l_1(p) = f_{14} \mu^4 - 2 f_{16} \mu^3 + (2 f_{12} + f_{18}) \mu^2 - 2 f_{26} \mu + f_{22}, \]
\[ l_2(p) = f_{15} \mu^4 - (f_{14} + f_{18}) \mu^3 + (f_{25} + f_{16}) \mu^2 - f_{24}, \]
\[ l_3(p) = g_{41} \mu^4 - (g_{21} + g_{44}) \mu^3 + (g_{12} + g_{26}) \mu - g_{22}, \]
\[ l_4(p) = f_{54} \mu^2 - 2 f_{45} \mu + f_{44}, \]
\[ l^*_1(p) = g_{41} \mu^2 - (g_{21} + g_{44}) \mu^2 + g_{24}, \]
\[ l^*_2(p) = -\beta_1 \mu^2 + 2 \beta_2 \mu - \beta_2 \]

Eqs. (2.24)–(2.26) can then be rewritten as
\[ F = 2 \text{Re} \left[ F_1^* + F_2^* + F_3^* + F_4^* \right], \]
\[ \Psi = 2 \text{Re} \left[ A_1 F_1^* + A_2 F_2^* + F_1^*/A_1 + A_1 F_1^* \right], \]
\[ V = 2 \text{Re} \left[ \Omega_1 F_1^* + \Omega_2 F_2^* + \Omega_2 F_3^* + F^*/\Omega_4 \right]. \tag{2.31} \]

With the aid of Eq. (2.31) we can obtain expressions for the stress and electric displacement components. Using Eqs. (2.17) and (2.31), we obtain

\[
\begin{cases}
\sigma_{11} = 2 \text{Re} \sum_{k=1}^{4} \left( \frac{\mu_k^2}{\mu_k} F_k^*(z_k) \right), \\
\sigma_{22} = 2 \text{Re} \sum_{k=1}^{4} \left( \frac{\mu_k^2}{-\mu_k} F_k^*(z_k) \right), \\
\sigma_{12} = -2 \text{Re} \left[ A_1 F_1^* + A_2 F_2^* + F_1^*/A_1 + A_1 F_1^* \right],
\end{cases} \tag{2.32}
\]

\[
\begin{align*}
\sigma_{23} &= -2 \text{Re} \left[ A_1 F_1^* + A_2 F_2^* + F_1^*/A_1 + A_1 F_1^* \right], \\
\sigma_{31} &= 2 \text{Re} \left[ p_1 A_1 F_1^* + p_2 A_2 F_2^* + p_3 F_1^*/A_3 + p_4 A_4 F_4^*/A_4 \right], \\
D_1 &= 2 \text{Re} \left[ p_1 \Omega_1 F_1^* + p_2 \Omega_2 F_2^* + p_3 \Omega_3 F_3^* + p_4 F^*/\Omega_4 \right], \\
D_2 &= -2 \text{Re} \left[ \Omega_1 F_1^* + \Omega_2 F_2^* + \Omega_3 F_3^* + F^*/\Omega_4 \right].
\end{align*} \tag{2.33}
\]

Finally, using the constitutive equations (2.14) in conjunction with Eqs. (2.32) and (2.33) allows us to find expressions for the strain and electric field. They are

\[
\begin{align*}
\varepsilon_{11} &= 2 \text{Re} \sum_{k=1}^{4} u_k s_k^* , \\
\varepsilon_{22} &= 2 \text{Re} \sum_{k=1}^{4} v_k \mu_k s_k^* , \\
2\varepsilon_{12} &= 2 \text{Re} \left[ \sum_{k=1}^{4} \left( \nu_k^* + \mu_k u_k^* \right) s_k^* \right], \\
2\varepsilon_{13} &= 2 \text{Re} \left[ \sum_{k=1}^{4} \left( w_k^* + \mu_k s_k^* \right) s_k^* \right], \\
2\varepsilon_{23} &= 2 \text{Re} \left[ \sum_{k=1}^{4} \left( w_k^* + \mu_k s_k^* \right) s_k^* \right], \\
E_i &= -2 \text{Re} \sum_{k=1}^{4} \Phi_k^* s_k^* , \\
E_i^* &= -2 \text{Re} \sum_{k=1}^{4} \Phi_k^* s_k^* ,
\end{align*} \tag{2.34}
\]

where

\[
\begin{align*}
\sigma_1 &= F_1^* , \quad s_2 = F_2^* , \quad s_3 = F_3^*/A_3 , \quad s_4 = F_4^*/\Omega_4 \\
u_k^* &= \begin{cases} \\
\mu_k f_1 + f_1 - f_1 A_k + f_1 A_k + f_1 \mu_k A_k - f_1 \mu_k A_k - g_{11} \mu_k A_k - g_{12} \Omega_k & \text{for } k=1,2 \\
A_k (\mu_k^2 f_1 + f_1 - f_1 A_k + f_1 A_k + f_1 \mu_k A_k - f_1 \mu_k A_k - g_{11} \mu_k A_k - g_{12} \Omega_k) - f_{11} + f_{15} \mu_k & \text{for } k=3 \\
\Omega_k (\mu_k^2 f_1 + f_1 - f_1 A_k + f_1 A_k + f_1 \mu_k A_k - f_1 \mu_k A_k + g_{11} \mu_k A_k - g_{12} \Omega_k) & \text{for } k=4 
\end{cases} \\
v_k^* &= \begin{cases} \\
\mu_k f_1 + (f_2 - f_2 A_k - g_{22} \Omega_k)/\mu_k + f_{23} A_k - f_2 + g_{12} \Omega_k & \text{for } k=1,2 \\
A_k (f_2 - f_2 A_k + f_2 A_k + f_2 /\mu_k - f_2 A_k - f_2 + g_{12} \Omega_k) - f_{21} + f_{25} \mu_k & \text{for } k=3 \\
\Omega_k (f_2 - f_2 A_k + f_2 A_k + f_2 /\mu_k + f_2 A_k - f_2 + g_{12} \Omega_k) & \text{for } k=4
\end{cases}
\end{align*} \tag{2.35}
\]
Substitution of Eq. (1.2) into Eq. (2.34), and then integration of the normal strains and the electric field $E = –\text{grad } \phi$ produces

$$u_i = 2 \text{Re} \left[ \sum_{k=1}^4 w_k s_k \right], \quad u_2 = 2 \text{Re} \left[ \sum_{k=1}^4 v_k s_k \right],$$

$$u_3 = 2 \text{Re} \left[ \sum_{k=1}^4 w_k s_k \right], \quad E_i = 2 \text{Re} \left[ \sum_{k=1}^4 \phi_k s_k \right]$$

The integrating constants, which represent the rigid body motions, are ignored here[7].

Recapitulating, based on the procedure above the generalized plane strain piezoelectric problem is reduced to one of finding four complex potentials, $s_i \ (i=1-4)$, in some region $\Omega$ of the material. Each potential is a function of a different generalized complex variable $z_k = x_1 + \mu_k x_2$.

### 2.3 Techniques of Fourier transformation

In this section we briefly examine the application of Fourier transform techniques to cracked piezoelectric materials. Yu and Qin [8,9] used Fourier transform techniques to study the crack-tip singularities and damage properties of thermopiezoelectric materials. They began with defining a Fourier transform pair

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\xi x} \, dx, \quad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{-i\xi x} \, d\xi$$

and by introducing the shorthand notation given by Barnett and Lothe [10]. With this shorthand notation, the governing equation (1.10) and the constitutive relationship (1.6) can be rewritten as

$$\Pi_{\alpha,\beta} = f_{\beta}$$

where \( f_i = Q_i \), and

\[
\Pi_{ij} = E_{iJKm} \sigma_{iJKm}
\]

(2.43)

where \( f_i = Q_i \), and

\[
\Pi_{ij} = \begin{cases} 
\sigma_{ij}, & i, J = 1, 2, 3 \\
D_{ij}, & J = 4; \ i, 1, 2, 3 
\end{cases}
\]

(2.44)

\[
U_K = \begin{cases} 
u_k, & K = 1, 2, 3 \\
\phi, & K = 4 
\end{cases}
\]

(2.45)

\[
E_{iJKm} = \begin{cases} 
c_{ijkm}, & i, J, K, m = 1, 2, 3 \\
e_{mij}, & K = 4; \ i, J, m = 1, 2, 3 \\
e_{ikw}, & J = 4; \ i, K, m = 1, 2, 3 \\
\kappa_{im}, & J = K = 4; \ i, m = 1, 2, 3 
\end{cases}
\]

(2.46)

For generalized two-dimensional deformations in which \( U = \{u_1, u_2, u_3, \phi\}^T \) depends on \( x_1 \) and \( x_2 \) only, where the superscript “\( T \)” denotes the transpose, a general solution can be obtained by applying the transform to Eq. (2.42) over \( x_1 \). This gives

\[
\nabla^2 \hat{U} + i \xi (\mathbf{R} + \mathbf{R}^T) \frac{\partial \hat{U}}{\partial x_2} - \mathbf{T} \frac{\partial^2 \hat{U}}{\partial x_2^2} = 0
\]

(2.47)

in which we assume \( f_j = 0 \) in Eq. (2.42) for the sake of simplicity. The matrices \( \mathbf{Q} \), \( \mathbf{R} \), and \( \mathbf{T} \) are 4x4 real matrices whose components are

\[
Q_{jk} = E_{1jk1}, \quad R_{jk} = E_{1jk2}, \quad T_{jk} = E_{2jk2}
\]

(2.48)

The solution of Eq. (2.47) can be obtained by considering an arbitrary eigenfunction of the form

\[
\hat{U} = \mathbf{a} e^{i\eta x_2}
\]

(2.49)

Substituting Eq. (2.49) into Eq. (2.47), it is found that

\[
[\mathbf{Q} \xi^2 + i \xi (\mathbf{R} + \mathbf{R}^T) + \eta^2 \mathbf{T}] \mathbf{a} = 0
\]

(2.50)

Letting \( p = \eta \xi \), we have eight eigenvalues \( p \) from Eq. (2.50), which consists of four pairs of complex conjugates [11]. Denote

\[
\eta_M = \begin{cases} 
p_M \xi, & \xi > 0 \\
p_M^* \xi, & \xi < 0 
\end{cases}
\]

(2.51)

where \( M = 1, 2, 3, 4 \). It is obvious that \( \text{Im}(p_M) > 0 \) for all \( \xi \). Such a definition is expedient for development of the subsequent derivation. A general solution of Eq. (2.47) is obtained from a linear combination of the eight eigensolutions, say \( F_i \) and \( G_i \),
(i=1–4), which are obtained by replacing \( \eta \) in Eq. (2.49) with \( \eta_M \) \( (M=1–4) \), when the roots \( p_{\eta} \) are distinct. The result is

\[
\hat{U} = \sqrt{2\pi}(\hat{A}\hat{F}f + \hat{A}\hat{G}g)H(\xi) + \sqrt{2\pi}(\hat{A}\hat{F}f + \hat{A}\hat{G}g)H(-\xi)
\]

(2.52)

where \( H(\xi) \) is the Heaviside step function, and

\[
\mathbf{F}(\xi, x_2) = \{ F_\alpha(\xi, x_2) \} = \{ e^{-i\eta_\alpha x_2} \}, \quad \mathbf{G}(\xi, x_2) = \{ G_\alpha(\xi, x_2) \} = \{ e^{+i\eta_\alpha x_2} \}
\]

(2.53)

Note that \( \eta = \rho \xi \), \( \mathbf{f} \) and \( \mathbf{g} \) are two vector functions of \( \xi \) to be determined from the electroelastic boundary conditions of a given problem.

The transformed stress and electric displacements follow from the constitutive relation of Eq. (2.43):

\[
\hat{\Pi}_1 = i\xi \sqrt{2\pi}(\hat{B}\hat{P}\hat{F}f + \hat{B}\hat{P}\hat{G}g)H(\xi) + i\xi \sqrt{2\pi}(\hat{B}\hat{P}\hat{F}f + \hat{B}\hat{P}\hat{G}g)H(-\xi)
\]

(2.54)

\[
\hat{\Pi}_2 = -i\xi \sqrt{2\pi}(\hat{B}\hat{F}f + \hat{B}\hat{G}g)H(\xi) - i\xi \sqrt{2\pi}(\hat{B}\hat{F}f + \hat{B}\hat{G}g)H(-\xi)
\]

(2.55)

The traction-charge vector on a surface with normal \( \mathbf{n}=(n_1, n_2, 0) \) can be found from Eqs. (2.54) and (2.55) as follows:

\[
\mathbf{t} = \hat{\Pi}_1 n_1 + \hat{\Pi}_2 n_2 = i\xi \sqrt{2\pi}[\hat{B}(n_1\mathbf{P} - n_2\mathbf{I})\hat{F}f + \hat{B}(n_1\mathbf{P} - n_2\mathbf{I})\hat{G}g]H(\xi)
\]

\[
+ i\xi \sqrt{2\pi}[\hat{B}(n_1\mathbf{P} - n_2\mathbf{I})\hat{F}f + \mathbf{B}(n_1\mathbf{P} - n_2\mathbf{I})\hat{G}g]H(-\xi)
\]

(2.56)

where \( \mathbf{I} \) is the unit matrix.

Equations (2.52), (2.54) and (2.55) represent the solution for the elastic and electric fields in the Fourier transform space. The general solution for an electroelastic field in real space is obtained by applying the inverse Fourier transform to Eqs. (2.52), (2.54)–(2.56). The results are

\[
U(x_1, x_2) = \int_{-\infty}^{0} [\hat{A}\hat{F}f + \hat{A}\hat{G}g]e^{-ix_1\xi}d\xi + \int_{0}^{\infty} [\hat{A}\hat{F}f + \hat{A}\hat{G}g]e^{-ix_1\xi}d\xi
\]

(2.57)

\[
\Pi_1(x_1, x_2) = i\int_{0}^{\infty} [\hat{B}\hat{P}\hat{F}f + \hat{B}\hat{P}\hat{G}g]e^{-ix_1\xi}d\xi + i\int_{-\infty}^{0} [\hat{B}\hat{P}\hat{F}f + \hat{B}\hat{P}\hat{G}g]e^{-ix_1\xi}d\xi
\]

(2.58)

\[
\Pi_2(x_1, x_2) = i\int_{0}^{\infty} [\hat{B}\hat{F}f + \hat{B}\hat{G}g]e^{-ix_1\xi}d\xi + i\int_{-\infty}^{0} [\hat{B}\hat{F}f + \hat{B}\hat{G}g]e^{-ix_1\xi}d\xi
\]

(2.59)

\[
\mathbf{t}(x_1, x_2) = i\int_{0}^{\infty} \xi[B(n_1\mathbf{P} - n_2\mathbf{I})\hat{F}f + \hat{B}(n_1\mathbf{P} - n_2\mathbf{I})\hat{G}g]e^{-ix_1\xi}d\xi
\]

\[
+ i\int_{-\infty}^{0} \xi[B(n_1\mathbf{P} - n_2\mathbf{I})\hat{F}f + \mathbf{B}(n_1\mathbf{P} - n_2\mathbf{I})\hat{G}g]e^{-ix_1\xi}d\xi
\]

(2.60)

For a given boundary value problem, the eight functions \( \mathbf{f} \) and \( \mathbf{g} \) are determined from the appropriate boundary conditions.
2.4 Trefftz finite element method

The solution methods discussed in the preceding sections are mostly based on analytical approaches. For a complex structure, however, a powerful numerical method is required to obtain a meaningful solution for electroelastic crack problems. Of all the numerical methods, the finite element method (FEM) and boundary element method (BEM) may be the most versatile computational tools to treat piezoelectric problems. Particularly, the Trefftz FEM has recently received attention from researchers in the field of solid mechanics. In the literature there are only a few papers addressing the application of Trefftz FEM to piezoelectric problems. Qin [12,13] introduced the Trefftz FEM for piezoelectric problems in 2003. Wang et al. [14] used Trefftz FEM and computed eigensolutions to determine singular electroelastic fields in piezoelectricity. In this section, the application of Trefftz FEM to piezoelectric problems is briefly examined.

2.4.1 Basic equations

Consider a linear piezoelectric material in which the constitutive relations, the differential governing equations and boundary conditions are given in Eqs. (1.5), (1.10)–(1.12), respectively. Moreover, in the Trefftz finite element form, Eqs. (1.2), (1.5), (1.10)–(1.12) should be completed by the following inter-element continuity requirements:

\[ u_e = u_f, \quad \phi_e = \phi_f \quad \text{(on } \Gamma_e \cap \Gamma_f, \text{ conformity)} \]  \hfill (2.61)

\[ t_e + t_f = 0, \quad D_{ae} + D_{af} = 0 \quad \text{(on } \Gamma_e \cap \Gamma_f, \text{ reciprocity)} \]  \hfill (2.62)

where “e” and “f” stand for any two neighboring elements. The equations mentioned above are taken as the basis to establish the modified variational principle for Trefftz finite element analysis of piezoelectric materials [12].

2.4.2 Assumed fields

The main idea of the Trefftz FEM is to establish a FE formulation whereby intra-element continuity is enforced on a non-conforming internal displacement field chosen so as to a priori satisfy the governing differential equation of the problem under consideration [12]. With the Trefftz FEM the solution domain \( \Omega \) is subdivided into elements, and over each element “e,” the assumed intra-element fields are
\[ U = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \phi \end{bmatrix} = \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \\ \hat{\phi} \end{bmatrix} + \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \mathbf{c} = \tilde{U} + \sum_{j=1}^{N_e} N_j \mathbf{c}_j = \tilde{U} + \mathbf{N} \mathbf{c} \quad (2.63) \]

where \( \mathbf{c} \) stands for undetermined coefficient, and \( \tilde{U} (= [\hat{u}_1, \hat{u}_2, \hat{u}_3, \hat{\phi}]^T) \) and \( \mathbf{N} \) are known functions. If the governing differential equation (2.42) is rewritten in a general form

\[ 9 \mathbf{R} \mathbf{U}(\mathbf{x}) + f(\mathbf{x}) = 0, \quad (\mathbf{x} \in \Omega_e) \quad (2.64) \]

where \( \mathbf{R} \) stands for the differential operator matrix for Eq. (2.42), \( \mathbf{x} \) for the position vector, \( f (= [f_1, f_2, f_3, f_4]^T) \) for the known right-hand side term, the overhead bar indicates the imposed quantities, and \( \Omega_e \) stands for the \( e \)th element sub-domain, then \( \tilde{U} = \mathbf{U}(\mathbf{x}) \) and \( \mathbf{N} = \mathbf{N}(\mathbf{x}) \) in Eq. (2.63) must be chosen such that

\[ 9 \mathbf{R} \hat{\mathbf{U}} + f = 0 \quad \text{and} \quad 9 \mathbf{N} = 0 \quad (2.65) \]

everywhere in \( \Omega_e \). A complete system of homogeneous solutions \( \mathbf{N}_j \) can be generated by way of the solution in Stroh formalism

\[ \mathbf{U} = 2 \text{Re} \{ \mathbf{A} \langle f(z_i) \rangle \mathbf{c} \} \quad (2.66) \]

where “Re” stands for the real part of a complex number, \( \mathbf{A} \) is the material eigenvector matrix which has been well defined in the literature (see pp. 17-18 of [11]), \( \langle f(z_i) \rangle = \text{diag}[[f(z_1), f(z_2), f(z_3), f(z_4)] \) is a diagonal 4x4 matrix, and \( f(z_i) \) is an arbitrary function with argument \( z_i = x_1 + \mu_i x_2 \). \( \mu_i \) (i=1-4) are the material eigenvalues[11].

The unknown coefficient \( \mathbf{c} \) may be calculated from the conditions on the external boundary and/or the continuity conditions on the inter-element boundary. Thus various Trefftz element models can be obtained by using different approaches to enforce these conditions. In the majority of cases a hybrid technique is used, whereby the elements are linked through an auxiliary conforming displacement frame which has the same form as in the conventional FE method. This means that, in the Trefftz finite element approach, a conforming electric potential and displacement (EPD) field should be independently defined on the element boundary to enforce the field continuity between elements and also to link the coefficient \( \mathbf{c} \), appearing in Eq. (2.63), with nodal EPD \( \mathbf{d} (= [d_i]) \). The frame is defined as

\[ \tilde{\mathbf{U}}(\mathbf{x}) = \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_3 \\ \tilde{\phi} \end{bmatrix} = \begin{bmatrix} \tilde{N}_1 \\ \tilde{N}_2 \\ \tilde{N}_3 \\ \tilde{N}_4 \end{bmatrix} \mathbf{d} = \tilde{\mathbf{N}} \mathbf{d} \quad (\mathbf{x} \in \Gamma_e) \quad (2.67) \]
where the symbol “~” is used to specify that the field is defined on the element boundary only, \( \mathbf{d} = \mathbf{d}(\mathbf{e}) \) stands for the vector of the nodal displacements which are the final unknowns of the problem, \( \Gamma_e \) represents the boundary of element \( e \), and \( \mathbf{N} \) is a matrix of the corresponding shape functions which are the same as those in conventional FE formulation.

Using the above definitions the generalized boundary forces and electric displacements can be derived from Eqs. (1.11) and (2.63), and denoted as

\[
\begin{bmatrix}
    \mathbf{T} \equiv \\
    \begin{bmatrix}
        t_1 \\
        t_2 \\
        t_3 \\
        \mathbf{D}_n
    \end{bmatrix}
\end{bmatrix} = \begin{bmatrix}
    \sigma_{1,i} n_j \\
    \sigma_{2,i} n_j \\
    \sigma_{3,i} n_j \\
    D_{nj}
\end{bmatrix} = \begin{bmatrix}
    \mathbf{Q}_1 \\
    \mathbf{Q}_2 \\
    \mathbf{Q}_3 \\
    \mathbf{Q}_4
\end{bmatrix} = \mathbf{T} + \mathbf{Q} \mathbf{e} \quad \text{(2.68)}
\]

where \( t_i \) and \( \mathbf{D}_n \) are derived from \( \mathbf{U} \).

### 2.4.3 Element stiffness equation

Based on the two independent assumed fields, Eqs. (2.63) and (2.67), presented above, the element matrix equation can be generated by a variational approach [15]. For a three-dimensional piezoelectric problem, the variational functional can be constructed as [12]

\[
\Psi_{we} = \frac{1}{2} \int_{\Omega} \mathbf{U}^T \mathbf{d} \Omega - \int_{\Gamma_e} \mathbf{T}^T (\mathbf{U} - \mathbf{U}/2) d \Gamma + \int_{\Gamma_{we}} \mathbf{U}^T \mathbf{T} d \Gamma \quad \text{(2.69)}
\]

Substituting the expressions given in Eqs. (2.63), (2.67), and (2.68) into (2.69) produces

\[
\Psi_{we} = \frac{1}{2} \mathbf{e}^T \mathbf{H}_e \mathbf{e} - \mathbf{e}^T \mathbf{G}_e \mathbf{d} + \mathbf{e}^T \mathbf{h}_e + \mathbf{e}^T \mathbf{g}_e + \text{terms without } \mathbf{e} \text{ or } \mathbf{d} \quad \text{(2.70)}
\]

in which the matrices \( \mathbf{H}_e, \mathbf{G}_e \) and the vectors \( \mathbf{h}_e, \mathbf{g}_e \) are as follows:

\[
\mathbf{H}_e = \int_{\Gamma_e} \mathbf{Q}^T \mathbf{N} d \Gamma = \int_{\Gamma_e} \mathbf{N}^T \mathbf{Q} d \Gamma \quad \text{(2.71)}
\]

\[
\mathbf{G}_e = \int_{\Gamma_e} \mathbf{Q}^T \mathbf{N} d \Gamma \quad \text{(2.72)}
\]

\[
\mathbf{h}_e = \frac{1}{2} \int_{\Omega_e} \mathbf{N}^T \mathbf{d} \Omega + \int_{\Gamma_{we}} (\mathbf{Q}^T \mathbf{U} + \mathbf{N}^T \mathbf{T}) d \Gamma \quad \text{(2.73)}
\]

\[
\mathbf{g}_e = \int_{\Gamma_{we}} \mathbf{N}^T \mathbf{T} d \Gamma - \int_{\Gamma_e} \mathbf{N}^T \mathbf{T} d \Gamma \quad \text{(2.74)}
\]
To enforce inter-element continuity on the common element boundary, the unknown vector \( \mathbf{c} \) should be expressed in terms of nodal DOF \( \mathbf{d} \). An optional relationship between \( \mathbf{c} \) and \( \mathbf{d} \) in the sense of variation can be obtained from

\[
\frac{\partial \Psi_{me}}{\partial \mathbf{c}} = \mathbf{H}_e \mathbf{c} - \mathbf{G}_e \mathbf{d} + \mathbf{h}_e = 0
\]

(2.75)

This leads to

\[
\mathbf{c} = \mathbf{H}_e^{-1} (\mathbf{G}_e \mathbf{d} - \mathbf{h}_e)
\]

(2.76)

and then straightforwardly yields the expression of \( \Psi_{me} \) only in terms of \( \mathbf{d} \) and other known matrices:

\[
\Psi_{me} = -\frac{1}{2} \mathbf{d}^T (\mathbf{G}_e^T \mathbf{H}_e^{-1} \mathbf{G}_e) \mathbf{d} + \mathbf{d}^T (\mathbf{G}_e^T \mathbf{H}_e^{-1} \mathbf{h}_e + \mathbf{g}_e) + \text{terms without } \mathbf{d}
\]

(2.77)

Therefore, the element stiffness matrix equation can be obtained by taking the vanishing variation of the functional \( \Psi_{me} \) as

\[
\frac{\partial \Psi_{me}}{\partial \mathbf{d}} = 0 \Rightarrow \mathbf{K}_e \mathbf{d} = \mathbf{P}_e
\]

(2.78)

where \( \mathbf{K}_e = \mathbf{G}_e^T \mathbf{H}_e^{-1} \mathbf{G}_e \) and \( \mathbf{P}_e = \mathbf{G}_e^T \mathbf{H}_e^{-1} \mathbf{h}_e + \mathbf{g}_e \) are, respectively, the element stiffness matrix and the equivalent nodal flow vector. The expression (2.78) is the elemental stiffness-matrix equation for Trefftz finite element analysis.

### 2.5 Integral equations

An integral equation is, mathematically, an equation in which an unknown function appears under an integral sign. It is noted that most crack and stress singularity problems in piezoelectric structures and materials can be formulated in terms of a certain type of integral equation such as Fredholm, Volterra, and Abel integral equations. In order to provide fundamental knowledge and to enhance understanding of these integral equations which appear in coming chapters, a brief review of Fredholm, Volterra, and Abel integral equations is presented in this section.

#### 2.5.1 Fredholm integral equations

A homogeneous Fredholm integral equation of the first kind is written as [16]:

\[
\int_a^b K(x,y) \phi(y) dy = f(x) \quad (a \leq x \leq b)
\]

(2.79)
where the continuous kernel function $K(x,y)$ and the inhomogeneous term $f(x)$ are known functions. The equation is to be satisfied for $x$ in the interval $a \leq x \leq b$, the same as the interval of integration. It is typical to find the unknown function $\phi(y)$.

An inhomogeneous Fredholm equation of the second kind has the form

$$\lambda \phi(x) = \int_a^b K(x,y) \phi(y) dy + f(x) \quad (a \leq x \leq b) \quad (2.80)$$

where $\lambda$ is a known constant. Given the kernel $K(x,y)$, and the function $f(x)$, the problem is to determine the function $\phi(y)$. A standard approach to solving Eq. (2.80) is called an integral equation Neumann series, which may be described as follows. Take

$$\phi_0(x) = f(x),$$
$$\phi_1(x) = f(x) + \lambda \int_a^b K(x,y) f(y) dy,$$
$$\phi_2(x) = f(x) + \lambda \int_a^b K(x,y_1) f(y_1) dy_1 + \lambda^2 \int_a^b \int_a^b K(x,y_1) K(y_1, y_2) f(y_2) dy_1 dy_2,$$

... 

$$\phi(x) = \sum_{i=0}^{\infty} \lambda^i u_i(x) \quad (2.81)$$

where

$$u_0(x) = f(x),$$
$$u_1(x) = \int_a^b K(x, y_1) f(y_1) dy_1,$$
$$u_2(x) = \int_a^b \int_a^b K(x, y_1) K(y_1, y_2) f(y_2) dy_1 dy_2,$$

... 

The Neumann series solution is then

$$\phi(x) = \lim_{n \to \infty} \phi_n(x) = \lim_{n \to \infty} \sum_{i=0}^{n} \lambda^i u_i(x) \quad (2.83)$$

Alternatively, if the kernel $K(x,y)$ is separable, i.e., it can be written in the form

$$K(x,y) = \sum_{i=1}^{n} M_i(x) N_i(y) \quad (2.84)$$

Eq. (2.80) may be solved as follows. Let

$$\phi(x) = f(x) + \int_a^b K(x,t) \phi(t) dt$$

$$= f(x) + \lambda \sum_{j=1}^{n} M_j(x) \int_a^b N_j(t) \phi(t) dt = f(x) + \lambda \sum_{j=1}^{n} c_j M_j(x) \quad (2.85)$$
2.5 Integral equations

where

\[ c_j = \int_a^b N_j(t)\phi(t)dt \quad (2.86) \]

Now multiply both sides of Eq. (2.85) by \( N_i(x) \) and integrate over \( dx \), and we have

\[ \int_a^b \phi(x)N_i(x)dx = \int_a^b f(x)N_i(x)dx + \lambda \sum_{j=1}^n \int_a^b M_j(x)N_i(x)dx \quad (2.87) \]

By Eq. (2.86), the first term of Eq. (2.87) is just \( c_i \). Now define

\[ b_i = \int_a^b N_i(x)f(x)dx, \quad a_j = \int_a^b N_i(x)M_j(x)dx \quad (2.88) \]

So Eq. (2.87) becomes

\[ c_i = b_i + \lambda \sum_{j=1}^n a_j c_j \quad (2.89) \]

Eq. (2.89) can be written in matrix form as

\[ C = B + \lambda AC \quad (2.90) \]

So we have

\[ (1 - \lambda A)C = B, \quad C = (1 - \lambda A)^{-1}B \quad (2.91) \]

2.5.2 Volterra integral equations

It is noted from Eq. (2.79) that the integration limits of a Fredholm equation are constants. A Volterra integral equation of the first kind is obtained by replacing the upper integration limit \( b \) in Eq. (2.79) with the variable \( x \):

\[ \int_a^x K(x,y)\phi(y)dy = f(x) \quad (a \leq x) \quad (2.92) \]

Thus, for any fixed range of \( x \), say \( 0 \leq x \leq h \), it is the same as a Fredholm equation with a kernel that vanishes for \( y > x \). Consequently, all results for the Fredholm equation are still valid.

Like the definition of the Fredholm equation above, a Volterra integral equation of the second kind is an integral equation of the form

\[ \phi(x) = \lambda \int_a^x K(x,y)\phi(y)dy + f(x) \quad (a \leq x) \quad (2.93) \]

where \( K(x,y) \) is again a known integral kernel, \( f(x) \) is a specified function, and \( \phi(x) \) is the function to be determined.
As a special type of Volterra equation of the first kind, Volterra’s singular equation

\[ \int_a^x \frac{N(x,y)}{(x-y)^\alpha} \phi(y) dy = f(x) \]  

(2.94)

has received wide application in the field of fracture mechanics and computational engineering, where \( N(x,y) \) is a specified bounded function and the exponent \( \alpha \) is a positive number less than 1: \( 0 < \alpha < 1 \).

Equation (2.94) can be solved by reducing it to an equation of the corresponding Volterra equation of the first kind with a bounded kernel. To this end, multiplying both sides of Eq. (2.94) by the function \( \frac{1}{(z-x)^{1-\alpha}} \) and integrating with respect to \( x \) from \( a \) to \( z \), we obtain

\[ \int_a^z \frac{1}{(z-x)^{1-\alpha}} \left[ \int_a^x \frac{N(x,y)}{(x-y)^\alpha} \phi(y) dy \right] dx = \int_a^z \frac{f(x) dx}{(z-x)^{1-\alpha}} \]  

(2.95)

or, upon application of the Dirichlet transformation the equation

\[ \int_a^z \frac{N(x,y) dy}{(z-x)^{1-\alpha} (x-y)^\alpha} \phi(y) dy = f_1(z) \]  

(2.96)

This is already a Volterra equation of the first kind with a bounded kernel

\[ K(z,y) = \int_y^z \frac{N(x,y) dx}{(z-x)^{1-\alpha} (x-y)^\alpha} \]  

(2.97)

where the known function \( f_1(z) \) is

\[ f_1(z) = \int_a^z \frac{f(x) dx}{(z-x)^{1-\alpha}} \]  

(2.98)

The solution \( \phi \) of Eq. (2.94) obviously satisfies the transformed equation (2.96).

### 2.5.3 Abel’s integral equation

Abel’s integral equation

\[ \int_a^z \frac{\phi(y)}{(z-y)^\alpha} dy = f(x) \quad (0 < \alpha < 1) \]  

(2.99)

is a particular case of the integral equation (2.94) when \( N=1 \). The integral equation
may be solved explicitly in the following way: when \( N = 1 \) the kernel of the transformed equation (2.96) has the constant value

\[
\frac{1}{\sqrt{\pi \alpha}} \int_0^1 \frac{dt}{(1-t)^{1/\alpha} t^{1/2}} = \frac{\pi}{\sin \alpha \pi}
\]  

(2.100)

in which \( t = (x-y)/(z-y) \) has been used. Let \( F(y) \) be any function which is continuous and has a continuous derivative throughout the solution domain \( I \). Multiply Eq. (2.100) by \( F'(y) \) and integrate from \( a \) to \( z \). That gives

\[
\frac{\pi}{\sin \alpha \pi} [F(z) - F(a)] = \int_a^z \int_a^z \frac{F'(y) \ dy \ dx}{(z-x)^{1/\alpha} (x-y)^{1/2}}
\]  

(2.101)

Applying Dirichlet’s generalized formula to the second term of Eq. (2.101), we obtain

\[
F(z) - F(a) = \frac{\sin \alpha \pi}{\pi} \int_a^z \left[ \frac{1}{(z-x)^{1/\alpha}} \int_a^z \frac{F'(y) \ dy}{(x-y)^{1/2}} \right] \ dx
\]  

(2.102)

Multiply Eq. (2.99) by \( 1/(z-x)^{1/\alpha} \) \( dx \) and integrate and the equation takes the simple form \( a \) to \( z \), thus obtaining

\[
\int_a^z \frac{f(x) \ dx}{(z-x)^{1/\alpha}} = \int_a^z \frac{1}{(z-x)^{1/\alpha}} \int_a^z \frac{\phi(y) \ dy}{(x-y)^{1/2}} \ dx
\]  

(2.103)

If in Eq. (2.102) we let

\[
F(x) = \int_a^x \phi(y) \ dy
\]  

(2.104)

it will be seen that the preceding equation reduces to

\[
\frac{\pi}{\sin \alpha \pi} \int_a^z \phi(y) \ dy = f_1(z)
\]  

(2.105)

By differentiating Eq. (2.105), we obtain the value of this solution

\[
\phi(z) = \frac{\sin \alpha \pi}{\pi} \frac{d}{dz} \int_a^z \frac{f(x) \ dx}{(z-x)^{1/\alpha}}
\]  

(2.106)

Equation (2.106) can be further written in the form

\[
\phi(z) = \frac{\sin \alpha \pi}{\pi} \left( \frac{f(a)}{z^{1/\alpha}} + \int_a^z \frac{df(x)/dx \ dx}{(z-x)^{1/2}} \right)
\]  

(2.107)
2.6 Shear-lag model

The term shear-lag has been widely used to study strengthening mechanisms through the load transfer from matrix to reinforcement in composite materials. The shear-lag model was originally proposed by Cox [17] and subsequently modified by many researchers. It is assumed that the load transfer from matrix to fiber occurs via shear stresses on the surface between them. Cox’s shear-lag model can be obtained by considering the free-body diagram of a differential element of the fiber, as shown in Fig. 2.1. For static equilibrium of the forces acting along the \( x \) direction, we have

\[
(\sigma_f + d\sigma_f)\pi r^2 - \sigma_i \pi r^2 + \tau_i (2\pi r) dx = 0
\]  

(2.108)

where \( r = d/2 \), \( \sigma_f \) is the fiber normal stress along the \( x \) direction at a distance from the end of fiber, \( \tau_i \) is the interfacial shear stress at a distance from the end of fiber, and \( x \) is the coordinate along the fiber length.

![Fig. 2.1 Free-body diagram of a differential element of a fiber.](image)

Equation (2.108) can be simplified to

\[
\frac{d\sigma_f}{dx} = -\frac{2\tau_i}{r}
\]

(2.109)

Equation (2.109) is referred to as the basic shear-lag equation.

Cox further assumed that the total shear forces on the neighboring annuli remain constant. That assumption leads to the following relationships:

\[
2\pi r_k \tau_k dx = \text{constant} \quad (k = 1, 2, \cdots, n)
\]

(2.110)

where \( r_k \) and \( \tau_k \) are defined in Fig. 2.2. Eq. (2.110) can be rewritten in the form
2.6 Shear-lag model

Fig. 2.2 Distribution of stresses and geometry of $r_k$ and $\rho$.

Thus, the shear stress $\tau$ in the matrix at any radius $\rho$ is related to the interfacial shear stress, $\tau_i$, of the fiber and fiber radius $r$ by the following relation:

$$\tau = \frac{\tau_i}{\rho}$$  \hspace{1cm} (2.112)

Using Eq. (2.112), the shear strain of the matrix near the fiber, which is a function of the displacement of the matrix, can be expressed as

$$\frac{du}{d\rho} = \gamma = \frac{\tau}{G_m} = \frac{\tau_i}{G_m} \left( \frac{r}{\rho} \right)$$  \hspace{1cm} (2.113)

The difference between the displacement at $R$ and that at $r$ or the fiber surface at any point $x$ can be obtained by integrating Eq. (2.113) with respect to $\rho$:

$$(u_R - u_r) = \int_{u_r}^{u_R} du = \frac{\tau_i}{G_m} \int_r^R \frac{1}{\rho} d\rho = \frac{\tau_i}{G_m} \ln \left( \frac{R}{r} \right)$$  \hspace{1cm} (2.114)

where $R=D/2$.

Substituting Eq. (2.114) into Eq. (2.109), we have

$$\frac{d\sigma_f}{dx} = \frac{2G_m(u_R - u_R)}{r^2 \ln(R/r)}$$  \hspace{1cm} (2.115)

To determine the stress built up along the fiber, we need to establish the relationship of $u_R$ and $u_r$ with the fiber stress or strain. To simplify the following derivation, assume the fiber has no shear deformation, then $u_r = u_f$ for any position $r$. Therefore
\[ \frac{du}{dx} = \epsilon_i = \frac{\sigma_i}{E_i}, \quad \frac{du}{dx} = \epsilon_m \equiv \epsilon_i \] (2.116)

where \( \epsilon_m \) and \( \epsilon_i \) are respectively the longitudinal strains in the matrix and in the composite.

Differentiating Eq. (2.115) and using the relation in Eq. (2.116), we have

\[ \frac{d^2 \sigma_i}{dx^2} = \beta^2 (\sigma_i - E_i \epsilon_i) \] (2.117)

where

\[ \beta^2 = \frac{2G_m}{r^2 E_i \ln(D/d)} = \frac{2\pi G_m}{A_i E_i \ln(D/d)} \] (2.118)

with \( A_i \) being the area of fiber cross-section, \( E_i \) the Young’s modulus of the fiber, \( G_m \) the matrix shear modulus.

The solution of Eq. (2.117) is of the form:

\[ \sigma_i = (\sigma_i)_h + (\sigma_i)_p \] (2.119)

where \( (\sigma_i)_p \) is the particular solution and \( (\sigma_i)_h \) the homogeneous solution. They are

\[ (\sigma_i)_p = E_i \epsilon_i, \quad (\sigma_i)_h = A \sinh(\beta x) + B \cosh(\beta x) \] (2.120)

The coefficients \( A \) and \( B \) can be determined from the boundary conditions:

\[ \sigma_i = 0 \quad \text{at} \ x = \pm L/2 \] (2.121)

Substituting the boundary conditions (2.121) into Eq. (2.119) and after some mathematical manipulation, the resulting fiber and interfacial shear stresses are

\[ \sigma_i = E_i \epsilon_m \left( 1 - \frac{\cosh(\beta x)}{\cosh(\beta L/2)} \right), \]
\[ \tau_i = \frac{r}{2} E_i \epsilon_m \beta \frac{\sinh(\beta x)}{\cosh(\beta L/2)} \] (2.122)

where \( L \) is the fiber length.

### 2.7 Hamiltonian method and symplectic mechanics

The strategy of simplifying a mechanical problem by exploiting symmetry so as to reduce the number of variables is one of classical mechanics’ grand themes. It is theoretically deep, practically important, and recurrent in the history of the subject.
The best-known general approach using the strategy is undoubtedly the symplectic Hamiltonian method \[18\], which uses displacements and associated general stresses as dual variables so that the boundary conditions are satisfied without any assumption of displacement or shape functions. Thus the complete solution space covering all kinds of boundary conditions along the edges can be obtained.

To illustrate the symplectic Hamiltonian method, we begin with considering the Principle of Virtual Work. It is one of the oldest principles in physics, which may find its origin in the work of Aristotle (384–322 B.C.) on the static equilibrium of levers. The principle of virtual work was written in its current form in 1717 by Jean Bernoulli (1667–1748) and states that a system composed of \(N\) particles is in static equilibrium if the virtual work

\[
\delta W = \sum_{i=1}^{N} \mathbf{F}_i \cdot \delta \mathbf{x}_i = 0
\]

(2.123)

for all virtual displacements \((\delta \mathbf{x}_1, \cdots, \delta \mathbf{x}_N)\) that satisfy physical constraints, where \(\mathbf{F}_i\) is the force acting on the particle \(i\). Given the commonness of systems of \(N\) particles with constraints, it is natural to seek a description of mechanics relevant only in the subset of 3D Euclidean space accessible to the system. The number of generalized coordinates required to specify completely the configuration of the system is called the number of degrees of freedom of the system. Typically, if a system of \(N\) particles, each having mass \(m_i\) and Cartesian coordinate \(\mathbf{x}_i, i = 1, \cdots, N\), is subjected to \(k\) holonomic constraints,

\[
f_j(x_1, x_2, \cdots, x_N, t) = 0 \quad (j = 1, 2, \cdots, k)
\]

(2.124)

we have \(n = 3N-k\) generalized coordinates, \(q_i\), which are independent.

It was Jean Le Rond d’Alembert (1717–1783) who generalized the principle of virtual work (in 1742) by including within it the accelerating force \(-m \frac{d^2 \mathbf{x}_i}{dt^2}\)

(2.123):

\[
\delta W = \sum_{i=1}^{N} (\mathbf{F}_i - m_i \frac{d^2 \mathbf{x}_i}{dt^2}) \cdot \delta \mathbf{x}_i = 0
\]

(2.125)

so that the equations of dynamics could be obtained.

To obtain the Lagrangian function of the system we need the mapping from the \(n = 3N-k\) generalized coordinates to the usual Cartesian coordinates on \(\mathbb{R}^3\) for each particle:

\[
x_i = x_i(q^1, q^2, \cdots, q^n, t)
\]

\[
\cdots
\]

\[
x_N = x_N(q^1, q^2, \cdots, q^n, t)
\]

(2.126)
Note that this collection of mappings (2.126) is equivalent to a (single) time-parameterized mapping from the $3N-k$ generalized coordinates $(q^1, \cdots, q^n)$ to the Euclidean hyperspace $\mathbb{R}^{3N}$ with $3N$ coordinates $(x_1, y_1, z_1, \ldots, x_N, y_N, z_N)$. Performing a Taylor expansion of the mapping, Eq. (2.126), about the point $(q^1, \cdots, q^*)$ (i.e., expanding $x_i(q^1 + \delta q^1, \cdots, q^n + \delta q^n, t)$ about $(q^1, \cdots, q^n)$) at a fixed time $t$ we obtain

$$\frac{\partial x_i}{\partial q^j} = \sum_{j=1}^n \frac{\partial x_i}{\partial q^j} \delta q^j$$

(2.127)

The quantity $\frac{\partial x_i}{\partial q^j}$ is analogous to the Jacobian of the transformation from $(q^1, \cdots, q^n)$ \(\Rightarrow (x_1, y_1, z_1, \ldots, x_N, y_N, z_N)$. Equation (2.127) can be used to cast D’Alembert’s principle (2.125) in terms of the generalized coordinates.

Making use of the relations

$$r_i = \frac{dx_i}{dt} = \frac{\partial x_i}{\partial t} + \sum_{j=1}^n \frac{\partial x_i}{\partial q^j} \delta q^j, \quad \frac{\partial r_i}{\partial q^j} = \frac{\partial x_i}{\partial q^j}$$

(2.129)

and the definition of the kinetic energy of the system $K = \sum_{i=1}^N \frac{1}{2} m_i r_i \cdot r_i$, Eq. (2.128) leads to

$$\sum_{i=1}^n \left[ \frac{d}{dt} \left( \frac{\partial K}{\partial q^j} \right) - \frac{\partial K}{\partial q^j} - Q_j \right] \delta q^j = 0$$

(2.130)

which is D’Alembert’s principle in configuration space. Since the system is, by hypothesis, holonomic, the $q^j$ form a set of independent coordinates. Any virtual displacement $\delta q^j$ is independent of $\delta q^k \text{ (} k \neq j \text{)}$ and, therefore, for Eq. (2.130) to hold, each term in the sum must separately vanish. For non-trivial $\delta q^j$ this can only happen if each coefficient vanishes, or, equivalently

$$\frac{d}{dt} \left( \frac{\partial K}{\partial q^j} \right) - \frac{\partial K}{\partial q^j} - Q_j = 0 \quad (j = 1, 2, \cdots, n)$$

(2.131)

Equation (2.131) are frequently referred to as Lagrange’s equations, in which we
note that the generalized force $Q_j$ is associated with any active (conservative or nonconservative) force $F_j$. Hence, for a conservative active force derivable from a scalar potential function $V$ (i.e., $F = -\nabla U(q^1, \ldots, q^n, t)$), the $i$th component of the generalized force is $Q_i = -\frac{\partial U}{\partial q_i}$, and Lagrange’s equation (2.131) becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q^i}\right) - \frac{\partial L}{\partial q^j} = 0 \quad (j = 1, 2, \ldots, n)$$  \hspace{1cm} (2.132)

where the Lagrangian is defined as: $L = K - V$.

The $n$ second-order Euler-Lagrange equations (2.132) can be written as $2k$ first-order differential equations, known as Hamilton’s equations (William Rowan Hamilton, 1805–1865), in a $2n$-dimensional phase space with coordinates $z = (q^1, \ldots, q^n; p^1, \ldots, p^n)$, where the dual variable of $p$ according to Legendre’s transformation is

$$p_j (q, q, t) = \frac{\partial L}{\partial q^j} (q, q, t)$$  \hspace{1cm} (2.133)

In terms of these new coordinates, the Euler-Lagrange equations (2.132) are transformed into Hamilton’s canonical equations

$$\frac{dq^j}{dt} = \frac{\partial H}{\partial p^j}, \quad \frac{dp^j}{dt} = -\frac{\partial H}{\partial q^j}$$  \hspace{1cm} (2.134)

where the Hamiltonian function $H$ is defined from the Lagrangian function $L$ by the Legendre transformation (Adrien-Marie Legendre, 1752–1833):

$$H(q, p, t) = p \cdot \dot{q} (q, p, t) - L(q, \dot{q}(q, p, t), t)$$  \hspace{1cm} (2.135)

Using the definition of state vector $\nu = [q, p]^T$, Eq. (2.134) can be expressed as

$$\dot{\nu} = H\nu + h$$  \hspace{1cm} (2.136)

where $H$ is the Hamiltonian matrix and $h$ is a $2n$-vector [18,19]. The Hamiltonian matrix $H$ satisfies the matrix equation

$$JHJ = H^T$$  \hspace{1cm} (2.137)

where $J$ is a symplectic matrix defined as

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}, \quad J^2 = -I_{2n}, \quad J^T = -J$$  \hspace{1cm} (2.138)

With matrix $J$, a symplectic matrix $S$ can be defined as

$$S^TJS = J$$  \hspace{1cm} (2.139)

As an application of symplectic mechanics we consider a plane stress problem with the strip domain $V$ ( $0 \leq z \leq l$, $-h \leq x \leq h$) as shown in Fig. 2.3 [20]. The
force equilibrium, constitutive, and boundary equations of the problem are respectively

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + f_x = 0, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + f_y = 0
\] (2.140)

\[
\sigma_y = F_{y1}(z), \quad \tau_{xy} = F_{y2}(z) \quad (y = -h)
\]

\[
\sigma_y = F_{y2}(z), \quad \tau_{xy} = F_{y2}(z) \quad (y = h)
\] (2.141)

\[
\sigma_x = \frac{E}{1-v^2}(\varepsilon_x + \nu \varepsilon_y), \quad \sigma_y = \frac{E}{1-v^2}(\varepsilon_y + \nu \varepsilon_x), \quad \tau_{xy} = \frac{E}{2(1+v)} \gamma_{xy}
\] (2.142)

where \( F_x \) and \( F_y \) are the body forces. The relationship between strain and displacement is expressed as

\[
\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}
\] (2.143)

![Fig. 2.3 Configuration of the strip domain and loading condition.](image)

The corresponding potential variational functional and the strain energy density are then defined as

\[
\delta U_p = \delta \left\{ \left( \mu_x - uf_x - vf_y \right) dx dy - \int_0^l \left[ (uF_{y2} + vF_{y2})_{y=h} - (uF_{y1} + vF_{y1})_{y=-h} \right] dx \right\} = 0
\] (2.144)

\[
\mu_x = \frac{E}{2(1-v^2)} \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 + 2\nu \left( \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} \right) \right] + \frac{E}{4(1+v)} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2
\] (2.145)

To construct the corresponding Hamiltonian system, the \( x \)-coordinate is modeled as the time variable of the Hamiltonian system. If \( \dot{F}_{x1} = \dot{F}_{x1} = 0 \), the Lagrangian function of the problem is

\[
L(w, u, \dot{w}, \dot{u}) = \mu_x - uf_x - vf_y
\] (2.146)

The dual vectors \( q \) and \( p \) can then be defined as
\( q = (w \ u)^T, \quad p = (\sigma \ \tau)^T \) 

\[ (2.147) \]

with

\[ \sigma = \frac{\partial L}{\partial \dot{u}} = \frac{E}{1-\nu^2} (\dot{u} + \nu \frac{\partial v}{\partial y}), \quad \tau = \frac{\partial L}{\partial v} = \frac{E}{2(1+\nu)} (\dot{v} + \nu \frac{\partial u}{\partial y}) \] 

\[ (2.148) \]

Equations (2.148) yields

\[ \dot{u} = -\nu \frac{\partial v}{\partial y} + \frac{1-\nu^2}{E} \sigma, \quad \dot{v} = -\frac{\partial u}{\partial y} + \frac{2(1+\nu)}{E} \tau \] 

\[ (2.149) \]

Making use of Eqs. (2.140), (2.142), and (2.149), we have

\[ \dot{\sigma} = -\frac{\partial \tau}{\partial y} - f_x, \quad \dot{\tau} = -E \frac{\partial^2 \nu}{\partial y^2} - \nu \frac{\partial \sigma}{\partial y} - f_y \] 

\[ (2.150) \]

Equations (2.149) and (2.150) can be written in matrix as

\[ \begin{bmatrix} \dot{w} \\ \dot{u} \\ \dot{\sigma} \\ \dot{\tau} \end{bmatrix} = \begin{bmatrix} 0 & -\nu \frac{\partial}{\partial y} & \frac{1-\nu^2}{E} & 0 \\ -\frac{\partial}{\partial y} & 0 & 0 & \frac{2(1+\nu)}{E} \\ 0 & 0 & 0 & -\frac{\partial}{\partial y} \\ 0 & -E \frac{\partial^2 \nu}{\partial y^2} & -\nu \frac{\partial}{\partial y} & 0 \end{bmatrix} \begin{bmatrix} w \\ u \\ \sigma \\ \tau \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -f_x \\ -f_y \end{bmatrix} = Hv + h \] 

\[ (2.151) \]

The homogeneous solution of Eq. (2.151) can be obtained using the separation of the variable approach and the symplectic eigenfunction expansion. To this end, assume \( \psi \) in the form

\[ \psi(x, y) = \xi(x)\psi(y) \] 

\[ (2.152) \]

Substituting Eq. (2.152) into Eq. (2.151) with \( h=0 \) yields the solution for \( \xi(x) \):

\[ \xi(x) = e^{\mu x} \] 

\[ (2.153) \]

and the eigenvalue equation:

\[ H\psi(y) = \mu \psi(y) \] 

\[ (2.154) \]

in which \( \mu \) is an eigenvalue of the Hamiltonian operator matrix and \( \psi \) is given by

\[ \psi(y) = \begin{bmatrix} q(y) \\ p(y) \end{bmatrix} \] 

\[ (2.155) \]
Thus we have
\[ \psi(x, y) = e^{\mu y} \psi(y) \]  
(2.156)

It should be noted that the eigenvalue \( \mu \) appears, in general, in \( n \) equal and opposite pairs, \( \mu_i \) and \(-\mu_i\) \((i=1,2,\cdots,n)\) for a \((2n\times2n)\) Hamiltonian matrix \( H \), or else as \( \mu_i \) and \(1/\mu_i\) for a \((2n\times2n)\) symplectic matrix \( S \) [18,21]. Therefore, the \( 2n \) eigenvalues, when ordered appropriately, can be subdivided into the following two groups:

(a) \( \mu_i, \) with \( \text{Re} \mu_i < 0 \) or \( \text{Re} \mu_i = 0 \cap \text{Im} \mu_i < 0 \) \((i=1,2,\cdots,n)\) for a \((2n\times2n)\) Hamiltonian matrix \( H \), and

\[ \mu_i, \text{ with } |\mu_i| < 1 \quad (i=1,2,\cdots,n) \]  
(2.157)

for a \((2n\times2n)\) Hamiltonian matrix \( H \), and

(b) \( \mu_{ni} = -\mu_i \) \((i=1,2,\cdots,n)\) for a \((2n\times2n)\) symplectic matrix \( S \);

\[ \mu_{ni} = 1/\mu_i, \text{ with } |\mu_{ni}| > 1 \quad (i=1,2,\cdots,n) \]  
(2.159)

for a \((2n\times2n)\) Hamiltonian matrix \( H \), and

\[ \mu_{ni} = 1/\mu_i, \text{ with } |\mu_{ni}| > 1 \quad (i=1,2,\cdots,n) \]  
(2.160)

for a \((2n\times2n)\) symplectic matrix \( S \).

Further, Zhong and Williams [18] pointed out that the eigenvectors of \( H \) (or \( S \)) are related by the adjoint symplectic orthogonality relationship. Suppose that \( \psi_1 \) and \( \psi_2 \) are two eigenvectors of \( H \) (or \( S \)), with corresponding eigenvalues \( \mu_1 \) and \( \mu_2 \) which are unequal, we have

\[ \langle \psi_1, \psi_2 \rangle = \int_{-h}^{h} \psi_1^* J \psi_2 dx = 0 \]  
(2.161)

### 2.8 State space formulation

The idea of state space was used initially in system engineering and control theory. With state space representation, a system of linear differential equations for an engineering system can be described as

\[
\begin{align*}
\dot{X}(t) & = A(t)X(t) + B(t)u(t), \\
Y(t) & = C(t)X(t) + D(t)u(t)
\end{align*}
\]  
(2.162)

where \( X(\cdot) \) is called the “state vector”, \( Y(\cdot) \) is the “output vector”, \( u(\cdot) \) is the “input (or control) vector”, \( A(\cdot) \) is the “state matrix”, \( B(\cdot) \) is the “input matrix”, \( C(\cdot) \) is the “output matrix”, and \( D(\cdot) \) is the “feedthrough (or feedforward) matrix” (see Fig. 2.4). For simplicity, \( D(\cdot) \) is often chosen to be the zero matrix, i.e., the system is designed to have no direct feed-through. Notice that in this general for-
In [22], Sosa and Castro considered a two-dimensional piezoelectric material whose constitutive equation, strain-displacement and electric field-electric potential relations are, respectively, defined by Eqs. (1.24) and (1.2). The governing equation (1.10) reduces to

$$\sigma_{xx, x} + \sigma_{zz, z} = 0, \quad \sigma_{zz, x} + \sigma_{zz, z} = 0, \quad D_{x, x} + D_{x, z} = 0 \quad (2.163)$$

in which for simplicity all body forces and the electric charge density are assumed to be zero.

The basic idea behind the state space formulation is to describe a given physical system in terms of the minimum possible number of variables. Sosa and Castro achieved this by eliminating $\sigma_{xx}$ and $D_{x, x}$ from Eqs. (1.2), (1.24), and (2.163), presenting the following system of differential equations:

$$u_x = -w_x + \left(\sigma_{xx} - e_{33} \phi_x\right)/c_{33},$$
$$w_x = (-\alpha u_x + \kappa_{33} \sigma_{zz} + e_{33} D_{x, z})/\gamma,$$
$$\sigma_{zz, x} = -\sigma_{zz, x, x},$$
$$\sigma_{zz, z} = \left(\frac{c_{33}^2 - c_{11}}{c_{33}} - \frac{\beta^2}{\gamma c_{33}}\right) u_{xx} + \left(\frac{\beta e_{33}}{\gamma c_{33}} - \frac{\epsilon_{15}}{c_{33}}\right) \sigma_{zz, x} - \frac{\beta}{\gamma} D_{x, z}, \quad (2.164)$$
$$\phi_x = (-\beta u_x + e_{33} \sigma_{zz} - e_{33} D_{x, z})/\gamma,$$
$$D_{x, z} = (-\epsilon_{15} \sigma_{zz, x} + \kappa \phi_{x, x})/c_{55}$$

where

$$\alpha = c_{33} \kappa_{33} + e_{31} e_{33}, \quad \beta = e_{33} e_{33} - c_{33} e_{31},$$
$$\gamma = c_{33} \kappa_{33} + e_{33}^2, \quad \kappa = c_{55} \kappa_{33} + e_{15}^2 \quad (2.165)$$
They then applied the Fourier transform (2.43) to Eq. (2.164), yielding

\[
\begin{bmatrix}
\hat{u} \\
\hat{w} \\
\hat{\sigma}_{zz} \\
\hat{\phi} \\
\hat{D}_z
\end{bmatrix} = \begin{bmatrix}
0 & i\xi & 0 & c_{35}^{-1} & e_{15}c_{35}^{-1}i\xi & 0 \\
\alpha i\xi / \gamma & 0 & \kappa_{33} / \gamma & 0 & 0 & e_{33} / \gamma \\
0 & 0 & 0 & i\xi & 0 & 0 \\
\beta i\xi / \gamma & 0 & e_{33} / \gamma & 0 & 0 & -c_{33} / \gamma \\
0 & 0 & 0 & e_{15}c_{35}^{-1}i\xi & -\kappa c_{35}^{-1}e_{15} & 0
\end{bmatrix} \begin{bmatrix}
\hat{u} \\
\hat{w} \\
\hat{\sigma}_{zz} \\
\hat{\phi} \\
\hat{D}_z
\end{bmatrix}
\]

(2.166)

in which the assumptions are made that quantities \(u, u_x, w, \sigma_{zz}, \sigma_{z\zeta}, \phi, \phi_z, \) and \(D_z\) tend to zero as \(|x| \to \infty\), and

\[
a_{41} = c_{11} + \frac{\beta^2}{\gamma c_{33}} - \frac{c_{13}^2}{c_{33}}, \quad a_{41} = \frac{c_{13}}{c_{33}} - \frac{\beta e_{33}}{\gamma c_{33}}
\]

(2.167)

Introducing the transformed state vector, \(\tilde{S}(\xi,z) = \begin{bmatrix} \hat{u} \quad \hat{w} \quad \hat{\sigma}_{zz} \quad \hat{\phi} \quad \hat{D}_z \end{bmatrix}^T\), Eq. (2.166) becomes

\[
\frac{d\tilde{S}(\xi,z)}{dz} = A(\xi)\tilde{S}(\xi,z)
\]

(2.168)

where \(A\) is a 6x6 matrix appearing in Eq. (2.166), whose only feature is having zeros in its main diagonal. The solution to Eq. (2.168) is given by [22]

\[
\tilde{S}(\xi,z) = \exp[zA(\xi)]\tilde{S}(\xi,0)
\]

(2.169)

in which the exponential matrix is the transfer matrix that propagates the initial transformed state vector on the bounding surface into the field at depth \(z\). Consequently, the remaining task is to evaluate the transfer matrix \(\exp[zA]\), explicitly. Sosa obtained the solution by the following two steps:

1. The eigenvalues \(\lambda\) of \(A\) are found from the associated characteristic equation:

\[
\lambda^6 + p\xi^2\lambda^4 + q\xi^4\lambda^2 + r\xi^6 = 0
\]

(2.170)

where the coefficients \(p, q,\) and \(r,\) as functions of the material properties, are given by

\[
p = \frac{1}{\gamma c_{33}} \left[ ac_{35} + \frac{\beta}{c_{33}} (2c_{33}e_{13} - c_{33}c_{55}) - \beta + \frac{\gamma}{c_{33}} (c_{13}c_{35} + c_{13}^2 - c_{13}c_{33}) - \kappa c_{33} \right],
\]

\[
q = \frac{1}{\gamma c_{55}} \left[ \frac{c_{33}e_{11} - c_{13}^2}{c_{33}} \left( c_{55}K_{33} + 2e_{13}c_{33} + \frac{\kappa}{c_{55}} - \frac{e_{13}^2c_{33}}{c_{55}} - \kappa c_{33} + \frac{c_{13}}{c_{33}} (ac_{35} + \beta e_{33}) \right) \right]
\]

Sosa obtained the solution by the following two steps:
2.8 State space formulation

\[
\begin{align*}
-\frac{\kappa}{\gamma}(ae_{33} + \beta e_{33}) + \frac{\alpha \beta}{\gamma}(e_{15} - \frac{e_{33}e_{55}}{e_{33}}) + \frac{\beta^2}{\gamma e_{33}}(e_{55}k_{33} + \epsilon_{15}e_{33})
\end{align*}
\]

\[
\begin{align*}
r = \frac{\kappa}{\gamma} e_{55} \left[ (e_{13}^3 - e_{33}e_{11}) \left( \frac{e_{55}^2}{e_{33}} + \frac{e_{13}^2}{e_{33}} - \alpha e_{13} - \frac{\beta e_{33}e_{13}}{e_{33}} \right) \right]
\end{align*}
\]

(2.171)

Sosa indicated that the roots of Eq. (2.170) can be found analytically. They can always be expressed in the following form:

\[
\lambda_{i,1} = \pm a|\varepsilon| \\ \lambda_{i,2,5} = \pm (b + ic)|\varepsilon| \\ \lambda_{i,6} = \pm (b - ic)|\varepsilon|
\]

(2.172)

where \(a, b,\) and \(c\) are real numbers depending on the material properties.

(2) The matrix exponential is expanded into a matrix polynomial as

\[
\exp[zA] = a_0I + a_1A + a_2A^2 + a_3A^3 + a_4A^4 + a_5A^5
\]

(2.173)

where no higher powers of \(A\) are needed on account of the Cayley-Hamilton theorem, namely,

\[
A^6 + pA^2 + qA + rI = 0
\]

(2.174)

The coefficients \(a_0, \ldots, a_5\) in Eq. (2.173) are determined in terms of the eigenvalues of \(A\) by noting that each \(\lambda\) satisfies

\[
\exp[z\lambda] = a_0 + a_1\lambda + a_2\lambda^2 + a_3\lambda^3 + a_4\lambda^4 + a_5\lambda^5
\]

(2.175)

Using Eq. (2.175) six times, each for each eigenvalue, generates an algebraic system of six equations with unknowns \(a_0, \ldots, a_5\), whose solution is written as

\[
a_i = \frac{1}{2} \sum_{j=0}^{5} A_{ij} \left[ e^{\lambda_j z} - (-1)^i e^{-\lambda_j z} \right] \quad (i = 1 - 5)
\]

(2.176)

where

\[
A_{ik} = \sum_{j=k+1}^{k+5} \frac{\lambda_j^2}{d_k}, \\ A_{ik} = \prod_{j=k+1}^{k+5} \frac{\lambda_j}{d_k}, \\ A_{2k} = \sum_{j=k+1}^{k+5} \frac{\lambda_j^2}{d_k}, \\ A_{3k} = \sum_{j=k+1}^{k+5} \frac{\lambda_j^3}{d_k}, \\ A_{4k} = \frac{1}{d_k}, \\ A_{5k} = \frac{1}{\lambda_k d_k}, \\ d_k = (\lambda_k^2 - \lambda_k^2)(\lambda_k^2 - \lambda_k^2) \quad (k = 1, 2, 3)
\]

(2.177)

Knowledge of the eigenvalues and, therefore, \(a_i\) from Eq. (2.176), together with the various powers of \(A\) provides the complete determination of the exponential matrix \(\exp[zA]\). Letting the exponential matrix be denoted by \(B(M, \xi, z)\), where the argument \(M\) emphasizes the dependence on the various material constants, one can write Eq. (2.169) as
Thus, Eq. (2.178) gives the state vector consisting of the transformed stresses, displacements, electric potential, and electric displacement at an arbitrary depth \( z \) in the solution domain. Finally, solution (2.178) must be inverted to find the physical variables. Finding the inverse Fourier transform of Eq. (2.43) depends heavily on the problem under consideration. Sosa and Castro in [23] presented a detailed illustration of how to conduct the inverse Fourier transform of Eq. (2.178).

In this chapter, we have briefly introduced techniques of potential function, solution with Lekhnitskii formalism, techniques of Fourier transformation, Trefftz finite element method, integral equations, shear-lag model, symplectic mechanics, and state space method, which are all used in later chapters.

References


[13] Qin QH: Solving anti-plane problems of piezoelectric materials by the Trefftz finite


