Chapter 3 Green’s function for thermoelectroelastic problems

3.1 Introduction
In the previous chapter we described Green’s functions of piezoelectric material without or with defects such as cracks, holes, bimaterial interface, half-plane boundary, and inclusions. However, the widespread use of piezoelectric materials in structural applications has generated renewed interest in thermoelastic behaviour. In particular, information about thermal stress concentrations around material or geometrical defects in piezoelectric solids has wide application in composite structures. Unlike in the cases of anisotropic elasticity and piezoelectricity, relatively little work has been reported regarding Green’s functions in thermoelastic and thermopiezoelectric solids. Using Stroh’s formalism and one-to-one mapping methods, Qin [1] obtained Green’s functions in closed form for an infinite piezoelectric plate with an elliptic hole induced by temperature discontinuity. Qin and Mai [2,3] also investigated thermoelectroelastic Green’s functions for half-plane and bimaterial problems. The studies in [4-6] further investigated thermoelectroelastic Green’s functions for piezoelectric materials with various openings or an elliptic inclusion.

In this chapter, we begin with a discussion of Green’s functions in free space of thermopiezoelectricity. The results are then extended to cases of infinite thermopiezoelectricity with a half-plane boundary, bimaterial interface, holes of various shapes, and an elliptic inclusion.

3.2 Green’s function in free space

3.2.1 General solution in Stroh formalism
Consider an anisotropic piezoelectric solid in which all fields are assumed to depend on in-plane coordinates $x_1$ and $x_2$ only. The general solutions of stress and electric displacement (SED) $\mathbf{\Pi}$, elastic displacement and electric potential (EDEP) vector $\mathbf{U}$, temperature $T$, and heat flux $h_i$ in the solid can be expressed in terms of complex analytic functions as follows [3]:

$$
T = 2 \text{Re}[g(z_i)], \quad h_i = -\vartheta_2, \quad h_2 = \vartheta_1,
$$

$$
\mathbf{U} = 2 \text{Re}[\mathbf{A} \langle f(z_\alpha) \rangle \mathbf{q} + \mathbf{c} g(z_\alpha)], \quad \mathbf{\Pi}_i = -\varphi_2, \quad \mathbf{\Pi}_2 = \varphi_1
$$

(3.1)

where

$$
\vartheta = 2k \text{Im}[g'(z_i)], \quad \varphi = 2 \text{Re}[\mathbf{B} \langle f(z_\alpha) \rangle \mathbf{d} g(z_\alpha)],
$$

$$
z_i = x_i + p_i x_2, \quad z_\alpha = x_i + p_\alpha x_2
$$

(3.2)

where $\vartheta$ is the heat flux function, $p_i$ represents the eigenvalue of heat conduction equation [7], $\langle f(z_\alpha) \rangle$ is defined in Eq (1.131), but it is a 4×4 matrix in this chapter rather than the original 5×5 matrix, an overbar denotes the complex conjugate, a prime represents the differentiation with respect to the argument, $\mathbf{q}$ is a constant vector to be determined by the boundary conditions, $i = \sqrt{-1}$, $k = \sqrt{k_{11}k_{22} - k_{12}^2}$, $k_{ij}$ are the coefficients of heat conduction, and $f(z_i)$ and $g(z_\alpha)$ are arbitrary functions with complex arguments $z_i$ and $z_\alpha$, respectively. $\mathbf{A}$, $\mathbf{B}$, $\mathbf{c}$, and $\mathbf{d}$ are associated with material constants and are well defined in the literature (see [7], for example).

3.2.2 Green’s function in free space
For an infinite space subjected to a line heat source \( h^* \) and a line temperature discontinuity \( \hat{T} \) both located at \( z_{00} = (x_{10} + p_{1}x_{20}) \), the function \( g'(z_i) \) can be chosen in the form [3]

\[
g'(z_i) = q_0 \ln(z_i - z_{10}) \quad (3.3)
\]

where \( q_0 \) is a complex constant which can be determined from the conditions

\[
\int_C dT = \hat{T} \quad \text{for any closed curve } C \text{ enclosing the point } z_{10},
\]

and

\[
\int_C d\vartheta = -h^* \quad \text{for any closed curve } C \text{ enclosing the point } z_{10}
\]

Substitution of Eq (3.3) into Eqs (3.1) and (3.2), and later into Eqs (3.4) and (3.5), yields

\[
q_0 = \frac{\hat{T}}{4\pi i} - \frac{h^*}{4\pi k} \quad (3.6)
\]

The function \( g(z_i) \) in Eq (3.1) or (3.2) can be obtained by integrating Eq (3.3) with respect to \( z_i \), which yields

\[
g(z_i) = f^*(z_i - z_{10}) q_0 \quad (3.7)
\]

where \( f^*(x) = x(\ln x - 1) \).

Substituting Eq (3.7) into Eqs (3.1) and (3.2) and noting that \( q = 0 \) for the current thermal problem, the Green’s function for electroelastic field in free space is derived as

\[
U = 2 \text{ Re}[g^*(z_i - z_{10})q_0], \quad \varphi = 2 \text{ Re}[df^*(z_i - z_{10})q_0] \quad (3.8)
\]

### 3.3 Half-plane solid

In this section the Green’s function in a half-plane piezoelectric solid subjected to loadings \( h^* \) and \( \hat{T} \), both located at \( z_{00} = (x_{10} + p_{1}x_{20}) \), are presented based on the concept of perturbation [8].

#### 3.3.1 Green’s function for thermal fields

If the infinite straight boundary of the half-plane solid is in the thermally insulated condition, we have

\[
\vartheta = 0 \quad (3.9)
\]

To satisfy the thermally insulated condition (3.9), the general solution of the thermal field can be assumed in the form from the concept of perturbation [8],

\[
T = 2 \text{ Re}[f_0(z_i) + f_1(z_i)], \quad \vartheta = 2k \text{ Im}[f_0(z_i) + f_1(z_i)] \quad (3.10)
\]

The function \( f_0 \) in Eq (3.10) can be chosen to represent the solutions associated with the unperturbed thermal fields (or the solution for an infinite homogeneous solid), and \( f_1 \) is the function corresponding to the perturbed field of the solid due to defects (a half-plane boundary in this Section). \( f_1 = 0 \) if there is no defect in a homogeneous solid. Therefore, \( f_0 \) is given by Eq (3.3) as

\[
f_0(z_i) = g'(z_i) = q_0 \ln(z_i - z_{10}) \quad (3.11)
\]
Making use of Eq (3.10)₂, the thermally insulated condition on the surface \( x_2 = 0 \), i.e. \( \vartheta = 0 \), provides

\[
f_i(z_i) = \overline{q}_0 \ln(z_i - \overline{z}_{i0})
\]

Further, if \( T = 0 \), rather than \( \vartheta = 0 \), on the surface \( x_2 = 0 \), we have

\[
f_i(z_i) = -\overline{q}_0 \ln(z_i - \overline{z}_{i0})
\]

Having obtained the solutions of \( f_0 \) and \( f_1 \), the function \( g'(z_i) \) can now be written as

\[
g'(z_i) = q_0 \ln(z_i - z_{i0}) \pm \overline{q}_0 \ln(z_i - \overline{z}_{i0})
\]

where the symbol “+” before \( \overline{q}_0 \) stands for the condition \( \vartheta = 0 \) on the \( x_1 \)-axis, and “-” stands for the condition \( T = 0 \) on \( x_2 = 0 \). For the sake of clarity, we always use the symbol “+” in the following derivation. It should be understood that the symbol “+” is replaced by “-” if the boundary condition \( T = 0 \) on \( x_2 = 0 \) is involved. Substitution of Eq (3.14) into Eq (3.10) yields the Green’s functions of temperature and heat flux:

\[
T(z_i) = 2 \text{Re}\{q_0 \ln(z_i - z_{i0}) + \overline{q}_0 \ln(z_i - \overline{z}_{i0})\},
\]

\[
\vartheta(z_i) = 2k \text{Im}\{q_0 \ln(z_i - z_{i0}) + \overline{q}_0 \ln(z_i - \overline{z}_{i0})\}
\]

3.3.2 Green’s function for electroelastic fields

For linear thermopiezoelectric problems, the solution (3.2)₂ can be assumed to consist of a particular part \( \varphi_p \), attributable to thermal loading, and a modified part \( \varphi_m \), to ensure satisfaction of the half-plane boundary condition, as \( \varphi = \varphi_m + \varphi_p \). \( \varphi_p \) can be determined from the function \( g(z_i) \) defined Eq (3.14). Therefore, the remaining task is to find the modified part \( \varphi_m \), which is related to \( \varphi_p \) through the traction-charge condition on the boundary \( x_2 = 0 \):

\[
\varphi = \varphi_p + \varphi_m = 0 \quad \text{on} \quad x_2 = 0
\]

On the basis of the expressions (3.1)₄ and (3.2)₂, the particular solutions \( U_p \) and \( \varphi_p \) for electroelastic fields can be written as

\[
U_p = 2 \text{Re}\{c g(z_i)\}, \quad \varphi_p = 2 \text{Re}\{d g(z_i)\}
\]

where subscript “p” refers to the particular solution. The function \( g(z_i) \) in Eq (3.18) can be obtained by integrating Eq (3.14) with respect to \( z_i \), which yields

\[
g(z_i) = q_0 f^+(z_i - z_{i0}) + \overline{q}_0 f^+(z_i - \overline{z}_{i0})
\]

The particular solution \( \varphi_p \) induced by \( g(z_i) \) [see Eq (3.18)] does not, generally, satisfy the traction-charge free condition (3.17) along the axis \( x_2 = 0 \). We therefore need to find a modified isothermal solution \( \varphi_m \) for a given problem such that when it is superposed on the particular electroelastic solution the traction-charge free condition along the axis \( x_2 = 0 \) will be satisfied. Owing to the fact that \( f(z_k) \) and \( g(z_i) \) have the same rule affecting the traction-charge in Eq (3.2)₂, possible function forms come from the partition of \( g(z_i) \). This is
\[ \phi_m = 2 \text{Re}[\mathcal{B}(f(z_0))q] \]  

where

\[ f(z_0) = q_0 f^*(z_{01}) + \overline{q_0} f^*(z_{02}) \]  

Substitution of Eqs (3.18) and (3.20) into Eq (3.17) yields

\[ q = -\mathbf{B}^{-1} \mathbf{d} \]  

Likewise, if \( U = 0 \) on \( x_2 = 0 \), then

\[ q = -\mathbf{A}^{-1} \mathbf{c} \]  

Thus, the resulting Green’s function for the half-plane problem can be written as

\[ T = 2 \text{Re}\{q_0 \ln y + \overline{q_0} \ln y^*\}, \]  

\[ U = 2 \text{Re}\{-\mathbf{A} \langle f(z_0) \rangle \mathbf{B}^{-1} \mathbf{d} + \mathbf{c}[q_0 f^*(y) + \overline{q_0} f^*(y^*)]\}, \]  

\[ \varphi = 2 \text{Re}\{-\mathbf{B} \langle f(z_0) \rangle \mathbf{B}^{-1} \mathbf{d} + \mathbf{d}[q_0 f^*(y) + \overline{q_0} f^*(y^*)]\} \]

where \( f(z_0) \) is defined by Eq (3.21) and

\[ y_i = z_i - z_{01}, \quad y_i^* = z_i - \overline{z}_{02} \]  

3.4 Bimaterial solid

Consider a bimaterial solid in which the upper half-plane \( (x_2 > 0) \) is occupied by material 1 and the lower half-plane \( (x_2 < 0) \) by material 2 (see Fig. 2.5). They are rigidly bonded together so that

\[ T^{(1)} = T^{(2)}, \quad \vartheta^{(1)} = \vartheta^{(2)}, \quad \text{at} \quad x_2 = 0 \]  

\[ U^{(1)} = U^{(2)}, \quad \varphi^{(1)} = \varphi^{(2)}, \quad \text{at} \quad x_2 = 0 \]

where the superscripts (1) and (2) label the quantities relating to materials 1 and 2, respectively. In the rest of this section we present Green’s functions of the bimaterial solid described above due to a line heat source \( h^* \) and a line temperature discontinuity \( \hat{T} \) both located at \( z_{i0}^{(1)} = x_{i0} + p_i^{(1)} x_{20} \) in material 1.

3.4.1 Green’s function for temperature fields

On the basis of the concept of perturbation given by Stagni [8], the general solution for the bimaterial solid can be assumed in the form [7]

\[ T^{(1)} = 2 \text{Re}[f_0(z_{i1}^{(1)}) + f_1(z_{i1}^{(1)})], \quad \vartheta^{(1)} = 2k^{(1)} \text{Im}[f_0(z_{i1}^{(1)}) + f_1(z_{i1}^{(1)})], \quad x_2 > 0, \]  

\[ T^{(2)} = 2 \text{Re}[f_2(z_{i2}^{(2)})], \quad \vartheta^{(2)} = 2k^{(2)} \text{Im}[f_2(z_{i2}^{(2)})], \quad x_2 < 0 \]

where the function \( f_0 \) is defined by Eq (3.11), \( f_1 \) and \( f_2 \) are functions corresponding to the perturbed field of the solid due to the existence of the bimaterial interface, and

\[ z_{i1}^{(1)} = x_i + p_i^{(1)} x_2, \quad z_{i2}^{(2)} = x_i + p_i^{(2)} x_2, \]

\[ k^{(1)} = \sqrt{k_{11}^{(1)} k_{22}^{(1)} - (k_{12}^{(1)})^2}, \quad k^{(2)} = \sqrt{k_{11}^{(2)} k_{22}^{(2)} - (k_{12}^{(2)})^2} \]
To satisfy the interface conditions (3.28), the functions $f_1$ and $f_2$ can be assumed in the form

$$ f_1(z_i^{(1)}) = q_i \ln y_1^{(1)} , \quad f_2(z_i^{(2)}) = q_2 \ln y_1^{(2)} $$  (3.33)

where $y_1^{(1)} = z_i^{(1)} - \tau_1^{(1)}$, $y_1^{(2)} = z_i^{(2)} - \tau_1^{(2)}$, and $q_1$ and $q_2$ are two constants to be determined. Substitution of Eq (3.33) into Eqs (3.30) and (3.31), and later into Eq (3.28), yields

$$ q_i = -b_i \overline{y}_0, \quad q_2 = b_2 q_0, \quad b_1 = \frac{k^{(2)} - k^{(1)}}{k^{(2)} + k^{(1)}}, \quad b_2 = \frac{2k^{(1)}}{k^{(2)} + k^{(1)}} $$  (3.34)

Therefore the Green’s function for thermal fields can be written as

$$ T^{(1)} = 2 \text{Re}[q_0 \ln(z_i^{(1)} - \tau_1^{(1)}) + q_1 \ln y_1^{(1)}], \quad x_2 > 0 $$  (3.35)

$$ g^{(1)} = 2k^{(1)} \text{Im}[q_0 \ln(z_i^{(1)} - \tau_1^{(1)}) + q_1 \ln y_1^{(1)}], $$

$$ T^{(2)} = 2 \text{Re}[q_2 \ln y_1^{(2)}], \quad g^{(2)} = 2k^{(2)} \text{Im}[q_2 \ln y_1^{(2)}], \quad x_2 < 0 $$  (3.36)

### 3.4.2 Green’s function for electroelastic fields

To use the interfacial condition (3.29) we first consider the particular solution due to the thermal field. This can be done by substituting Eq (3.33) into Eqs (3.30) and (3.31), and then integrating the result with respect to $z$, yielding the function $g(z)$, which is required when determining $U$ and $\phi$ in Eqs (3.1) and (3.2), in the form

$$ g_i(z_i^{(1)}) = q_0 f^*(y_1^{(1)}) - b_i \overline{y}_0 f^*(y_2^{(1)}) $$  (3.37)

for $x_2 > 0$, and

$$ g_i(z_i^{(2)}) = b_i q_0 f^*(y_2^{(1)}) $$  (3.38)

for $x_2 < 0$, where $y_1^{(1)} = z_i^{(1)} - \tau_1^{(1)}$. The particular solution for electroelastic fields can then be given by

$$ U_p^{(1)}(z_i^{(1)}) = 2 \text{Re}[\mathbf{c}^0_i g_i(z_i^{(1)})] = 2 \text{Re}[\mathbf{c}^0_i q_0 f^*(y_1^{(1)}) - b_i \overline{y}_0 f^*(y_2^{(1)})], $$  (3.39)

$$ \phi_p^{(1)}(z_i^{(1)}) = 2 \text{Re}[\mathbf{d}^0_i g_i(z_i^{(1)})] = 2 \text{Re}[\mathbf{d}^0_i q_0 f^*(y_1^{(1)}) - b_i \overline{y}_0 f^*(y_2^{(1)})] $$  (3.40)

for $x_2 > 0$, and

$$ U_p^{(2)}(z_i^{(2)}) = 2 \text{Re}[\mathbf{c}^0_i g_i(z_i^{(2)})] = 2b_2 \text{Re}[\mathbf{c}^0_i q_0 f^*(y_1^{(2)})], $$  (3.41)

$$ \phi_p^{(2)}(z_i^{(2)}) = 2 \text{Re}[\mathbf{d}^0_i g_i(z_i^{(2)})] = 2b_2 \text{Re}[\mathbf{d}^0_i q_0 f^*(y_1^{(2)})] $$  (3.42)

for $x_2 < 0$. The solutions (3.39)-(3.42) do not generally satisfy the interface condition (3.29). Therefore, development of a modified solution is needed such that when it is superposed on the particular solutions (3.39)-(3.42) the interface condition (3.29) will be satisfied.

Owing to the fact that $f(z)$ and $g(z)$ have the same rule affecting $U$ and $\phi$ in Eqs (3.1) and (3.2), possible function forms come from the partition of solution $g(z)$. This is
\[ \left\langle f(z^{(i)}_a) \right\rangle = \text{diag}[f(y^{(i)}_1), f(y^{(i)}_2), f(y^{(i)}_3), f(y^{(i)}_4)] \] (3.43)

where
\[ y^{(j)}_i = z^{(j)} - z^{(j)}_{i0}, \quad i=1-4, \quad j=1, 2 \] (3.44)

Thus the resulting expressions of \( U^{(i)} \) and \( \varphi^{(i)} \) can be given as
\[ U^{(i)} = 2 \text{Re}\{A^{(i)} \left\langle f(z^{(i)}_a) \right\rangle q_i + c^{(i)}[q_0 f(y^{(i)}_1) - b \bar{q}_0 f(y^{(i)}_2)]\}, \] (3.45)
\[ \varphi^{(i)} = 2 \text{Re}\{B^{(i)} \left\langle f(z^{(i)}_a) \right\rangle q_i + d^{(i)}[q_0 f(y^{(i)}_1) - b \bar{q}_0 f(y^{(i)}_2)]\} \] (3.46)

for \( x_2 > 0 \), and
\[ U^{(2)} = 2 \text{Re}[A^{(2)} \left\langle f(z^{(2)}_a) \right\rangle q_2 + b_2 q_0 c^{(2)} f(y^{(2)}_1)], \] (3.47)
\[ \varphi^{(2)} = 2 \text{Re}[B^{(2)} \left\langle f(z^{(2)}_a) \right\rangle q_2 + b_2 q_0 d^{(2)} f(y^{(2)}_1)] \] (3.48)

for \( x_2 < 0 \). The substitution of Eqs (3.45)-(3.48) into Eq (3.29) yields
\[ q_1 = q_0[B^{(1)} - B^{(2)} A^{(2)^{-1}} A^{(1)}]^{-1} \times \{[b_2 d^{(2)} + b_1 \bar{d}^{(1)} - d^{(1)}] \]
\[ - B^{(2)} A^{(2)^{-1}}[b_2 c^{(2)} + b_1 \bar{c}^{(1)} - c^{(1)}] \}, \] (3.49)
\[ q_2 = q_0[B^{(1)} A^{(1)^{-1}} A^{(2)} - B^{(2)^{-1}} A^{(1)^{-1}}]^{-1} \times \{[b_2 d^{(2)} + b_1 \bar{d}^{(1)} - d^{(1)}] \]
\[ - B^{(1)} A^{(1)^{-1}}[b_2 c^{(2)} + b_1 \bar{c}^{(1)} - c^{(1)}] \} \] (3.50)

Thus, the thermoelectroelastic Green’s functions can be obtained by substituting Eqs (3.49) and (3.50) into Eqs (3.45)-(3.48).

3.5 Elliptic hole problems

3.5.1 Green’s function of thermal field for insulated hole problems

Consider an infinite piezoelectric plate containing an elliptic hole subjected to loadings \( h^* \) and \( \hat{T} \) at \( z_{t0} \), as shown in Fig. 2.7 (it is noted that the loading vector here is \( (h^*, \hat{T}) \) rather than \( (q_0, b) \)). If the hole is thermally insulated, the boundary conditions can be written as
\[ \vartheta = 0, \quad \text{at infinity}, \] (3.51)
\[ h_t \to 0, \quad \text{at infinity}, \] (3.52)
in addition to the balance conditions (3.4) and (3.5).

The geometric equation of the elliptic hole \( \Gamma \) (see Fig. 2.7) is still defined by Eq (2.161), and the mapping function is given by Eq (2.162). Following the treatment in Section 2.8, a suitable function satisfying the boundary conditions (3.51) and (3.52) can be given in the form [7]:
\[ \vartheta = 2k \text{Im}[g'(z_t)] = 2k \text{Im}[q_0 \ln(z_t - \zeta_{t0}) + q_t \ln(\zeta_t^{-1} - \bar{\zeta}_{t0})] \] (3.53)

where \( \zeta_t \) and \( \zeta_{t0} \) are related to \( z_t \) and \( z_{t0} \) by
\[ z_t = c_t \zeta_t + d_t \zeta_t^{-1}, \quad z_{t0} = c_t \zeta_{t0} + d_t \zeta_{t0}^{-1} \] (3.54)

with
\[ c_1 = (a + i p_1 b) / 2, \quad d_1 = (a + i p_1 b) / 2 \] (3.55)

It should be mentioned that the function \( \ln(\zeta_i - \overline{\zeta}_{00}) \) is single valued since \( \zeta_i^{-1} \) is always located inside the circle and \( \overline{\zeta}_{00} \) is a point outside the unit circle. When \( x_1 \) and \( x_2 \) are given by Eq (2.161), \( \zeta_i = e^{i \theta} \), and angle \( \theta \) is defined in Fig. 2.7. Hence, the substitution of Eq (3.53) into Eq (3.51) yields

\[ \theta = 2k \text{Im}[g_z(z_i)] = 2k \text{Im}[q_0 \ln(e^{i \theta} - \overline{\zeta}_{00}) + q_i \ln(e^{-i \theta} - \overline{\zeta}_{00})] = 0 \] (3.56)

The first term in Eq (3.56) can be replaced by the negative of its complex conjugate, that is,

\[ \text{Im}[q_0 \ln(e^{i \theta} - \overline{\zeta}_{00})] = - \text{Im}[\overline{q}_0 \ln(e^{-i \theta} - \overline{\zeta}_{00})] \] (3.57)

Substituting Eq (3.57) into (3.56), we have

\[ q_i = \overline{q}_0 \] (3.58)

### 3.5.2 Green’s functions for electroelastic fields

Noting Eqs (3.1) and (3.2), the particular solution for the electroelastic field induced by the line heat source \( h^* \) and the line temperature discontinuity \( \hat{T} \) can be written as

\[ u_p = 2 \text{Re}[e g(z_i)] \quad \psi_p = 2 \text{Re}[d g(z_i)] \] (3.59)

The function \( g(z_i) \) in Eq (3.59) above can be obtained by integrating \( g_z(z_i) \) in Eq (3.53) with respect to \( z_0 \), which yields

\[ g(z_i) = q_0 \{ c_1 F_1(\zeta_i) + d_1 F_2(\zeta_i) \} + \overline{q}_0 \{ c_2 F_4(\overline{\zeta}_i) + d_2 F_3(\overline{\zeta}_i) \} \] (3.60)

where

\[ F_1(s) = (s - \overline{\zeta}_{00})[\ln(s - \overline{\zeta}_{00}) - 1], \quad F_2(s) = (s^{-1} - \overline{\zeta}_{00}^{-1}) \ln(s - \overline{\zeta}_{00}) + \overline{\zeta}_{00}^{-1} \ln s, \]
\[ F_3(s) = (s^{-1} - \overline{\zeta}_{00}^{-1})[\ln(s^{-1} - \overline{\zeta}_{00}^{-1}) - 1], \quad F_4(s) = (s - \overline{\zeta}_{00}^{-1}) \ln(s^{-1} - \overline{\zeta}_{00}^{-1}) - \overline{\zeta}_{00}^{-1} \ln s \] (3.61)

For a zero traction-charge hole, the electroelastic boundary condition will be satisfied if

\[ \psi = 0 \] (3.62)

To satisfy condition (3.62) along the hole boundary, possible function forms of \( f_z(z_\alpha) \) come from the partition of \( g(z_i) \), for the same reason as that stated for Eq (3.43). They are

\[ f_i(z_\alpha) = F_i(\zeta_\alpha), \quad (i=1-4) \] (3.63)

where \( \zeta_\alpha \) is related to \( z_\alpha \) by Eq (2.162).

The Green’s functions for the electroelastic field can thus be chosen as

\[ U = 2 \text{Re} \sum_{k=1}^{4} [A f_k(\zeta_\alpha) q_k + e g(\zeta_\alpha)] \] (3.64)
\[ \psi = 2 \text{Re} \sum_{k=1}^{4} [B f_k(\zeta_\alpha) q_k + d g(\zeta_\alpha)] \] (3.65)
Noting that \( \zeta_\alpha = \zeta_\tau = e^{i\theta} \) on the hole boundary, we have
\[
q_i = -q_0 c_i B^{-1} d, \quad q_2 = -q_0 c_i B^{-1} d, \quad q_3 = -\overline{q_0} c_i B^{-1} d, \quad q_4 = -\overline{q_0} d_i B^{-1} d
\]
Substituting Eq (3.66) into Eqs (3.64) and (3.65), the Green’s functions can then be further written as
\[
\begin{align*}
U &= -2 \text{Re}[A] q_0 c_i \{ F_1(\zeta_\alpha) \} + q_0 d_i \{ F_2(\zeta_\alpha) \} + \overline{q_0} c_i \{ F_4(\zeta_\alpha) \} + \overline{q_0} d_i \{ F_3(\zeta_\alpha) \}] B^{-1} d \\
\varphi &= -2 \text{Re}[B] q_0 c_i \{ F_1(\zeta_\alpha) \} + q_0 d_i \{ F_2(\zeta_\alpha) \} + \overline{q_0} c_i \{ F_4(\zeta_\alpha) \} + \overline{q_0} d_i \{ F_3(\zeta_\alpha) \}] B^{-1} d
\end{align*}
\]
\[(3.67)\]
\[(3.68)\]

3.6 Arbitrarily shaped hole problems

In Section 2.9, applications of Green’s function to a piezoelectric plate containing an arbitrarily shaped hole were presented. Extension of the procedure to include thermal effects is described in this section. Green’s functions in closed form for an infinite thermopiezoelectric plate with various holes induced by thermal loads are derived using Stroh formalism. The loads may be a point heat source or a temperature discontinuity.

3.6.1 Basic equations

As was treated in Section 2.9, we consider a plane strain analysis where the material is transversely isotropic and where coupling occurs between in-plane stresses and in-plane electric fields. For a Cartesian coordinate system \( Oxyz \), choose the \( z \)-axis as the poling direction, and denote the coordinates \( x \) and \( z \) by \( x_1 \) and \( x_2 \) in order to obtain a compacted notation. The plane strain constitutive equations are expressed in matrix form as
\[
\begin{align*}
\sigma_{11} &= c_{11} e_{11} + 2 c_{12} e_{12} + \frac{1}{2} c_{33} e_{33} \\
\sigma_{22} &= c_{12} e_{12} + c_{22} e_{22} + \frac{1}{2} c_{33} e_{33} \\
\sigma_{12} &= 0 \\
D_1 &= e_{21} e_{22} \\
D_2 &= e_{21} e_{22} - \kappa_{11} e_{33}
\end{align*}
\]
\[
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{12} \\
D_1 \\
D_2
\end{bmatrix} =
\begin{bmatrix}
c_{11} & c_{12} & 0 & 0 & e_{21} \\
c_{12} & c_{22} & 0 & 0 & e_{22} \\
0 & 0 & c_{33} & e_{33} & 0 \\
e_{21} & e_{22} & 0 & 0 & -\kappa_{11} \\
e_{21} & e_{22} & 0 & 0 & -\kappa_{22}
\end{bmatrix}
\begin{bmatrix}
e_{11} \\
\epsilon_{22} \\
2 e_{12} \\
E_1 \\
E_2
\end{bmatrix} = -
\begin{bmatrix}
\lambda_{11} \\
\lambda_{22} \\
2 \epsilon_{12} \\
-\kappa_{11} \\
-\kappa_{22}
\end{bmatrix} T
\]
\[(3.69)\]
\[(3.70)\]

or inversely
\[
q_i = \rho g h_j,
\]
\[
\begin{align*}
e_{11} &= f_{11} f_{12} 0 0 g_{21} \\
e_{22} &= f_{12} f_{22} 0 0 g_{22} \\
2 e_{12} &= 0 0 f_{33} g_{13} 0 \\
-E_1 &= 0 0 g_{13} -\beta_{11} 0 \\
-E_2 &= g_{21} g_{22} 0 0 -\beta_{22}
\end{align*}
\]
\[
\begin{bmatrix}
e_{11} \\
\epsilon_{22} \\
2 e_{12} \\
-E_1 \\
-E_2
\end{bmatrix} =
\begin{bmatrix}
f_{11} & f_{12} & 0 & 0 & g_{21} \\
f_{12} & f_{22} & 0 & 0 & g_{22} \\
0 & 0 & f_{33} & g_{13} & 0 \\
0 & 0 & g_{13} & -\beta_{11} & 0 \\
g_{21} & g_{22} & 0 & 0 & -\beta_{22}
\end{bmatrix}
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{12} \\
D_1 \\
D_2
\end{bmatrix} +
\begin{bmatrix}
\alpha_{11} \\
\alpha_{22} \\
0 \\
0 \\
0
\end{bmatrix} T
\]
\[(3.71)\]
\[(3.72)\]

where \( \lambda_2 \) and \( \kappa_2 \) are pyroelectric constants, \( \lambda_{ij} \) and \( \alpha_{ij} \) are stress-temperature constants and thermal expansion constants respectively, \( q_i \) and \( h_i \) are heat intensity and heat flux,
$k_{ij}$ and $\rho_{ij}$ the coefficients of heat conductivity and heat resistivity.

The boundary conditions at the rim of the hole can be written as

$$\emptyset = \varphi = 0,$$  \hspace{1cm} (3.73)

if the hole is thermal-insulated, traction- and charge-free along the hole boundary. Besides, the infinite boundary condition

$$h \to 0, \ \Pi \to 0 \ \text{at} \ \infty$$  \hspace{1cm} (3.74)

and the balance conditions

$$\int_C dT = \hat{T}$$  \hspace{1cm} and \hspace{1cm} $$\int_C d\emptyset = -h^*$$ \hspace{1cm} for any closed curve $C$ enclosing the point $\zeta_{i0}$  \hspace{1cm} (3.75)

should also be satisfied.

3.6.2 Green’s function for thermal fields

Based on the one-to-one mapping described in Section 2.9 and the concept of perturbation [8], the general solution for temperature and heat-flow function can be assumed in the form

$$T = 2 \text{Re}[g'(z_t)] = 2 \text{Re}[f_0(\zeta_t) + f_1(\zeta_t)],$$  \hspace{1cm} (3.76)

$$\emptyset = 2k \text{Im}[g'(z_t)] = 2k \text{Im}[f_0(\zeta_t) + f_1(\zeta_t)]$$  \hspace{1cm} (3.77)

where $z_t$ and $\zeta_t$ are related by the mapping function

$$z_t = a(a_1, \zeta_t + a_2 \zeta_t^{-1} + e m_1 a_3 \zeta_t^m + e m_1 a_4 \zeta_t^{-m})$$  \hspace{1cm} (3.78)

with

$$a_1 = (1-i p_1 e) / 2, \quad a_2 = (1+i p_1 e) / 2,$$

$$a_3 = \gamma (1+i p_1 e) / 2, \quad a_4 = \gamma (1-i p_1 e) / 2$$  \hspace{1cm} (3.79)

For an infinite piezoelectric plate containing a hole subjected to loadings $h^*$ and $\hat{T}$ at $z_{i0}$ (see Fig. 2.9), the function $f_0$ can be chosen in the form

$$f_0(\zeta_t) = q_0 \ln(\zeta_t - \zeta_{i0})$$  \hspace{1cm} (3.80)

where $q_0$ is given in Eq (3.6).

Thus the boundary condition (3.73) requires that

$$f_1(\zeta_t) = \mathcal{G}_0 \ln(\zeta_t - \zeta_{i0})$$  \hspace{1cm} (3.81)

The function $g$ in Eq (3.2) can thus be obtained by integrating the functions $f_0$ and $f_1$ with respect to $z_t$, which leads to

$$g(z_t) = a a_1 [q_0 F_1(\zeta_t, z_{i0}) + \mathcal{G}_0 F_2(\zeta_t^{-1}, z_{i0})] + a a_2 [q_0 F_2(\zeta_t, z_{i0}) + \mathcal{G}_0 F_1(\zeta_t^{-1}, z_{i0})]$$

$$+ \mathcal{A}_2 [q_0 F_3(\zeta_t, z_{i0}) + \mathcal{G}_0 F_4(\zeta_t^{-1}, z_{i0})] + a a_4 [q_0 F_4(\zeta_t, z_{i0}) + \mathcal{G}_0 F_3(\zeta_t^{-1}, z_{i0})]$$  \hspace{1cm} (3.82)
where

\[ F_1(\zeta, \zeta_{\text{ref}}) = (\zeta - \zeta_{\text{ref}})[\ln(\zeta - \zeta_{\text{ref}}) - 1], \] (3.83)

\[ F_2(\zeta, \zeta_{\text{ref}}) = (\zeta^{-1} - \zeta_{\text{ref}}^{-1})\ln(\zeta - \zeta_{\text{ref}}) + \zeta_{\text{ref}}^{-1}\ln \zeta, \] (3.84)

\[ F_3(\zeta, \zeta_{\text{ref}}) = (\zeta^m - \zeta_{\text{ref}}^m)\ln(\zeta - \zeta_{\text{ref}}) - \zeta_{\text{ref}}^{-m}\sum_{n=1}^{m-1} \frac{1}{n} \left( \frac{\zeta}{\zeta_{\text{ref}}} \right)^n, \] (3.85)

\[ F_4(\zeta, \zeta_{\text{ref}}) = (\zeta^{-m} - \zeta_{\text{ref}}^{-m})\ln(\zeta - \zeta_{\text{ref}}) + \zeta_{\text{ref}}^{-m}\ln \zeta - \zeta_{\text{ref}}^{-m}\sum_{n=1}^{m-1} \frac{1}{n} \left( \frac{\zeta}{\zeta_{\text{ref}}} \right)^n. \] (3.86)

### 3.6.3 Green’s functions for electroelastic fields.

From Eqs (3.14) and (3.2) the particular solution for electroelastic fields induced by thermal loading can be written as

\[ U_p = 2 \Re[cg(z_r)], \quad \varphi_p = 2 \Re[\text{dg}(z_r)] \] (3.87)

To satisfy the condition (3.73) along the hole boundary, possible function forms of \( f(z_k) \) should come from the partition of \( g(z_t) \), for the same reason as that stated for Eq (3.43). They are

\[ f_1(z_k) = d[q_0F_1(\zeta_k, \zeta_{\text{ref}}) + q_0F_2(\zeta_k, \zeta_{\text{ref}}) + \bar{q}_0F_1(\zeta_k^{-1}, \zeta_{\text{ref}}^{-1}) + \bar{q}_0F_2(\zeta_k^{-1}, \zeta_{\text{ref}}^{-1})]/2 \]
\[ + \epsilon_{j1} a \epsilon_1[q_0F_3(\zeta_k, \zeta_{\text{ref}}) + q_0F_4(\zeta_k, \zeta_{\text{ref}}) + \bar{q}_0F_3(\zeta_k^{-1}, \zeta_{\text{ref}}^{-1}) + \bar{q}_0F_4(\zeta_k^{-1}, \zeta_{\text{ref}}^{-1})]/2, \] (3.88)

\[ f_2(z_k) = ip_k a \epsilon_1[q_0F_3(\zeta_k, \zeta_{\text{ref}}) - q_0F_4(\zeta_k, \zeta_{\text{ref}}) - \bar{q}_0F_3(\zeta_k^{-1}, \zeta_{\text{ref}}^{-1}) - \bar{q}_0F_4(\zeta_k^{-1}, \zeta_{\text{ref}}^{-1})]/2 \]
\[ + i\epsilon_{j1} p_k a \epsilon_1[q_0F_3(\zeta_k, \zeta_{\text{ref}}) - q_0F_4(\zeta_k, \zeta_{\text{ref}}) - \bar{q}_0F_3(\zeta_k^{-1}, \zeta_{\text{ref}}^{-1}) - \bar{q}_0F_4(\zeta_k^{-1}, \zeta_{\text{ref}}^{-1})]/2. \] (3.89)

where functions \( f_1 \) and \( f_2 \) represent two different possible functions.

The Green’s functions for electroelastic fields can thus be chosen as

\[ U = 2 \Re \left\{ \sum_{k=1}^{2} [A f_1(z_a)]q_1 + cg(z_r) \right\}, \quad \varphi = 2 \Re \left\{ \sum_{k=1}^{2} [B f_2(z_a)]q_k + \text{dg}(z_r) \right\} \] (3.90)

Substitution of Eqs (3.88) and (3.89) into Eq (3.90) and later into Eq (3.73) leads to

\[ q_1 = -B^{-1}\bar{d}, \quad q_2 = -P^{-1}B^{-1}\bar{d}p_1^*. \] (3.91)

Substituting Eq (3.91) into Eq (3.90), the Green’s functions can then be written as

\[ U = 2 \Re \left\{ -A[f_1(z_a)] + f_2(z_a)P^{-1}p_1^* \right\} B^{-1}\bar{d} + cg(z_r) \], \quad (3.92)
\[
\varphi = 2 \text{Re} \{-B[\langle f_1(z_a) \rangle + \langle f_2(z_a) \rangle P^{-1} \tilde{p}_1 B^{-1} \tilde{d} + d g(z_c)]\} \tag{3.93}
\]

### 3.7 Elliptic inclusion problems

In this section fundamental solutions for thermal loading applied outside, inside or on the surface of an elliptic piezoelectric inclusion in an infinite piezoelectric matrix are presented. By combining the method of Stroh’s formalism, the technique of one-to-one mapping, the concept of perturbation, and the method of analytical continuation, Green’s functions are derived for an elliptic piezoelectric inclusion embedded in an infinite piezoelectric matrix subjected to a line heat source \(h^\ast\) and a line temperature discontinuity \(\hat{T}\) (Fig. 3.1).

![Diagram of three loading cases](image)

**Fig. 3.1** Three loading cases: (a) outside the inclusion; (b) inside; (c) on the interface

To make the derivation tractable, consider again the one-to-one mapping defined by Eq (2.162). It will map the region outside the elliptic inclusion onto the exterior of a unit circle in the \(\zeta_k\)-plane. Further, the transformation (2.162) has been proved in Section 2.8 to be single valued outside the ellipse, since the roots of equation (2.167) are located inside the unit circle \(|\zeta_k| < 1\), where \(\zeta_k\) is related to \(z_k\) by Eq (2.162). However, the mapping (2.162) is not single valued inside the ellipse, because the roots of Eq (2.167) are located inside the unit circle, as indicated in Section 2.8.
circumvent this problem, the mapping of $\Omega_2$ (see Fig. 2.8) is done by excluding a slit, $\Gamma_0$, which represents a circle of radius $\sqrt{m_2} = \left| \frac{a_{2k}^{(2)}}{a_{1k}^{(2)}} \right|$ in the $\zeta_\kappa$-plane, from the ellipse [9]. In this case the function (2.162) will transform $\Gamma$ and $\Gamma_0$ into a ring of outer and inner circles with radii $r_{ow} = 1$ and $r_a = \sqrt{m_2}$, respectively. Besides, anywhere inside the ellipse and on the slit $\Gamma_0$ a function $f(\zeta_\kappa)$ must satisfy the condition [6]

$$f[\sqrt{m_2} \sigma(0)] = f[\sqrt{m_2} e^{2\pi i \theta} / \sigma(0)]$$

(3.94)

to ensure that the field is single valued [9], where $\sigma(0) = e^{\theta}$ stands for a point located on the unit circle in the $\zeta_\kappa$-plane, and $\theta$ is a polar angle defined in Fig. 2.7.

3.7.1 Green’s functions of thermal loading applied outside the inclusion
Consider an elliptic inclusion embedded in an infinite piezoelectric matrix subjected to loadings $h^*$ and $\hat{T}$ at a point $(x_{10}, x_{20})$ which is outside the inclusion (see Fig. 3.1a). If the inclusion and matrix are assumed to be perfectly bonded along the interface, the temperature, heat flow ($h_n$), elastic displacements, electric potential, stress and electric displacement ($\Pi_n$) across the interface should be continuous, i.e.,

$$T^{(1)} = T^{(2)}, \quad G^{(1)} = G^{(2)}, \quad U^{(1)} = U^{(2)}, \quad \Phi^{(1)} = \Phi^{(2)}, \quad (3.95)$$

along the interface.

3.7.1.1 Green’s functions for thermal fields. On the basis of the one-to-one mapping (3.54) and the concept of perturbation given by Stagni [8], the general solution for temperature and heat-flow function can be assumed in the form

$$T^{(1)} = f_{10}(\zeta_1^{(1)}) + f_0(\zeta_1^{(1)}) + f_1(\zeta_1^{(1)}) + f_{10}(\zeta_1^{(1)}) \quad \zeta_1^{(1)} \in \Omega_1, \quad (3.96)$$

$$G^{(1)} = -ik_1f_0(\zeta_1^{(1)}) + ik_1f_{00}(\zeta_1^{(1)}) - ik_1f_1(\zeta_1^{(1)}) + ik_1f_{10}(\zeta_1^{(1)})$$

$$T^{(2)} = f_{20}(\zeta_2^{(2)}) + f_2(\zeta_2^{(2)}) \quad \zeta_2^{(2)} \in \Omega_2. \quad (3.97)$$

$$G^{(2)} = -ik_2f_{20}(\zeta_2^{(2)}) + ik_2f_{20}(\zeta_2^{(2)})$$

When an infinite space is subjected to loadings $h^*$ and $\hat{T}$ at $(x_{10}, x_{20})$, the function $f_0$ can be chosen in the form

$$f_0(\zeta_1^{(1)}) = q_0 \ln(\zeta_1^{(1)} - \zeta_{10}^{(1)}) \quad (3.98)$$

where $\zeta_1^{(1)}$ and $\zeta_{10}^{(1)}$ are related to the complex arguments $z_1^{(1)}$ and $z_{10}^{(1)} = x_{10} + p_{10}^{(1)}x_{20}$ through the following transformation functions:

$$\zeta_1^{(i)} = \frac{z_1^{(i)} + \sqrt{z_1^{(i)}} - a^2 - p_{10}^{(i)}b^2}{a - ip_{10}^{(i)}b}, \quad \zeta_{10}^{(i)} = \frac{z_{10}^{(i)} + \sqrt{z_{10}^{(i)}} - a^2 - p_{10}^{(i)}b^2}{a - ip_{10}^{(i)}b}, \quad (i=1,2) \quad (3.99)$$

and $q_0$ has the same form as that defined in Eq (3.6) in which $k$ is replaced by $k^{(1)}$.

As for the function $f_2$, noting that it is holomorphic in the annular ring (see Fig. 2.8), it can be represented by Laurent’s series,
\[ f_2(\zeta_{(2)}) = \sum_{j=-\infty}^{\infty} c_j \zeta_{(2)}^j, \quad (3.100) \]

whose coefficients \( c_j \) can be related to \( c_j \) by means of Eq (3.94) in the following manner:

\[ c_{-j} = \Gamma^*_j c_j, \quad \Gamma_j^* = \left(\frac{a + ibp_{(2)}^j}{a - ibp_{(2)}^j}\right) \quad (3.101) \]

Inserting Eqs (3.98) and (3.100) into Eqs (3.96) and (3.97) and later into Eqs (3.95)(1,2) yields

\[ f_i(\sigma) + \bar{f}_0(\sigma) - \sum_{j=1}^{\infty} [\bar{c}_j + \Gamma^*_j c_j] \sigma^{-j} = \sum_{j=1}^{\infty} [c_j + \Gamma^*_j \bar{c}_j] \sigma^j - f_i(\sigma) - f_0(\sigma), \quad (3.102) \]

\[ -f_i(\sigma) + \bar{f}_0(\sigma) - \frac{k_i}{k^{(2)}} \sum_{j=1}^{\infty} [c_j - \Gamma^*_j \bar{c}_j] \sigma^{-j} = -\frac{k_i}{k^{(2)}} \sum_{j=1}^{\infty} [c_j - \Gamma^*_j \bar{c}_j] \sigma^j + f_i(\sigma) - \bar{f}_0(\sigma) \quad (3.103) \]

One of the important properties of holomorphic functions used in the method of analytic continuation is that if the function \( f(\zeta) \) is holomorphic in \( \Omega_1 \) (or \( \Omega_2 \)), then \( \bar{f}(1/\zeta) \) is holomorphic in \( \Omega_1 + \Omega_2 \) (or \( \Omega_2 \)), \( \Omega_0 \) denoting the region inside the circle of radius \( \sqrt{m} \). Hence, put

\[ \omega(\zeta) = \begin{cases} 
  f_i(\zeta) + \bar{f}_0(1/\zeta) - \sum_{j=1}^{\infty} [\bar{c}_j + \Gamma^*_j c_j] \zeta^{-j}, & \zeta \in \Omega_1, \\
  -\bar{f}_i(1/\zeta) - f_0(\zeta) + \sum_{j=1}^{\infty} [c_j + \Gamma^*_j \bar{c}_j] \zeta^j, & \zeta \in \Omega_0 + \Omega_2, 
\end{cases} \quad (3.104) \]

where the function \( \omega(\zeta) \) is holomorphic and single valued in the whole plane. By Liouville’s theorem, we have \( \omega(\zeta) = \text{constant} \). However, constant function \( f \) does not produce stress and electric displacement (SED), which may be neglected. Thus, letting \( \omega(\zeta) = 0 \), we have

\[ \sum_{j=1}^{\infty} [\bar{c}_j + \Gamma^*_j c_j] \zeta^{-j} = f_i(\zeta) + \bar{f}_0(1/\zeta), \quad \zeta \in \Omega_1, \quad (3.105) \]

\[ \sum_{j=1}^{\infty} [c_j + \Gamma^*_j \bar{c}_j] \zeta^j = f_i(1/\zeta) + f_0(\zeta), \quad \zeta \in \Omega_0 + \Omega_2. \]

It should be mentioned that the superscripts (1) and (2) are omitted in Eqs (3.104) and (3.105). To further simplify subsequent writing, we shall omit them again in the related expressions when the distinction is unnecessary. Similar to the procedures in Eq (3.102), we can derive from Eq (3.103) that
The equations (3.105) and (3.106) provide

\[
\begin{align*}
\kappa^{(2)} & \frac{k^{(2)}}{k^{(1)}} \sum_{j=1}^{\infty} [\bar{c}_j - \Gamma_j c_j][\zeta^{-j}] = -f_i(\zeta) + f_0(1/\zeta), \quad \zeta \in \Omega_1, \\
\kappa^{(2)} & \frac{k^{(2)}}{k^{(1)}} \sum_{j=1}^{\infty} [c_j - \bar{\Gamma}_j c_j][\zeta^j] = -f_i(1/\zeta) + f_0(\zeta), \quad \zeta \in \Omega_0 + \Omega_2.
\end{align*}
\]  

(3.106)

The equations (3.105)_2 and (3.106)_2 provide

\[
f_0(\zeta) = \frac{1}{2} \sum_{j=1}^{\infty} [(1 + k^{(2)}/k^{(1)})c_j + (1 - k^{(2)}/k^{(1)})\bar{\Gamma}_j c_j][\zeta^{-j}].
\]  

(3.107)

With the use of the series representation

\[
f_0(x) = \sum_{k=1}^{\infty} e_k x^k, \quad e_k = \frac{f_0^{(k)}(0)}{k!} = \frac{1}{2\pi i} \int_c \frac{f_0(x)}{x^{k+1}} \, dx,
\]  

(3.108)

the function \( f_0(\zeta) \) given in Eq (3.98) can be expressed as

\[
f_0(\zeta) = \sum_{j=1}^{\infty} e_j \zeta^j, \quad e_j = -\frac{\delta_0 \zeta_0^{(i)-j}}{j}.
\]  

(3.109)

where \( \delta_{ij} = 1 \), when \( i = j \); \( \delta_{ij} = 0 \), when \( i \neq j \).

Comparing the coefficients of corresponding terms in Eqs (3.107) and (3.109), one obtains

\[
c_j = (G_0 - \bar{\Gamma}_j G_j / G_0)(e_j - \bar{\Gamma}_j c_j / G_0), \quad (j = 1, 2, \ldots, \infty)
\]  

(3.110)

where \( G_0 = (1 + k_2/k_1)/2 \), \( G_j = (1 - k_2/k_1)\Gamma_j^*/2 \).

With the solution obtained for \( c_k \), the functions \( f_i, g_i' \) and \( g_i'' \) can be further written as

\[
f_i(\zeta) = \sum_{j=1}^{\infty} [\bar{c}_j + \Gamma_j^* c_j][\zeta^{(i)-j}] - \bar{q}_0 \ln(\zeta_0^{(i)-1} - \zeta_0^{(i)}),
\]  

(3.111)

\[
g_i'(\zeta^{(i)}) = \bar{q}_0 \ln(\zeta^{(i)} - \zeta_0^{(i)}) - \bar{q}_0 \ln(\zeta_0^{(i)-1} - \zeta_0^{(i)}) + \sum_{j=1}^{\infty} [\bar{c}_j + \Gamma_j^* c_j][\zeta^{(i)-j}],
\]  

(3.112)

\[
g_i''(\zeta^{(i)}) = \sum_{j=1}^{\infty} c_j[\zeta^{(2)i} + \Gamma_j^* c_j^{(2)-j}].
\]  

(3.113)

Substitution of Eqs (3.112) and (3.113) into Eqs (3.1) and (3.2) yields Green’s functions for thermal fields.

**3.7.1.2 Green’s functions for electroelastic fields.** From Eqs (3.1)_4 and (3.2)_2, the particular piezoelectric solution induced by the thermal loadings \( \hat{h}^* \) and \( \hat{T} \) can be written as

\[
U_p^{(i)} = 2 \text{Re}[c^{(i)} g_i(\zeta^{(i)})], \quad q_p^{(i)} = 2 \text{Re}[d^{(i)} g_i(\zeta^{(i)})], \quad (i=1, 2),
\]  

(3.114)

The function \( g(z_i) \) in Eq (3.114) can be obtained by integrating Eqs (3.112) and (3.113) with respect to \( z_n \), which yields
\[ g_1(\zeta^{(1)}_t) = a_{11}^{(1)}[q_0 F_1(\zeta^{(1)}_t, \zeta^{(1)}_r) - \tilde{q}_0 F_2(\zeta^{(1)}_t, \tilde{\zeta}^{(1)}_r)] + a_{21}^{(1)}[q_0 F_2(\zeta^{(1)}_t, \zeta^{(1)}_r) - \tilde{q}_0 F_1(\zeta^{(1)}_t, \tilde{\zeta}^{(1)}_r)] + \alpha_1 \ln \zeta^{(1)}_t + \sum_{j=1}^{\infty} G_{1j}(\zeta^{(1)}_t) \ln \zeta^{(1)}_t, \]  

(3.115)

\[ g_2(\zeta^{(2)}_t) = \sum_{j=1}^{\infty} [G_{2j}(\zeta^{(1)}_t) + G_{3j}(\zeta^{(2)}_t)] \]  

(3.116)

where

\[ F_1(\zeta_t, \zeta_{r0}) = (\zeta_t - \zeta_{r0})[\ln(\zeta_t - \zeta_{r0}) - 1], \]  

\[ F_2(\zeta_t, \zeta_{r0}) = (\zeta_t^{-1} - \zeta_{r0}^{-1}) \ln(\zeta_t - \zeta_{r0}) + \zeta_{r0}^{-1} \ln \zeta_t, \]

\[ a_{1k}^{(k)} = (a - ip^{(k)b})/2, \quad a_{2k}^{(k)} = (a + ip^{(k)b})/2, \quad (k = 1, 2), \]

\[ G_{ij} = -[\tilde{\zeta}_{j+1} + \Gamma_j^c c_{j+1}] a_{1j}^{(1)} - a_{2j}^{(1)}(\tilde{\zeta}_{j+1} + \Gamma_j^c c_{j+1}) s_{j1}/j, \]

(3.117)

\[ G_{2j} = (a_{1c}^{(2)} c_{j+1} - a_{2c}^{(2)} c_{j+1}) / j, \]

\[ G_{3j} = -[a_{1c}^{(2)} \Gamma_j^c c_{j+1} - a_{2c}^{(2)} \Gamma_j^c c_{j+1}] / j, \]

and \( s_{ji} = 1 \) for \( i \neq j \), \( s_{jj} = 0 \) for \( i = j \).

To satisfy the condition (3.95)\(_{3,4}\) along the interface, the function \( f(\zeta_m) \) can be assumed in the form [6]:

\[ f_{1m}^{(1)}(\zeta^{(1)}_m) = a[q_0 F_1(\zeta^{(1)}_m, \zeta^{(1)}_r) + q_0 F_2(\zeta^{(1)}_m, \zeta^{(1)}_r) - \tilde{q}_0 F_1(\zeta^{(1)}_m, \tilde{\zeta}^{(1)}_r) - \tilde{q}_0 F_2(\zeta^{(1)}_m, \tilde{\zeta}^{(1)}_r)]/2, \quad (j=1, 2), \]  

(3.118)

\[ f_{2m}^{(1)}(\zeta^{(1)}_m) = ip^{(m)b}[q_0 F_1(\zeta^{(1)}_m, \zeta^{(1)}_r) + q_0 F_2(\zeta^{(1)}_m, \zeta^{(1)}_r) - \tilde{q}_0 F_1(\zeta^{(1)}_m, \tilde{\zeta}^{(1)}_r) - \tilde{q}_0 F_2(\zeta^{(1)}_m, \tilde{\zeta}^{(1)}_r)]/2, \]  

(3.119)

\[ f_{3m}^{(1)}(\zeta^{(1)}_m) = a_{1m}^{(1)} \ln \zeta^{(1)}_m, \quad (j=1, 2), \]  

(3.120)

where \( f_k^{(1)} \) are four component vectors, and \( r_k^{(1)} \) and \( s_k^{(1)} \) are constant vectors with four components to be determined. It should be pointed out that the vector \( s_k^{(1)} \) is different from the symbol \( s_{ij} \) appearing in Eq (3.117).

The Green’s functions for the electroelastic fields can thus be chosen as

\[ U^{(j)} = 2 \text{Re} \left\{ \sum_{k=1}^{3} A^{(j)}(\zeta^{(1)}_a) \left[ f_k^{(j)}(\zeta^{(1)}_a) \right] q_k^{(j)} + \sum_{k=1}^{5} A^{(j)}(\zeta^{(1)}_a) \left[ f_k^{(j)}(\zeta^{(1)}_a) \right] + c^{(j)} g_j(\zeta^{(1)}_t) \right\}, \]  

(3.121)

\[ \varphi^{(j)} = 2 \text{Re} \left\{ \sum_{k=1}^{3} B^{(j)}(\zeta^{(2)}_a) \left[ f_k^{(j)}(\zeta^{(2)}_a) \right] q_k^{(j)} + \sum_{k=1}^{5} B^{(j)}(\zeta^{(2)}_a) \left[ f_k^{(j)}(\zeta^{(2)}_a) \right] + d^{(j)} g_j(\zeta^{(2)}_t) \right\} \]  

(3.122)

The above two expressions, together with the interface condition (3.95)\(_{3,4}\), provide

\[ q^{(1)}_i = X_1 (A^{(2)}(\zeta^{(1)}_a) - B^{(2)}(\zeta^{(1)}_a)), \]  

(3.123)
\[ q_i^{(2)} = X_2 (B^{(i)} - A^{(i)}) c^{(1)}, \quad (3.124) \]
\[ q_i^{(1)} = \bar{p}_i^{(1)} p^{(1)} q_i^{(1)}, \quad (3.125) \]
\[ q_i^{(2)} = \bar{p}_i^{(2)} p^{(2)} q_i^{(2)}, \quad (3.126) \]
\[ q_i^{(1)} = \left\{ \alpha_i^{(1)} \right\}^{-1} \bar{a}_i^{(1)} \left( c_i + \bar{c}_i \right) q_i^{(1)}, \quad (3.127) \]
\[ q_i^{(2)} = \left\{ \alpha_i^{(2)} \right\}^{-1} \bar{a}_i^{(2)} \left( c_i + \bar{c}_i \right) q_i^{(2)}, \quad (3.128) \]
\[ A^{(i)} f_i^{(1)} (\sigma) + \bar{A}^{(i)} f_i^{(1)} (\sigma) = A^{(i)} f_i^{(2)} (\sigma) + \bar{A}^{(i)} f_i^{(2)} (\sigma) + \sum_{j=1}^{\infty} (c^{(i)} G_{ij} - \bar{c}^{(i)} G_{ij} - \bar{c}^{(2)} G_{ij}) \sigma^{-j} \]
\[ = A^{(i)} f_i^{(2)} (\sigma) + \bar{A}^{(i)} f_i^{(2)} (\sigma) = A^{(i)} f_i^{(1)} (\sigma) - \bar{A}^{(i)} f_i^{(1)} (\sigma) - \sum_{j=1}^{\infty} (c^{(i)} \bar{G}_{ij} - c^{(2)} G_{ij} - \bar{c}^{(2)} G_{ij}) \sigma^{j} \]
\[ (3.129) \]
\[ B^{(i)} f_i^{(1)} (\sigma) + \bar{B}^{(i)} f_i^{(1)} (\sigma) = B^{(i)} f_i^{(2)} (\sigma) - \bar{B}^{(i)} f_i^{(2)} (\sigma) + \sum_{j=1}^{\infty} (d^{(i)} G_{ij} - \bar{d}^{(i)} G_{ij} - d^{(2)} G_{ij}) \sigma^{-j} \]
\[ = B^{(i)} f_i^{(2)} (\sigma) + \bar{B}^{(i)} f_i^{(2)} (\sigma) = B^{(i)} f_i^{(1)} (\sigma) - \bar{B}^{(i)} f_i^{(1)} (\sigma) - \sum_{j=1}^{\infty} (d^{(i)} \bar{G}_{ij} - d^{(2)} G_{ij} - \bar{d}^{(2)} G_{ij}) \sigma^{j} \]
\[ (3.130) \]

where
\[ X_1 = (A^{(i)} - B^{(i)} B^{(i)})^{-1}, \quad X_2 = (A^{(i)} - B^{(i)} B^{(i)})^{-1}. \quad (3.131) \]

Therefore, by Liouville’s theorem, Eqs (3.129) and (3.130) yield
\[ A^{(i)} f_i^{(1)} (\sigma) + \bar{A}^{(i)} f_i^{(1)} (\sigma) = A^{(i)} f_i^{(2)} (\sigma) + \bar{A}^{(i)} f_i^{(2)} (\sigma) - \sum_{j=1}^{\infty} (c^{(i)} G_{ij} - \bar{c}^{(i)} G_{ij} - \bar{c}^{(2)} G_{ij}) \sigma^{-j}, \quad (3.132) \]
\[ A^{(i)} f_i^{(1)} (\sigma) + \bar{A}^{(i)} f_i^{(1)} (\sigma) = A^{(i)} f_i^{(2)} (\sigma) + \bar{A}^{(i)} f_i^{(2)} (\sigma) - \sum_{j=1}^{\infty} (c^{(i)} \bar{G}_{ij} - c^{(2)} G_{ij} - \bar{c}^{(2)} G_{ij}) \sigma^{j}, \quad (3.133) \]
\[ B^{(i)} f_i^{(1)} (\sigma) + \bar{B}^{(i)} f_i^{(1)} (\sigma) = B^{(i)} f_i^{(2)} (\sigma) + \bar{B}^{(i)} f_i^{(2)} (\sigma) - \sum_{j=1}^{\infty} (d^{(i)} G_{ij} - \bar{d}^{(i)} G_{ij} - d^{(2)} G_{ij}) \sigma^{-j}, \quad (3.134) \]
\[ B^{(i)} f_i^{(1)} (\sigma) + \bar{B}^{(i)} f_i^{(1)} (\sigma) = B^{(i)} f_i^{(2)} (\sigma) + \bar{B}^{(i)} f_i^{(2)} (\sigma) - \sum_{j=1}^{\infty} (d^{(i)} \bar{G}_{ij} - d^{(2)} G_{ij} - \bar{d}^{(2)} G_{ij}) \sigma^{j} \quad (3.135) \]

The above four equations are not completely independent. For example, Eqs (3.132) and (3.134) can be obtained from Eqs (3.133) and (3.135). Thus only two of the equations are independent. However, there are four sets of constant vectors, i.e., \( r_i^{(j)} \) and \( s_i^{(j)} (j = 1, 2) \), to be determined. We need two more equations to make the solution unique. Through use of the relation (3.94) and Eqs (3.121) and (3.122), the
unknown vectors $r_j^{(2)}$ and $s_j^{(2)}$ appearing in $f_i^{(2)}$ and $f_s^{(2)}$ can be determined as follows:

$$r_j^{(2)} = (A^{(2)-1}\{\Gamma^{(2)}_{ja}\}A^{(2)} - B^{(2)-1}\{\Gamma^{(2)}_{ja}\}B^{(2)})^{-1}[A^{(2)-1}(c^{(2)}G_{j3} - \{\Gamma^{(2)}_{ja}\}c^{(2)}G_{j2})$$
$$- B^{(2)-1}(d^{(2)}G_{j3} - \{\Gamma^{(2)}_{ja}\}d^{(2)}G_{j2})],$$

$$s_j^{(2)} = B^{(2)-1}[\{\Gamma^{(2)}_{ja}\}(B^{(2)}r_j^{(2)} + d^{(2)}G_{j2}) - d^{(2)}G_{j3}]$$

$$j = 1, \ldots, \infty,$$  

(3.136)

where $\{\Gamma^{(2)}_{ja}\} = \left\{ \begin{array}{c} a + ibp_{\mu}^{(1)} \\ a - ibp_{\mu}^{(1)} \end{array} \right\}$.  

Once the constant vectors $r_j^{(2)}$ and $s_j^{(2)}$ are obtained, the unknown vectors $r_j^{(1)}$ and $s_j^{(1)}$ appearing in functions $f_i^{(1)}$ and $f_s^{(1)}$ can be determined from Eqs (3.132) and (3.134) [or(3.133) and (3.135)]. They are

$$r_j^{(1)} = iA^{(1)T}(M_2 + M_1)A^{(2)}r_j^{(2)} + (M_1 - M_2)\overline{A}^{(2)}s_j^{(2)}$$
$$- \overline{M}_1(c^{(1)}\overline{\zeta}_{j1} - c^{(2)}\overline{G}_{j2} - \overline{c}^{(2)}\overline{G}_{j3}) + i(\overline{d}^{(1)}\overline{\zeta}_{j1} - \overline{d}^{(2)}G_{j2} - \overline{d}^{(2)}G_{j3}),$$

(3.138)

$$s_j^{(1)} = iA^{(1)T}(M_2 + M_1)A^{(2)}s_j^{(2)} + (M_1 - M_2)\overline{A}^{(2)}r_j^{(2)}$$
$$- \overline{M}_1(c^{(1)}G_{j1} - c^{(2)}\overline{G}_{j2} - c^{(2)}G_{j3}) + i(\overline{d}^{(1)}G_{j1} - \overline{d}^{(2)}\overline{G}_{j2} - \overline{d}^{(2)}G_{j3})$$

(3.139)

where

$$M_k = -iB^{(k)}A^{(k)-1} = H^{(k)-1}(I + iS^{(k)}), \quad H^{(k)} = 2iA^{(k)}A^{(k)T}, \quad S^{(k)} = i(2A^{(k)}B^{(k)T} - I),$$

$$k = 1, 2$$

(3.140)

3.7.2 Green’s functions for thermal loads applied inside the inclusion

3.7.2.1 Green’s functions for thermal fields. For the case of thermal loads $h^*$ and $\hat{T}$ located at a point $(x_{10}, x_{20})$ inside the inclusion, the general solution for temperature and heat-flow function can be assumed in the form [5]

$$T^{(1)} = f_0(\zeta_{i1}^{(1)}) + f_1(\zeta_{i1}^{(1)}) + f_0(\zeta_{i2}^{(1)}) + f_1(\zeta_{i2}^{(1)}),$$

$$\vartheta^{(1)} = -ik^{(1)}f_0(\zeta_{i1}^{(1)}) - ik^{(1)}f_1(\zeta_{i1}^{(1)}) + ik^{(1)}f_0(\zeta_{i2}^{(1)}) + ik^{(1)}f_1(\zeta_{i2}^{(1)}),$$

(3.141)

$$T^{(2)} = f_0^{*}(\zeta_{i1}^{(2)}) + f_2(\zeta_{i1}^{(2)}) + f_0^{*}(\zeta_{i2}^{(2)}) + f_2(\zeta_{i2}^{(2)}),$$

$$\vartheta^{(2)} = -ik^{(2)}f_0^{*}(\zeta_{i1}^{(2)}) - ik^{(2)}f_2(\zeta_{i1}^{(2)}) + ik^{(2)}f_0^{*}(\zeta_{i2}^{(2)}) + ik^{(2)}f_2(\zeta_{i2}^{(2)}).$$

(3.142)

Here, $f_0$ and $f_0^*$ can be chosen to represent the solutions associated with the unperturbed thermal fields caused by thermal load. $f_1$ and $f_2$ are the functions corresponding to the perturbed field of matrix and inclusion, respectively. Since $(x_{10}, x_{20})$ is inside the inclusion, there is a branch cut extending from $(x_{10}, x_{20})$ to infinity. Thus the choice of $f_0$ and $f_0^*$ should take into account the discontinuity across this branch cut. A proper form of $f_0$ and $f_0^*$ may be chosen as [10]:

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\[
f_0'(\zeta^{(1)}) = d \ln \zeta^{(1)}, \quad f_0^*'(\zeta^{(2)}) = q_0 [\ln(z_2^{(2)} - z_{r_0}^{(2)}) - \ln a_{1r}^{(2)}] \quad (3.143)
\]
\[
f_2'(\zeta^{(2)}) = \sum_{j=-\infty}^{\infty} c_j \zeta^{(2j)}, \quad c_{-j} = \Gamma_j^* c_j \quad (3.144)
\]

where \( a_{1r}^{(k)} \) is defined in Eq (3.117), \( d \) and \( c_j \) are unknowns to be determined, \( \zeta^{(1)} \) and \( \zeta^{(2)} \) are related to the complex arguments \( z_i^{(1)} \) and \( z_2^{(2)} = x_{r_0} + p_2^{(2)} x_{20} \) by Eq (3.99), and \( q_0 \) is a complex number which can be determined from the conditions
\[
\int_C dT = \hat{T} \quad \text{for any closed curve } C \text{ enclosing the point } \zeta_{r_0}^{(2)} \quad (3.145)
\]
\[
\int_C d\theta = -h^* \quad \text{for any closed curve } C \text{ enclosing the point } \zeta_{r_0}^{(2)} \quad (3.146)
\]

Substitution of Eqs (3.143) and (3.144) into (3.141) and (3.142) and later into (3.145) and (3.146) yields
\[
q_0 = T_0 / 4\pi i - h^* / 4\pi k^{(2)} \quad (3.147)
\]

As for the unknowns \( d, c_j \) and the function \( f_1 \), they can be derived in the following way. Noting the relation
\[
\ln[(z_i - z_{r_0}) / a_{1r}] = \ln(\zeta_i - \zeta_{r_0}) + \ln(1 - a_{2r} / a_{1r}) \zeta_{r_0} \quad (3.148)
\]
where \( a_{1r} = (a - ip_1^* b) / 2 \), \( a_{2r} = (a + ip_1^* b) / 2 \), we have the series representation
\[
f_0^*'(\zeta^{(2)}) = q_0 [\ln \zeta^{(2)} - \sum_{j=1}^{\infty} e_j \zeta^{(2j)}] \quad \text{for } |\zeta^{(2)}| > |\zeta_{r_0}^{(2)}| \quad (3.149)
\]
in which \( e_j = (a_{2r}^{(2)} / a_{1r}^{(2)}) / \zeta_{r_0}^{(2)} / j). \) Imposition of the continuity conditions (3.95) on the interface leads to
\[
f_i(\sigma) + d \ln \sigma + f_i(\sigma) + d \ln \sigma = \sum_{j=1}^{\infty} [\bar{\sigma}_j + \Gamma_j^* c_j] \sigma^{-j} + \sum_{j=1}^{\infty} [c_j + \bar{\Gamma}_j^* \sigma_j] \sigma^j
\]
\[
+ q_0 [\ln \sigma - \sum_{j=1}^{\infty} e_j \sigma^{-j}] + \bar{q}_0 [\ln \sigma - \sum_{j=1}^{\infty} \bar{e}_j \sigma^j] \quad (3.150)
\]
\[
-f_i(\sigma) + d \ln \sigma + f_i(\sigma) - d \ln \sigma = \frac{k_1^{(2)}}{k_1^{(1)}} \sum_{j=1}^{\infty} [\bar{e}_j - \Gamma_j^* c_j] \sigma^{-j} - \frac{k_1^{(2)}}{k_1^{(1)}} \sum_{j=1}^{\infty} [c_j - \bar{\Gamma}_j^* \sigma_j] \sigma^j
\]
\[
- \frac{k_1^{(2)}}{k_1^{(1)}} q_0 [\ln \sigma - \sum_{j=1}^{\infty} e_j \sigma^{-j}] + \frac{k_1^{(2)}}{k_1^{(1)}} \bar{q}_0 [\ln \sigma - \sum_{j=1}^{\infty} \bar{e}_j \sigma^j] \quad (3.151)
\]

where \( \sigma = e^0 \) stands for a point located on the unit circle in the \( \zeta \)-plane, and \( \theta \) is a polar angle. Comparison of the coefficients of the “ln” terms on both sides of Eqs (3.150) and (3.151) yields
\[ d = \frac{1}{2} \left[ 1 + \frac{k^{(2)}}{k^{(1)}} \right] q_0 - \frac{1}{2} \left[ 1 - \frac{k^{(2)}}{k^{(1)}} \right] \bar{q}_0 \]  

(3.152)

In order to use the analytical continuation method, Eqs (3.150) and (3.151) are rewritten, by deleting the “ln” terms, as

\[ f_i(\sigma) - \sum_{j=1}^{\infty} [\bar{c}_j + \bar{\Gamma}_j^* c_j - q_0 e_j] \sigma^{-j} = -\bar{f}_i(\sigma) + \sum_{j=1}^{\infty} [c_j + \bar{\Gamma}_j^* c_j - \bar{q}_0 e_j] \sigma^j \]  

(3.153)

\[ -f_i(\sigma) - \frac{k^{(2)}}{k^{(1)}} \sum_{j=1}^{\infty} [\bar{c}_j - \bar{\Gamma}_j^* c_j + q_0 e_j] \sigma^{-j} = -\bar{f}_i(\sigma) - \frac{k^{(2)}}{k^{(1)}} \sum_{j=1}^{\infty} [c_j - \bar{\Gamma}_j^* c_j + \bar{q}_0 e_j] \sigma^j \]  

(3.154)

Similar to the treatment for the case of loading applied outside the inclusion, application of the method of analytical continuation to Eqs (3.153) and (3.154) yields

\[ f_i(\zeta) = \sum_{j=1}^{\infty} [\bar{c}_j + \bar{\Gamma}_j^* c_j - q_0 e_j] \zeta^{-j} \quad \zeta \in \Omega_1 \]  

(3.155)

\[ \bar{f}_i(\zeta) = \sum_{j=1}^{\infty} [c_j + \bar{\Gamma}_j^* c_j - \bar{q}_0 e_j] \zeta^j \quad \zeta \in \Omega_2 \]  

(3.156)

for Eq (3.153), and

\[ f_i(\zeta) = -\frac{k^{(2)}}{k^{(1)}} \sum_{j=1}^{\infty} [\bar{c}_j - \bar{\Gamma}_j^* c_j + q_0 e_j] \zeta^{-j} \quad \zeta \in \Omega_1 \]  

(3.157)

\[ \bar{f}_i(\zeta) = -\frac{k^{(2)}}{k^{(1)}} \sum_{j=1}^{\infty} [c_j - \bar{\Gamma}_j^* c_j + \bar{q}_0 e_j] \zeta^j \quad \zeta \in \Omega_2 \]  

(3.158)

for Eq (3.154). Solving Eqs (3.155) and (3.156) for \( c_j \) yields

\[ c_j = \frac{k^{(1)} - k^{(2)}}{2k^{(1)}} \left( G_0 - \bar{G}_j G_j / G_0 \right)^{-1} \left( \bar{q}_0 e_j - \bar{G}_j q_0 e_j / G_0 \right), \quad (j = 1, 2, \cdots, \infty) \]  

(3.159)

where

\[ G_0 = \left( 1 + k^{(2)} / k^{(1)} \right) / 2, \quad G_j = \left( 1 - k^{(2)} / k^{(1)} \right) \bar{\Gamma}_j^* / 2 \]  

(3.160)

Having obtained the solution of \( c_j \), the functions \( g_i \) in Eqs (3.1) and (3.2) can be given by

\[ g_1(\zeta_1) = q_0 \{ a_{1\zeta_1}(\zeta_1) (\ln \zeta_1 - 1) + a_{2\zeta_1}(1 + \ln \zeta_1) / \zeta_1 \} + (\bar{c}_1 + \bar{\Gamma}_1^* c_1 - q_0 e_1) a_{1\zeta_1}(\zeta_1) + \sum_{j=1}^{\infty} G_{1\zeta_1} \zeta_1^{-j} \]  

(3.161)

\[ g_2(\zeta_1) = q_0 \{ a_{2\zeta_1}(\zeta_1) (\ln \zeta_1 - 1) + a_{2\zeta_1}(1 + \ln \zeta_1) / \zeta_1 \} - q_0 e_1 a_{1\zeta_1}(\zeta_1) + \sum_{j=1}^{\infty} G_{2\zeta_1} \zeta_1^{-j} \]  

(3.162)

where
\[ G_{aj} = -[(\bar{c}_{j} + \Gamma_{j}^* c_{j+1} - q_0 e_{j+1})a_{1}^{(1)} - a_{2}^{(1)}(\bar{c}_{j} + \Gamma_{j}^* c_{j-1} - q_0 e_{j-1})]/j \]
\[ G_{sj} = (d_{1}^{(2)} c_{j}, t_{j}) - a_{2}^{(2)}(c_{j} - q_0 e_{j})]/j \]
\[ G_{6j} = [d_{1}^{(2)}(\Gamma_{j}^* c_{j+1} - q_0 e_{j+1}) - a_{2}^{(2)}(\Gamma_{j}^* c_{j-1} - q_0 e_{j-1})]/j \]
(3.161)

with \( t_{ij} = 1 \) for \( i \neq j \), \( t_{ij} = 0 \) for \( i = j \).

3.7.2.2 Green’s functions for electroelastic fields. From Eqs (3.1)4 and (3.2)2 the particular piezoelectric solution induced by thermal load can be written as

\[ \Phi^{(i)} = 2 \text{Re}[\mathbf{e}^{(i)} g_{i}(\zeta^{(i)})], \quad \Phi^{(j)} = 2 \text{Re}[\mathbf{d}^{(j)} g_{j}(\zeta^{(j)})] \quad (i=1,2) \]  
(3.162)

As stated previously, the particular solutions (3.162) do not generally satisfy the conditions (3.95)3,4 along the interface. We therefore need to find a corrective isothermal solution for a given problem so that, when it is superimposed on the particular thermoelastic solution, the interface conditions (3.95)3,4 will be satisfied. Owing to the fact that \( f(\zeta_{\alpha}) \) and \( g(\zeta_{\alpha}) \) have the same order to affect the SED in Eq (3.2)2, possible function forms may be taken from the partition of \( g(\zeta_{\alpha}) \) in order to satisfy the boundary conditions considered. They are

\[ f_{1}^{(j)}(\zeta_{\alpha}^{(j)}) = a_{1}^{(j)} \ln \zeta_{\alpha}^{(j)}, \quad f_{2}^{(j)}(\zeta_{\alpha}^{(j)}) = a_{2}^{(j)} \ln \zeta_{\alpha}^{(j)}, \quad (j=1,2) \]  
(3.163)

\[ f_{k}^{(j)}(\zeta_{\alpha}^{(j)}) = \sum_{k=1}^{\infty} \left( \zeta_{\alpha}^{(j)} \right)^{k} r_{k}^{(j)}, \quad f_{k}^{(j)}(\zeta_{\alpha}^{(j)}) = \sum_{k=1}^{\infty} \left( \zeta_{\alpha}^{(j)} \right)^{k} s_{k}^{(j)} \quad (j=1,2) \]  
(3.164)

where \( r_{k}^{(j)} \) are 4-component vectors, \( s_{k}^{(j)} \) are constant vectors with 4-components to be determined, and

\[ F_{1}(\zeta_{\alpha}) = \zeta_{\alpha}^{-1}(1 + \ln \zeta_{\alpha}) - \zeta_{\alpha}^{-1}(1 - \ln \zeta_{\alpha}), \]
\[ F_{2}(\zeta_{\alpha}) = \zeta_{\alpha}^{-1}(1 + \ln \zeta_{\alpha}) + \zeta_{\alpha}^{-1}(1 - \ln \zeta_{\alpha}), \]  
(3.165)

The Green’s functions for the electroelastic fields can thus be chosen as

\[ \Phi^{(j)} = 2 \text{Re} \left[ \sum_{k=1}^{3} \mathbf{A}^{(j)} \zeta_{\alpha}^{(j)} \left( f_{k}^{(j)}(\zeta_{\alpha}^{(j)}) \right) q_{k}^{(j)} + \sum_{k=4}^{5} \mathbf{A}^{(j)} f_{k}^{(j)}(\zeta_{\alpha}^{(j)}) + \mathbf{c}^{(j)} g_{j}(\zeta_{\alpha}^{(j)}) \right] \]  
(3.166)

\[ \Phi^{(j)} = 2 \text{Re} \left[ \sum_{k=1}^{3} \mathbf{B}^{(j)} \zeta_{\alpha}^{(j)} \left( f_{k}^{(j)}(\zeta_{\alpha}^{(j)}) \right) q_{k}^{(j)} + \sum_{k=4}^{5} \mathbf{B}^{(j)} f_{k}^{(j)}(\zeta_{\alpha}^{(j)}) + \mathbf{d}^{(j)} g_{j}(\zeta_{\alpha}^{(j)}) \right] \]  
(3.167)

Substituting Eqs (3.166) and (3.167) into Eq (3.95) provides

\[ q_{1}^{(1)} = q_{0} X_{1} \left[ \mathbf{A}^{(2)-1} (\mathbf{c}^{(2)} - \mathbf{c}^{(1)}) - \mathbf{B}^{(2)-1} (\mathbf{d}^{(2)} - \mathbf{d}^{(1)}) \right], \]  
(3.168)

\[ q_{1}^{(2)} = q_{0} X_{2} \left[ \mathbf{A}^{(1)-1} (\mathbf{c}^{(1)} - \mathbf{c}^{(2)}) - \mathbf{B}^{(1)-1} (\mathbf{d}^{(1)} - \mathbf{d}^{(2)}) \right], \]  
(3.169)

\[ q_{2}^{(1)} = q_{0} P^{0-1} X_{1} \left[ \mathbf{A}^{(2)-1} (p_{1}^{(2)} \mathbf{c}^{(2)} - p_{1}^{(1)} \mathbf{c}^{(1)}) - \mathbf{B}^{(2)-1} (p_{1}^{(2)} \mathbf{d}^{(2)} - p_{1}^{(1)} \mathbf{d}^{(1)}) \right], \]  
(3.170)
\[ q^{(2)}_3 = q_0 P^{(2)-1} X_2 [A^{(1)-1}(p_1^{(1)}e^{(1)} - p_1^{(2)}e^{(2)}) - B^{(1)-1}(p_1^{(1)}d^{(1)} - p_1^{(2)}d^{(2)})], \] (3.171)

\[ q^{(1)}_3 = \left( a_{(k)}^{(1)} \right)^{-1} X_3 [A^{(2)-1}(b_1^{(1)}e^{(1)} - b_2^{(1)}e^{(2)}) - B^{(2)-1}(b_1^{(1)}d^{(1)} - b_2^{(2)}d^{(2)})], \] (3.172)

\[ q^{(2)}_3 = \left( a_{(k)}^{(2)} \right)^{-1} X_3 [A^{(1)-1}(b_2^{(1)}e^{(2)} - b_1^{(2)}e^{(1)}) - B^{(2)-1}(b_1^{(2)}d^{(2)} - b_2^{(1)}d^{(1)})], \] (3.173)

\[ A^{(1)}f^{(1)}_2(\sigma) + \bar{A}^{(1)}f^{(1)}_4(\sigma) - A^{(2)}f^{(2)}_2(\sigma) - \bar{A}^{(2)}f^{(2)}_4(\sigma) + \sum_{j=1}^{\infty} (c^{(1)}G_{sj} - c^{(2)}G_{sj} - c^{(2)}G_{sj}^*) \sigma^{-j} \] (3.174)

\[ B^{(1)}f^{(1)}_2(\sigma) + \bar{B}^{(1)}f^{(1)}_4(\sigma) - B^{(2)}f^{(2)}_2(\sigma) - \bar{B}^{(2)}f^{(2)}_4(\sigma) + \sum_{j=1}^{\infty} (d^{(1)}G_{sj} - d^{(2)}G_{sj} - d^{(2)}G_{sj}^*) \sigma^{-j} \] (3.175)

where

\[ P^{(j)} = \text{diag}[p_1^{(j)}, p_2^{(j)}, p_3^{(j)}, p_4^{(j)}], \quad b_1^{(1)} = a_1^{(1)}(c_1 + \Gamma_1^{*} c_1 - q_0 e_1), \quad b_1^{(2)} = -q_0 a_1^{(2)} e_1 \] (3.176)

It is found that Eqs (3.175) and (3.176) can be obtained from Eqs (3.129) and (3.130) by replacing \( G_{1j} \) by \( G_{4j} \), \( G_{2j} \) by \( G_{5j} \), and \( G_{3j} \) by \( G_{6j} \). Therefore, the procedure for determining the four sets of constant vectors, i.e., \( r^{(j)}_k \) and \( s^{(j)}_k \) \((j=1,2)\), operates in the same way as that treated in Section 3.7.1.2. The results of \( r^{(j)}_k \) and \( s^{(j)}_k \) in Eq (3.164) have the same form as those in Eqs (3.136)-(3.139), but \( G_{1j} \), \( G_{2j} \), \( G_{3j} \) in those equations are now replaced by \( G_{4j} \), \( G_{5j} \), and \( G_{6j} \).

### 3.7.3 Green’s functions for thermal loads applied on the interface

#### 3.7.3.1 Green’s functions for thermal fields

When \((x_{10}, x_{20})\) is on the interface, say \( x_{10} = a \cos \theta_0, \quad x_{20} = b \sin \theta_0 \), \( \zeta^{(1)}_0 = \zeta^{(2)}_0 = e^{\theta_0} \), the proper choice of \( f_0 \) and \( f_0^* \) should reflect the singularity properties of both inclusion and matrix. By referring to the Eq (20) given in Yen et al [10], they may be assumed in the form

\[ f_0(\zeta^{(1)}_0) = q_0 \ln(\zeta^{(1)}_0 - e^{\theta_0}) \] (3.177)

\[ f_0(\zeta^{(2)}_0) = \sum_{j=1}^{\infty} a_j \zeta^{(1)-j}_0 + q_1 \ln(\zeta^{(1)-1}_0 - e^{\theta_0}) \] (3.178)

\[ f_0^*(\zeta^{(2)}_0) = q_2 \ln[(\zeta^{(2)}_0 - \zeta^{(2)}_0^*) / a^{(2)}_0] \] (3.179)

With the use of the series representation

\[ \ln(1 - x) = -\sum_{j=1}^{\infty} x^j / j \] (3.180)

and the relation (3.148), Eq (3.179) may be rewritten as

\[ f_0^*(\zeta^{(2)}_0) = q_2 \ln(\zeta^{(2)}_0 - e^{\theta_0}) - \sum_{j=1}^{\infty} \left[ (a_2^{(2)} e^{-\theta_0} / a^{(2)}_0)^{j-1} \right] / j \] (3.181)
The continuity conditions (3.95)\textsubscript{1,2} on the interface give

\[ q_{1} = \frac{k^{(1)} - k^{(2)}}{k^{(1)} + k^{(2)}} q_{0}, \quad q_{2} = \frac{2k^{(1)}}{k^{(1)} + k^{(2)}} q_{0} \]  \hspace{1cm} (3.182)

To determine the remaining unknown functions, \( f_{1} \) and \( f_{2} \), the method of analytical continuation presented in the previous sections will be used. As described in Eqs (3.105) and (3.106), the holomorphic properties provide

\[ f_{11}(\zeta) = f_{22}(\zeta) + \frac{f_{21}(\zeta)}{k^{(2)}}, \quad \zeta \in \Omega_{1} \]
\[ f_{11}(\zeta) = f_{22}(\zeta) + \frac{f_{21}(\zeta)}{k^{(2)}}, \quad \zeta \in \Omega_{2} \]
\[ f_{11}(\zeta) = k^{(1)} \left[ f_{22}(\zeta) - \frac{f_{21}(\zeta)}{k^{(1)}} \right], \quad \zeta \in \Omega_{1} \]
\[ f_{11}(\zeta) = k^{(1)} \left[ f_{22}(\zeta) - \frac{f_{21}(\zeta)}{k^{(1)}} \right], \quad \zeta \in \Omega_{2} \]  \hspace{1cm} (3.183)

where

\[ f_{11}(\zeta) = \sum_{j=1}^{\infty} a_{j} \zeta^{-j}, \quad f_{21}(\zeta) = \sum_{j=1}^{\infty} c_{j} \zeta^{-j}, \]  \hspace{1cm} (3.184)

where \( e_{j} = q_{2} (a_{2j} e^{-\theta_{j}} / a_{1j})^{j} / j \).

Solving Eq (3.183) yields

\[ a_{sj} = \bar{c}_{j} + \gamma_{j} c_{j} - e_{sj}, \]  \hspace{1cm} (3.185)
\[ c_{j} = \frac{k^{(1)} - k^{(2)}}{2k^{(1)}} (G_{0} - \bar{G}_{j} G_{j} / G_{0})^{-1} (\bar{c}_{j} - \bar{G}_{j} e_{j} / G_{0}), \quad j = 1, 2, \ldots, \infty \]  \hspace{1cm} (3.186)

Thus the corresponding functions \( g_{i} \) are given as

\[ g_{1}(\zeta_{i}^{(1)}) = \{ a_{i}^{(1)} [q_{0} F_{3}(\zeta_{i}^{(1)}, e^{-\theta_{i}}) + q_{1} F_{4}(\zeta_{i}^{(1)-1}, e^{-\theta_{i}})] + a_{2i}^{(1)} [q_{0} F_{4}(\zeta_{i}^{(1)}, e^{-\theta_{i}}) \} + \sum_{j=1}^{\infty} G_{j} \zeta_{i}^{(1)-j} \} \]  \hspace{1cm} (3.187)
\[ g_{2}(\zeta_{i}^{(2)}) = q_{2} \{ a_{i}^{(2)} F_{3}(\zeta_{i}^{(2)}, e^{\theta_{i}}) + a_{2i}^{(2)} F_{4}(\zeta_{i}^{(2)}, e^{\theta_{i}}) \} - e_{si} a_{i}^{(2)} \ln \zeta_{i}^{(2)} \]  \hspace{1cm} (3.188)

where

\[ F_{3}(x, y) = (x - y) \ln(x - y) - 1 \]  \hspace{1cm} (3.189)
\[ F_{4}(x, y) = \frac{1}{x} - \frac{1}{y} \ln(x - y) + \frac{1}{y} \ln x \]  \hspace{1cm} (3.190)
\[
G_{\tau j} = -\left[a_{1j}^{(1)}a_{\eta j-1}^{(1)} - t_{j1}a_{1j}^{(2)}a_{\eta j-1}^{(2)}\right]/j, \\
G_{8j} = (a_{1j}^{(2)}c_{j-1} - a_{2j}^{(2)}c_{j+1})/j = G_{5j}, \\
G_{9j} = -\left[a_{1j}^{(2)}(\Gamma_{j1} - e_{\eta j-1}) - a_{2j}^{(2)}(\Gamma_{j+1} - e_{\eta j-1})u_{j1}\right]/j. 
\]

For the same reason as given in relation to Eqs (3.166) and (3.167), the corresponding electroelastic solutions may be assumed in the form

\[
U^{(j)} = 2 \Re \left\{ \sum_{k=1}^{5} A^{(j)}(\zeta^{(j)}_{\alpha}) q_k^{(j)} + \sum_{k=6}^{7} A^{(j)}(\zeta^{(j)}_{\alpha}) q_k^{(j)} + c^{(j)}(\zeta^{(j)}_{\alpha}) g_j(\zeta^{(j)}_{\alpha}) \right\} 
\]

\[
\varphi^{(j)} = 2 \Re \left\{ \sum_{k=1}^{5} B^{(j)}(\zeta^{(j)}_{\alpha}) q_k^{(j)} + \sum_{k=6}^{7} B^{(j)}(\zeta^{(j)}_{\alpha}) q_k^{(j)} + d^{(j)}(\zeta^{(j)}_{\alpha}) g_j(\zeta^{(j)}_{\alpha}) \right\} 
\]

where

\[
f_{1}^{(j)}(\zeta^{(j)}_{\alpha}) = a[q_0F_3(\zeta^{(j)}_{\alpha}), e^{\phi_0} + q_0F_4(\zeta^{(j)}_{\alpha}), e^{\phi_0} + q_1F_3(\zeta^{(j)}_{\alpha}), e^{-\phi_0}] \\
f_{2}^{(j)}(\zeta^{(j)}_{\alpha}) = ip^{(j)}[q_0F_3(\zeta^{(j)}_{\alpha}), e^{\phi_0} + q_0F_4(\zeta^{(j)}_{\alpha}), e^{\phi_0} + q_1F_3(\zeta^{(j)}_{\alpha}), e^{-\phi_0}] \\
f_{3}^{(j)}(\zeta^{(j)}_{\alpha}) = a[q_2F_3(\zeta^{(j)}_{\alpha}), e^{\phi_0} + q_2F_4(\zeta^{(j)}_{\alpha}), e^{\phi_0}] \\
f_{4}^{(j)}(\zeta^{(j)}_{\alpha}) = ip^{(j)}[q_2F_3(\zeta^{(j)}_{\alpha}), e^{\phi_0} + q_2F_4(\zeta^{(j)}_{\alpha}), e^{\phi_0}] \\
f_{5}^{(j)}(\zeta^{(j)}_{\alpha}) = a_{\alpha}(j) \ln \zeta^{(j)}_{\alpha}
\]

\[
f_{6}^{(j)}(\zeta^{(j)}_{\alpha}) = \sum_{k=1}^{\infty} \zeta^{(j)}_{\alpha} r_k^{(j)}, \quad f_{7}^{(j)}(\zeta^{(j)}_{\alpha}) = \sum_{k=1}^{\infty} \zeta^{(j)}_{\alpha} s_k^{(j)}
\]

The continuity conditions (3.95)\textsubscript{3,4} now provide

\[
q_1^{(1)} = \mathbf{X}_1(\mathbf{B}^{(2)} - \mathbf{A}^{(2)} - \mathbf{B}^{(1)} - \mathbf{A}^{(1)}), \quad q_1^{(2)} = \mathbf{X}_2(\mathbf{A}^{(1)} - \mathbf{B}^{(1)}), \\
q_2^{(1)} = p_1^{(1)} \mathbf{P}^{(2)} q_1^{(1)}, \quad q_2^{(2)} = p_1^{(1)} \mathbf{P}^{(2)} q_1^{(2)}, \\
q_3^{(1)} = \mathbf{X}_3(\mathbf{A}^{(2)} - \mathbf{B}^{(2)}), \quad q_3^{(2)} = \mathbf{X}_4(\mathbf{B}^{(2)} - \mathbf{A}^{(2)}), \\
q_4^{(1)} = p_1^{(2)} \mathbf{P}^{(2)} q_1^{(1)}, \quad q_4^{(2)} = p_1^{(2)} \mathbf{P}^{(2)} q_1^{(2)}, \\
q_5^{(1)} = \langle a_k^{(2)} \rangle^{-1} \mathbf{X}_5[\mathbf{A}^{(2)} - \mathbf{B}^{(2)}], \quad q_5^{(2)} = \mathbf{X}_5[\mathbf{A}^{(2)} - \mathbf{B}^{(2)}], \\
q_6^{(1)} = \langle a_k^{(2)} \rangle^{-1} \mathbf{X}_6[\mathbf{A}^{(2)} - \mathbf{B}^{(2)}], \quad q_6^{(2)} = \mathbf{X}_6[\mathbf{A}^{(2)} - \mathbf{B}^{(2)}]
\]

while the remaining unknown coefficients have the same form as those in Eqs (3.136)–(3.139) except that \(G_{ij}, G_{2j}\) and \(G_{3j}\) should be replaced by \(G_{7j}, G_{8j}\) and \(G_{9j}\).

### 3.7.3 Green’s functions for an elliptic hole

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3.7.3.1 Boundary conditions. When the inclusion becomes an elliptic hole with finite permittivity and thermal conductivity but vanishing elastic stiffness, the Green's function can be derived based on the results in the last section. In this case, the thermal and electric fields are still continuous across the hole surface, but the hole is free of traction. The boundary conditions on the hole can thus be written as

\[ \mathbf{g}^{(1)} = \mathbf{g}^{(0)}; \quad t^{(n)}_n = [t_1, t_2, t_3, D_n]^T = \mathbf{t}_0 = \mathbf{I}_4 \mathbf{D}_0 \] (3.204)

where \( \mathbf{I}_4 = \{0 \ 0 \ 0 \ 1\}^T \), \( t_i \ (i=1,2,3) \) are the rectangular coordinates of surface force, \( \mathbf{t}_0 \) is surface traction-charge vector in the hole, the subscript "0" stands for the quantity associated with hole material, and \( \mathbf{D}_n \) and \( \mathbf{D}_0 \) are the normal components of electric displacement in the matrix and the hole, respectively. Noting that \( \mathbf{t}_n = \mathbf{f}_n \), the boundary condition (3.204) can be further written as

\[ \mathbf{f}^{(1)}|_{t} = \mathbf{I}_4 \int s \mathbf{D}_0 ds = \mathbf{I}_4 \int s (D_{20} dx_1 - D_{10} dx_2) \] (3.205)

By writing the electric potential \( \phi^{(0)}(z) \), where \( z_0 = x_1 + i x_2 \), inside the hole as

\[ \phi^{(0)}(z_0) = 2 \text{Re} \ f_0(z_0) \] (3.206)

the electric displacement \( D_{10} \) can be expressed as

\[ D_{10} = -2 \kappa_0 \text{Re}[df_0/dz_0], \quad D_{20} = 2 \kappa_0 \text{Im}[df_0/dz_0] \] (3.207)

where \( \kappa_0 \) is the dielectric constant of the hole material. Using the above relations, Eq (3.205) can be further written as

\[ \mathbf{f}^{(1)}|_{s} = 2 \kappa_0 \mathbf{I}_4 \text{Im}[f_0(z_0)] \] (3.208)

In addition, the continuous condition of electric potential across the hole surface requires

\[ \phi^{(1)} = 2 \text{Re} \ f_0(z_0) \] (3.209)

3.7.3.2 Green's function for holes. Since the interface condition for thermal field is the same as that in section 3.7.2, the Green's functions given in Eqs (3.187) and (3.188) are still available, but Eq (3.188) should be rewritten in the following form, using the subscript "0" here instead of the subscript "2":

\[ g_0(z)^{(0)} = q_2 \left\{ a_1^{(0)} F_1(z^{(0)}, e^{(0)}) + a_2^{(0)} F_4(z^{(0)}, e^{(0)}) \right\} - e_1 a_1^{(0)} \ln \xi^{(0)} + \sum_{j=1}^{[G_0]} \xi_j^{(0)} + \xi_j^{(0)} \right\} \] (3.210)

and \( \xi^{(0)} \) is related to \( z^{(0)}(x_1 + p^{(0)} x_2) \) by

\[ z_1^{(0)} = a_1^{(0)} \xi^{(0)} + a_2^{(0)} \xi^{(0)} \] (3.211)

The boundary conditions (3.208) and (3.209) together with Eqs (3.187) and (3.210) suggest that the solutions \( \phi^{(0)}, \phi^{(1)} \) and \( \mathbf{f}^{(1)} \) be chosen as
\[ \phi^{(0)}(\zeta_0) = 2 \text{Re} \left[ \sum_{k=1}^{5} f_{k0}(\zeta_0)q_{k0} + f_{i0}(\zeta_0) + f_{j0}(\zeta_0) + c_{4}^{(0)}g_{0}(\zeta_{10}) \right] \quad (3.212) \]

\[ \phi^{(1)}(\zeta) = 2 \text{Re} \left[ A_{4}^{(1)} \left( \sum_{k=1}^{5} \left( f_{k0}(\zeta_0) \right) q_{k}^{(1)} + f_{i}(\zeta) + f_{j}(\zeta) \right) + c_{4}^{(1)}g_{1}(\zeta_{1}) \right] \quad (3.213) \]

\[ \phi^{(1)} = 2 \text{Re} \left[ B_{1} \left( \sum_{k=1}^{5} \left( f_{k0}(\zeta_{1}) q_{k}^{(1)} + f_{i}(\zeta) + f_{j}(\zeta) \right) + d^{(1)}g_{1}(\zeta_{1}) \right) \right] \quad (3.214) \]

where \( \zeta_{1} = r_{1}^{(1)}(\zeta_{1}) \), \( q_{k0} \) and \( q_{k}^{(1)} = \{ q_{k1}, q_{k2}, q_{k3}, q_{k4} \}^{T} \) are constants to be determined, \( A_{4}^{(1)} = (A_{14}^{(1)} A_{24}^{(1)}, A_{34}^{(1)} A_{44}^{(1)}) \), and \( A_{ij} \) are the components of matrix \( A \), and

\[ f_{1a}(\zeta_{a}) = a[q_{0}F_{3}(\zeta_{a}, e^{a_{0}}) + q_{0}F_{4}(\zeta_{a}, e^{a_{0}})] + q_{1}F_{3}(\zeta_{a}, e^{-a_{0}}) + q_{1}F_{4}(\zeta_{a}, e^{-a_{0}})]/2 \] \( \alpha = 0-4 \) (3.215)

\[ f_{2a}(\zeta_{a}) = i p_{a}b[-q_{0}F_{3}(\zeta_{a}, e^{a_{0}}) + q_{0}F_{4}(\zeta_{a}, e^{a_{0}})] + q_{1}F_{3}(\zeta_{a}, e^{-a_{0}}) + q_{1}F_{4}(\zeta_{a}, e^{-a_{0}})]/2 \] \( \alpha = 0-4 \) (3.215)

\[ f_{3a}(\zeta_{a}) = a[q_{0}F_{3}(\zeta_{a}, e^{a_{0}}) + q_{2}F_{4}(\zeta_{a}, e^{a_{0}})]/2 \] \( \alpha = 0-4 \) (3.217)

\[ f_{4a}(\zeta_{a}) = i p_{a}b[-q_{2}F_{3}(\zeta_{a}, e^{a_{0}}) + q_{2}F_{4}(\zeta_{a}, e^{a_{0}})]/2 \] \( \alpha = 0-4 \) (3.217)

\[ f_{5a}(\zeta_{a}) = a_{1a}^{(1)} \ln \zeta_{a} \] \( f_{50}(\zeta_{0}) = \ln \zeta_{0} \) \( \alpha = 1-4 \) (3.218)

\[ f_{6}(\zeta) = \sum_{k=1}^{\infty} \left( r_{k}^{(1)} \right) + f_{7}(\zeta) = \sum_{k=1}^{\infty} \left( s_{k}^{(1)} \right) \] \( \alpha = 1-4 \) (3.219)

\[ f_{60}(\zeta_{0}) = \sum_{k=1}^{\infty} \zeta_{0}^{k} r_{k0}, \quad f_{70}(\zeta_{0}) = \sum_{k=1}^{\infty} \zeta_{0}^{k} s_{k0} \] \( \alpha = 1-4 \) (3.220)

where \( p_{m} = p_{m}^{(1)}, \quad p_{d} = i, \quad r_{k}^{(1)} = \{ r_{k1}, r_{k2}, r_{k3}, r_{k4} \}^{T} \) and \( s_{k}^{(1)} = \{ s_{k1}, s_{k2}, s_{k3}, s_{k4} \}^{T} \) are two four-component vectors to be determined.

The interface condition \( (3.209) \) provides

\[ q_{10} = A_{4}^{(1)} q_{1}^{(1)} + c_{4}^{(1)}, \quad \text{with} \quad q_{20} = \text{Re} \left[ \sum_{j=1}^{4} A_{4j}^{(1)} p_{j} q_{j}^{(1)} - i \tau_{1} c_{4}^{(1)} \right] \] \( \alpha = 0-4 \) (3.221)

\[ q_{30} = A_{4}^{(1)} q_{3}^{(1)} + c_{4}^{(1)}, \quad \text{with} \quad q_{40} = \text{Re} \left[ \sum_{j=1}^{4} A_{4j}^{(1)} p_{j} q_{j}^{(1)} + i \tau_{2} c_{4}^{(1)} \right] \] \( \alpha = 0-4 \) (3.222)

\[ q_{50} = A_{4}^{(1)} q_{5}^{(1)} + a_{a}^{(1)} a_{a}^{(1)} q_{a}^{(1)} + e_{a}^{(1)} a_{a}^{(1)} c_{4}^{(1)} \] \( \alpha = 0-4 \) (3.223)

The remaining unknowns \( r_{jo}, s_{jo}, q_{k}^{(1)}, r_{s}^{(1)} \) and \( s_{k}^{(1)} \) can also be determined through use of the boundary conditions \( (3.208), (3.209) \) and Eq \( (3.94) \), and we will omit those details, which are tedious and algebraic.

3.7.3.3 Cracks. By letting \( b \to 0 \) in Eq \( (3.55) \), the problem discussed above becomes an infinite piezoelectric solid containing a slit crack of length \( 2a \). In this case, Eqs
(3.187) and (3.188) are reduced to
\[
g_i(\zeta_i^{(m)}) = \frac{a}{2} \left\{ q_0 F_1(\zeta_i^{(m)}, e^{i\theta_0}) + q_1 F_4(\zeta_i^{(m)}, e^{-i\theta_0}) + q_0 F_4(\zeta_i^{(m)}, e^{i\theta_0}) \right\} \\
+ q_1 F_3(\zeta_i^{(m)}, e^{-i\theta_0}) + \alpha_1 \ln \zeta_i^{(m)} + \sum_{j=1}^{\infty} \frac{G_{ij}(\zeta_j^{(m)})}{j} \]

(3.224)
\[
g_0(\zeta_i^{(0)}) = \frac{a q_2}{2} \left\{ F_1(\zeta_i^{(0)}, e^{i\theta_0}) + F_4(\zeta_i^{(0)}, e^{-i\theta_0}) \right\} - \alpha_1 \ln \zeta_i^{(0)} \\
+ \sum_{j=1}^{\infty} \left[ \frac{G_{ij}(\zeta_j^{(0)})}{j} + G_{ij}(\zeta_j^{(0)}) \right] \]

(3.225)

Therefore the solutions \(\phi^{(0)}, \phi^{(1)}\) and \(\varphi^{(1)}\) should be chosen as
\[
\phi^{(0)}(\zeta_0) = 2 \Re \left\{ \sum_{k=1,3,5} f_{4k}(\zeta_0)q_{4k} + f_{60}(\zeta_0) + f_{70}(\zeta_0) + c_{4i}^{(0)} g_0(\zeta_i^{(0)}) \right\} \quad (3.226)
\]
\[
\phi^{(1)}(\zeta) = 2 \Re \left\{ A_{4i}^{(1)} \sum_{k=1,3,5} \left( f_{4k}(\zeta_0) \right) q_{4k}^{(1)} + f_6(\zeta) + f_7(\zeta) + c_{4i}^{(1)} g_1(\zeta_i^{(1)}) \right\} \quad (3.227)
\]
\[
\varphi^{(1)} = 2 \Re \left\{ B_{4i}^{(1)} \sum_{k=1,3,5} \left( f_{4k}(\zeta_0) \right) q_{4k}^{(1)} + f_6(\zeta) + f_7(\zeta) + d_{4i} g_1(\zeta_i^{(1)}) \right\} \quad (3.228)
\]

Similarly, the unknown constants in Eqs (3.226)-(3.228) can be determined by using the boundary conditions (3.208) and (3.209).

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