Chapter 8 Effective properties of thermopiezoelectricity

8.1 Introduction

In Chapters 3 and 4, we presented a linear theory of multi-field materials and its solutions for some special problems such as analytical expressions of a 2D thermopiezoelectric plate with a crack of finite length. Based on the theoretical results presented in those two chapters, micromechanics models including the generalized self-consistent method, differential approach, and Mori-Tanaka method are presented in this chapter to predict effective material properties of defective multi-field materials and heterogeneous materials.

It is well known that piezoelectric ceramics are brittle materials. Thus, they may develop various microdefects such as microcracks, delamination, and microvoids during the production process and service period. This drawback has driven the development of composites of piezoelectric ceramics combined with piezoelectrically inactive polymers or other ductile materials that exhibit higher toughness than the piezoelectric ceramic alone [1]. To accurately predict the effects of microdefects on material performance and to assist mechanical engineers in developing piezoelectric composites for electromechanical transducers and engineering smart material applications, it is valuable to develop reliable theories to predict the changes in material performance due to the presence of these microdefects and the effects of the material properties and microstructural geometry of the constituents on the effective electroelastic behaviour of the composite. Early in 1978, Newnham et al [1] presented a connectivity theory based on the combination of mechanics of material type parallel and series models to predict effective pyroelectric behaviour. Banno [2] generalized the connectivity theory to include the effects of a discontinuous reinforcement phase of particle reinforced piezocomposites. Grekov et al [3] further presented a concentric cylinder model for evaluating effective electroelastic properties of piezocomposites reinforced by long fibres. Dunn and Taya [4] studied the overall properties of piezoelectric composites containing interacting inhomogeneities using dilute method, self-consistent model, differential approach, and Mori-Tanaka method and obtained an explicit expression in a surface integral form for coupled electroelastic Eshelby tensors. With regard to the determination of effective thermal expansion and pyroelectric properties, Dunn [5] evaluated the effective pyroelectric properties of two-phase composites, again using the four micromechanics models mentioned above. Benveniste [6] showed that the effective thermal-stress constants and pyroelectric...
coefficients are related to the corresponding isothermal electroelastic moduli in two-phase media. For multiphase media, Benveniste [7] further indicated that the effective thermal-stress constants and pyroelectric coefficients follow from knowledge of the influence functions related to an electromechanical loading of the composite aggregate. Chen [8] obtained some formulae for the prediction of overall thermoelectroelastic moduli of multiphase fibrous composites, using the self-consistent and Mori-Tanaka methods. Qin and Yu [9] and Mai et al [10] presented effective thermoelectroelastic properties of cracked piezoelectric solids using the self-consistent and Mori-Tanaka methods. Benveniste and Dvorak [11] showed that for a two-phase system, exact connections can be obtained not only between the effective moduli, but also among the local pointwise fields induced by a uniform electromechanical loading. Later, the connections were generalized to study piezoelectric fibrous composites of three or four phases [12,13]. The phase boundaries are cylindrical but otherwise the microgeometry is totally arbitrary. Qin et al [14-16] developed a family of micromechanics models for evaluating defective thermopiezoelectric materials. Levin et al [17,18] developed self-consistent formulations for estimating the effective properties of piezocomposites with ellipsoidal inclusions. Using a generalized eigenstrain approach, Huang [19] obtained a unified explicit expression for the coupled electroelastic Eshelby tensors for piezoelectric ellipsoidal inclusions in a transversely isotropic medium. Based on the equivalent inclusion method and the Mori-Tanaka approach, Huang and Kuo [20] and Kuo and Huang [21] investigated the effective material behaviour of piezocomposite containing short fibres. They found that the longitudinal and in-plane shear moduli increased with fibre length, while the other moduli, piezoelectric and dielectric constants decreased. The method in [20,21] was later used to analyse effective material behaviour affected by microvoids [22] and to establish a statistical micromechanics model [23]. Using a method of 2-scale asymptotic expansions, Wojnar [24] analyzed the homogenization process of a piezoelectric periodic composite in which thermal effects are taken into account. Eduardo et al [25] used the 2-scale method to investigate anti-plane problems of thermopiezoelectric fibrous composites. Hori and Nemati-Nasser [26] generalized the Hashin–Shtrikman variational principle to the coupled problem of piezoelectricity and presented the upper and lower bounds for the effective moduli of heterogeneous piezoelectric materials. Based on the concept of a cell model, Poizat and Sester [27] studied 1-3 and 0-3 composites made of piezoceramic fibres embedded in a soft non-piezoelectric matrix; Beckert et al [28] estimated the relevant effective electromechanical parameters of composites continuously reinforced with coated piezoelectric fibres; Li et al [29] examined the influence of void volume fraction, void distribution, void shape and configuration on the effective properties of voided piezoelectric ceramics; Berger et al [30] presented an asymptotic homogenization method and its numerical model for 1–3 periodic composites made of piezoceramic fibres embedded in a soft non-piezoelectric matrix. Using the self-consistent approach, the orientation distribution function, and traditional Voigt-Reuss averages, Li [31] evaluated the effective electroelastic moduli of textured piezoelectric polycrystalline aggregates. Jiang et al [32] presented a generalized self-consistent method for analysing the effective electroelastic behaviour of anti-
plane fibrous piezocomposites by means of a three-phase confocal elliptical cylinder model. Recently, Wang et al [33] combined a micromechanics approach with a boundary node element to evaluate the effective electroelastic properties of transversely isotropic piezoelectric materials containing randomly distributed voids. Qin [34] developed a micromechanics–boundary element algorithm for predicting defective piezoelectric materials. Most of the developments in this field can also be found in [35-42]. This chapter, however, focuses on the results presented in [4,5,9,10,14-16,19,34].

8.2 Micromechanics model of thermopiezoelectricity with microcracks

8.2.1 Basic formulation of two-phase thermopiezoelectricity

It can be seen from the discussion in Section 4.2.4 that the resulting multifield theory is concerned with the piezoelectric analog of the uncoupled theory of thermoelasticity where the magnetic, electric, and elastic fields are fully coupled, but temperature enters the problem only through the constitutive equations. As a result of this, the effective conductivity and the effective electroelastic or magnetoelastoelectric constants can be determined independently, while evaluation of the effective thermal expansion and pyroelectric coefficients requires information about both of them. Accordingly, our derivation is divided into three major steps: First, develop formulations for effective conductivity; then find expressions for effective electroelastic (or magnetoelastic) constants; and finally, derive effective thermal expansion and pyroelectric coefficients based on the results obtained from the first two steps. To illustrate this process, we consider a two-dimensional piezoelectric plate weakened by microcracks. If this is a generalized plane stress problem, its thermoelastic constitutive relationship can be obtained by extending Eq (3.2.31) adding thermal terms. The addition rule is based on Eq (3.6.6):

\[
\begin{align*}
\sigma_1 &= c_{11} e_{11} + c_{13} e_{13} + c_{33} e_{33} + e_{31} e_{31} - \lambda_{11} \lambda_{11} e_{11} - \lambda_{13} \lambda_{13} e_{33} + \lambda_{13} \lambda_{13} e_{13} - \rho_1 T \\
\sigma_3 &= c_{11} e_{11} + c_{13} e_{13} + c_{33} e_{33} + e_{31} e_{31} - \lambda_{11} \lambda_{11} e_{11} - \lambda_{13} \lambda_{13} e_{33} + \lambda_{13} \lambda_{13} e_{13} - \rho_1 T \\
D_1 &= 0 \quad 0 \quad e_{33} \quad 0 \quad -\kappa_{11} \quad 0 \quad -E_i \\
D_3 &= 0 \quad 0 \quad e_{33} \quad 0 \quad -\kappa_{33} \quad 0 \\
\end{align*}
\]  

(8.2.1)

\[
\begin{align*}
\begin{bmatrix}
h_1 \\
h_3 \\
\end{bmatrix} &= \begin{bmatrix}
k_{11} & k_{13} \\
k_{33} & k_{33} \\
\end{bmatrix} \begin{bmatrix}
W_i \\
W_3 \\
\end{bmatrix}
\end{align*}
\]  

(8.2.2)

where heat intensity is defined by
If we choose heat flow $h$, stress $\sigma$, and electric displacement $D$ as independent variables, the constitutive equations (8.2.1) and (8.2.2) become:

\[
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_3 \\
\varepsilon_5 \\
-E_1 \\
-E_3
\end{bmatrix}
= \begin{bmatrix}
f_{11} & f_{13} & 0 & 0 & p_{31} \\
f_{13} & f_{33} & 0 & 0 & p_{33} \\
0 & 0 & f_{35} & p_{35} & 0 \\
0 & 0 & p_{15} & -\beta_{11} & 0 \\
0 & 0 & p_{33} & 0 & -\beta_{33}
\end{bmatrix}
\begin{bmatrix}
\sigma_1 \\
\sigma_3 \\
\sigma_5 \\
D_1 \\
D_3
\end{bmatrix}
+ \begin{cases}
\alpha_{11} \\
\alpha_{33}
\end{cases}
(8.2.4)
\]

\[
\begin{bmatrix}
H_1 \\
H_3
\end{bmatrix}
= \begin{bmatrix}
\rho_{11} & \rho_{13} \\
\rho_{33}
\end{bmatrix}
\begin{bmatrix}
q_1 \\
q_3
\end{bmatrix}
(8.2.5)
\]

where $\rho_j$ = heat resistivity. These equations can also be written in matrix form as:

\[
q = kH, \quad H = \rho q
\]

(8.2.6)

\[
\Pi = EZ - \Gamma T, \quad Z = F\Pi + \alpha T
\]

(8.2.7)

with

\[
q = \{q_1, q_3\}^T, \quad H = \{H_1, H_3\}^T
\]

(8.2.8)

\[
\Pi = \{\sigma_1, \sigma_3, \sigma_5, D_1, D_3\}^T
\]

(8.2.9)

\[
Z = \{Z_{11}, Z_{22}, Z_{12}, Z_{31}, Z_{33}\}^T = \{\varepsilon_1, \varepsilon_3, \varepsilon_5, -E_1, -E_3\}^T
\]

(8.2.10)

Generally, a crack may be viewed as an inclusion with zero mechanical stiffness. Thus, micromechanics theories of a cracked piezoelectric solid can be established based on some fundamental results in the theory of two-phase media. In the case of two-phase materials, the volume average of a physical variable $F$ is defined by:

\[
F = v_1F_1 + v_2F_2
\]

(8.2.11)

where subscripts ‘1’ and ‘2’ denote the matrix and inclusion phases, respectively, $v_1$ and $v_2$ their volume (or area) fractions, and overbar denotes the volume (or area for 2D analysis) average of a quantity over a representative volume element $\Omega$, i.e.,
\[
(\bar{\mathbf{w}}) = \frac{1}{\Omega} \int_{\Omega} \mathbf{w} \, d\Omega
\]  
(8.2.12)

The effective properties represented by the effective heat conductivity \( k'_e \) (or effective heat resistivity \( \rho'_e \)), the effective generalized stiffness \( E'_g \) (or generalized compliancy \( F'_g \)), and the effective generalized stress-thermal coefficients \( \Gamma'_g \) (or generalized thermal expansion \( \alpha'_g \)) of the cracked piezoelectric solid can be defined by the concept of the volume average (8.2.12) as

\[
\overline{q} = k' \bar{\mathbf{H}}, \quad \bar{\mathbf{f}} = \rho' \overline{q}
\]  
(8.2.13)

\[
\bar{\mathbf{f}} = E' \bar{\mathbf{Z}} - \Gamma' \bar{T}, \quad \bar{\mathbf{Z}} = F' \bar{\mathbf{f}} + \alpha' \bar{T}
\]  
(8.2.14)

Since the effective conductivity and the effective electroelastic constants can be determined independently, the applied remote temperature change \( T_e \) is set to be zero when we study effective electroelastic constants. Following the average energy theorem [43], we have

\[
\bar{T} = T_e = 0
\]  
(8.2.15)

In this case, Eq (8.2.14) becomes

\[
\bar{\mathbf{f}} = E' \bar{\mathbf{Z}}, \quad \bar{\mathbf{Z}} = F' \bar{\mathbf{f}}
\]  
(8.2.16)

Making use of Eq (8.2.11), the average generalized stress and strain can be written as:

\[
\bar{\mathbf{f}} = v_1 \mathbf{f}_1 + v_2 \mathbf{f}_2, \quad \bar{\mathbf{Z}} = v_1 \mathbf{Z}_1 + v_2 \mathbf{Z}_2
\]  
(8.2.17)

Substituting Eq (8.2.17) into Eq (8.2.16) and noting that \( \mathbf{E}_e = E_0 \mathbf{Z}_0 \), we obtain

\[
E' = E_1 + (E_2 - E_1) \mathbf{A}_1 v_2
\]  
(8.2.18)

\[
F' = F_1 + (F_2 - F_1) \mathbf{B}_1 v_2
\]  
(8.2.19)

where the symmetric tensors \( \mathbf{A}_2 \) and \( \mathbf{B}_2 \) are defined by the linear relations [15]:

\[
\mathbf{Z}_2 = \mathbf{A}_2 \mathbf{Z}_0, \quad \bar{\mathbf{f}}_2 = \mathbf{B}_2 \mathbf{Z}_0
\]  
(8.2.20)

with \( \mathbf{Z}_0 \) and \( \mathbf{H}_0 \) being the remote generalized stress and strain fields applied the effective medium.

Eq (8.2.20) cannot be used directly to analyze problems with voids or cracks due to the difficulty in evaluating \( \mathbf{Z}_2 \). To bypass this problem, we consider first the case when inclusions become voids. This implies that \( E_2 \to 0, \quad F_2 \to \infty \). In this case, the voids under consideration can be thought of as being filled with air,
which has a dielectric constant approximately three orders of magnitude smaller
than the dielectric constants of piezoelectric materials. The consequence of this
fact is that the boundary conditions on the hole boundary are given by $\mathbf{n} \cdot \mathbf{E} = 0$, where $\mathbf{n}$ is outward normal to the hole boundary. This is also equivalent to setting $\mathbf{E}_2 = 0$, where $\mathbf{E}_2$ stands for the material constants of the hole-phase. Then, Eqs (8.2.18) and (8.2.19) become

$$
\mathbf{E}^* = \mathbf{E}_1 (\mathbf{I} - \mathbf{A}_0 \mathbf{v}_2) \quad (8.2.21)
$$

$$
\mathbf{F}^* = \mathbf{F}_1 (\mathbf{I} + \mathbf{B}_0 \mathbf{v}_2) \quad (8.2.22)
$$

where $\mathbf{I}$ is the unit tensor, $\mathbf{A}_0$ is $A_2$ of Eq (8.2.18) for voids, and $\mathbf{B}_0$ is defined by [15]

$$
\mathbf{Z}_2 = \mathbf{F}_0 \mathbf{B}_0 \mathbf{I}_n
$$

The interpretation of $\mathbf{Z}_2$ in Eq (8.2.23) follows the average strain theorem [43]:

$$
(\bar{Z}_n)_{ij} = \frac{1}{\Omega_2} \int_{\Omega_2} Z_{ij} d\Omega_2 = \frac{1}{2\Omega_2} \int_{\partial\Omega_2} ([1 + H(i-3)]U_i n_j + U_j n_i) d\Omega_2 \quad (8.2.24)
$$

where $\Omega_2$ and $\partial\Omega_2$ are the total area and boundary of the voids, $\mathbf{n} = [n_1, n_2, 0]^T$ is the normal local to the void surface, $\mathbf{U} = [U_1, U_2, U_3]^T = [u_t, u_\phi]^T$, and $H(i)$ is the Heaviside step function.

Cracks are defined as very flat voids of vanishing height and thus also of vanishing area. Multiplying both sides of Eq (8.2.24) by $\mathbf{v}_2$ and considering the limit of flattening out in cracks, i.e., $v_2 \to 0$, one has

$$
\lim_{v_2 \to 0} (\bar{Z}_n)_{ij} v_2 = \frac{1}{2A} \int_L ([1 + H(i-3)]\Delta U_i n_j + \Delta U_j n_i) dl = X_g \quad (8.2.25)
$$

where $L = \cup_{l_1, \ldots, l_N}$, $l_i$ is the length of the $i$th crack, $N$ the number of cracks within the representative area element, $\Delta(*)$ stands for the jump of a quantity across the crack faces. For convenience, we define [44]

$$
\mathbf{P} = \lim_{v_2 \to 0} (\mathbf{A}_0 \mathbf{v}_2), \quad \mathbf{Q} = \lim_{v_2 \to 0} (\mathbf{B}_0 \mathbf{v}_2) \quad (8.2.26)
$$

Hence Eqs (8.2.21) and (8.2.22) can be rewritten as

$$
\mathbf{E}^* = \mathbf{E}_1 (\mathbf{I} - \mathbf{P}) \quad (8.2.27)
$$

$$
\mathbf{F}^* = \mathbf{F}_1 (\mathbf{I} + \mathbf{Q}) \quad (8.2.28)
$$

with the relation
\[ X = PZ_{\infty} = F^{\dagger}QI_{\infty} \]  

(8.2.29)

Thus, the estimation of the integral (8.2.25) and thus \( P \) (or \( Q \)) is the key to predicting the effective electroelastic moduli \( E^* \) and \( F^* \). The approximation of the integral (8.2.25) through use of various micromechanics models is the subject of the subsequent subsections.

### 8.2.2 Effective conductivity

It can be seen from the discussion in Subsection 8.2.1 that the key point for evaluating the effective properties of a cracked piezoelectric solid is to determine the concentration factors \( P \) and \( Q \), and thus to calculate the integral (8.2.25). For a cracked piezoelectric sheet subjected to a set of far fields \( W_{\infty} \) or \( h_{\infty} \), i.e.,

\[ T(s) = -W_{\infty}x_i \]  

(8.2.30)

or

\[ h_{\infty}(s) = h_{\infty}n_i \]  

(8.2.31)

where \( s \) stands for arc coordinate on boundary, subscript ‘\( \infty \)’ represents far field. Using the boundary conditions (8.2.30) and (8.2.31), the effective conductivity can be determined in the following way. For a cracked piezoelectric sheet, we have from the definition of average field and Eq (8.2.25) [9]

\[ \bar{W}_i = \langle \bar{W}_i \rangle_{id} + \frac{1}{A} \sum_{k=1}^{N} \int_{\Delta T} \Delta T_{n_i} dl \]  

(8.2.32)

\[ \bar{h}_i = \langle \bar{h}_i \rangle_{id} \]  

(8.2.33)

where ‘\( M \)’ represents the quantity associated with the matrix, and \( \Delta T \) stands for the jump of temperature field across crack faces:

\[ \Delta T(x) = T_{\infty}(x) - T_{\infty}(x) \]  

(8.2.34)

with subscripts ‘(U)’ and ‘(L)’ denoting the quantity associated with the upper and lower faces of the crack, respectively. If all cracks are assumed to have the same length and orientation, Eqs (8.2.32) and (8.2.33) can be further written as):

\[ k_{ij}W_{\infty} = k_{ij\infty}W_{\infty} - \frac{k_{ij\infty}}{A} \sum_{k=1}^{N} \int_{\Delta T} \Delta T_{n_i} dl = k_{ij\infty} (W_{\infty} - \bar{W}_i) \]  

(8.2.35)

\[ \rho_{ij}h_{\infty} = \rho_{ij\infty}h_{\infty} - \frac{1}{A} \sum_{k=1}^{N} \int_{\Delta T} \Delta T_{n_i} dl = \rho_{ij\infty} h_{\infty} + \bar{h}_i \]  

(8.2.36)
where subscript ‘c’ denotes the quantity associated with crack. By comparing Eq (8.2.36) with Eq (8.2.25), we see that the concentration factors \( P \) and \( Q \) can be expressed as follows:

\[
\lim_{v_{1} \to 0} \left( \frac{\tilde{W}_c}{v_2} \right) = \frac{1}{A} \int_{L} \Delta T ndl = \rho_{st} Q h_{w} = PW_w = X
\]  

(8.2.37)

It can be seen from Eq (8.2.36) that the solution of \( \Delta T \) along the crack line is required for calculating effective heat resistivity \( \rho^*_{ij} \). For a piezoelectric sheet with a number of cracks, it is very difficult to obtain an analytical solution of \( \Delta T \) when the interactions among cracks are taken into account. In the following, we show how to use micromechanics algorithms to evaluate \( \Delta T \), and then determine \( \rho^*_{ij} \) and \( k^*_{ij} \).

(i) Dilute method

In the dilute assumption we assume that the interaction among cracks in an infinite plate can be ignored. The concentration factor \( P \) is then obtained from the solution of the auxiliary problem of a single crack embedded in an infinite intact plate (see Fig. 8.1(a)). For an infinite plate with a horizontal crack and subject to the far field \( W_{2w} \), the temperature jump across the crack faces was obtained by Atkinson and Clements [45]:

\[
\Delta T(x) = \frac{4kW_{2w}}{k_{11}} (a^2 - x^2)^{1/2}
\]

(8.2.38)

where \( k = (k_{11} - k_{12})^{1/2} \), and \( a \) is the length of the crack. Since with the dilute method we assume that there is no interaction among cracks, the constants \( k \) and \( k_{11} \) in Eq (8.2.38) can be taken as \( k^*_{ij} \) and \( k^*_{11} \). Thus, the concentration factor \( P^*_{11} \) can be expressed as:

\[
P^*_{11} = P^*_{12} = P^*_{21} = 0, \quad P^*_{22} = \frac{2\pi k^*_{ij}e}{k^*_{11}}
\]

(8.2.39)

where superscript ‘DIL’ stands for the quantity associated with dilute method, and \( e = Na^7 / \Omega \) is the so-called crack density parameter. Substituting Eq (8.2.39) into Eqs (8.2.27) and (8.2.28) yields:

\[
k^*_{ij} = \begin{cases} 
  k^*_{11M} - k^*_{22M} & \text{for } i = j = 2 \\
  k^*_{11M} & \text{otherwise}
\end{cases}
\]

(8.2.40)

When \( e << 1 \), we have
\[
\frac{k_{22}^*}{k_{22M}} = 1 - 2\pi\varepsilon \frac{k_M}{k_{11M}} \approx \frac{1}{1 + 2\pi\varepsilon} \frac{k_M}{k_{11M}}
\]  
(8.2.41)

It should be pointed out that the formulation obtained above applies to problems in which all cracks have the same length and are in the horizontal direction.

(ii) Self-consistent method

In the self-consistent method [16], for each crack, the effect of crack interaction is taken into account approximately by embedding each crack directly in the effective medium (see Fig. 8.1b), i.e., the medium having the as yet unknown material properties of the cracked matrix. Obviously, with this method the same form is used as in Eq (8.2.39), except that the subscript ‘M’ is replaced by ‘*’, i.e.,

\[
P_{11}^{sc} = P_{12}^{sc} = P_{21}^{sc} = 0, \quad P_{22}^{sc} = \frac{2\pi\varepsilon}{k_{11}}
\]  
(8.2.42)

where superscript ‘SC’ stands for the quantity associated with self-consistent method.

(iii) Mori-Tanaka method

It can be seen from the discussion above that the dilute method is based on the solution of a single crack embedded in infinite matrix subjected to the far field \(W_{\infty}\) (see Fig. 8.1a). In this case, Eq (8.2.37) becomes

\[
\mathbf{X} = \mathbf{P}^{dil} \mathbf{W}_{\infty}
\]  
(8.2.43)

Alternatively, the self-consistent method is based on the solution of a single crack embedded in an infinite unknown effective material subjected also to the far field \(W_{\infty}\) (see Fig. 8.1b). With this method, Eq (8.2.37) becomes

\[
\mathbf{X} = \mathbf{P}^{sc} \mathbf{W}_{\infty}
\]  
(8.2.44)

In contrast, the Mori-Tanaka method [46] is based on the solution for a single crack embedded in an intact matrix subjected to an applied heat intensity equal to the as yet unknown average field \(\bar{\mathbf{W}}_{\infty}\) in the matrix (see Fig. 8.1c), which means that the introduction of cracks in the matrix results in a value of \(\mathbf{X}\) given by [16]:

\[
\mathbf{X} = \mathbf{P}^{MT} \mathbf{W}_{\infty} = \mathbf{P}^{dil} \bar{\mathbf{W}}_i
\]  
(8.2.45)

Thus, the key point is calculation of \(\bar{\mathbf{W}}_i\). One method is to use Eq (8.2.17)_2 and rewrite it in the form

\[
\bar{\mathbf{W}}_{i'\nu} = \mathbf{W}_{\infty} - \bar{\mathbf{W}}_{\nu'\nu}
\]  
(8.2.46)

For a cracked plate, noting that \(\nu_1 \to 1\) and \(\nu_2 \to 0\), we have
\[\lim_{\nu_1 \to 1}(\overline{W_1}) = \overline{W_1} = W_c - \lim_{\nu_2 \to 0}(\overline{W_2}) = (I - P_{MT})W_c \]  
(8.2.47)

where superscript ‘MT’ denotes the quantity associated with the Mori-Tanaka method. For example, \(P_{MT}\) stands for the concentration factor associated with the Mori-Tanaka method. Substituting Eq (8.2.47) into Eq (8.2.45) yields

\[P_{MT} = P_{DIL}^{-1} (I + P_{DIL})^{-1} \]  
(8.2.48)

Making use of Eq (8.2.39), we obtain

\[p_{12}^{MT} = p_{12}^{DIL} = p_{21}^{MT} = 0, \quad p_{22}^{MT} = \frac{p_{22}^{DIL}}{1 + p_{22}^{DIL}} \]  
(8.2.49)

It can be seen from Eq (8.2.49) that when \(\varepsilon << 1\), \(p_{22}^{MT} \approx p_{22}^{DIL}\).

\[\begin{array}{ccc}
\text{matrix material } k_{ijM} & \text{crack} & \text{effective material } k'_0 \\
\text{(a) Dilute method} & \text{crack} & \text{matrix material } k_{ijM} \\
\overline{W_1} & \overline{W_2} & \overline{W_3}
\end{array}\]

Another way to calculate \(\overline{W_1}\) is to use the results from the dilute method [Eq (8.2.41)]. To this end, assume the average heat intensity \(\overline{W}_{2M}\) in the matrix as:

\[\overline{W}_{2M} = W_{2e} + \overline{W}_{2p} \]  
(8.2.50)

where \(W_{2e}\) and \(\overline{W}_{2p}\) are, respectively, remote heat intensity perpendicular to the crack line and perturbed heat intensity due to the presence of the crack. With the assumption of the Mori-Tanaka method, \(\overline{W}_{2e}\) in Eq (8.2.35) can be written as [46]:

\[\overline{R}_{2e} = \lim_{\nu_1 \to 1}(\overline{H}_{2e}) = 2\pi\varepsilon(H_{2e} + \overline{H}_{2p}) \frac{k_{ijM}}{k_{ijM}} \]  
(8.2.51)

Substituting Eq (8.2.51) into Eq (8.2.47) yields

\[\begin{array}{ccc}
\text{matrix material } k_{ijM} & \text{crack} & \text{effective material } k'_0 \\
\text{(a) Dilute method} & \text{crack} & \text{matrix material } k_{ijM} \\
\overline{W_1} & \overline{W_2} & \overline{W_3}
\end{array}\]

Fig. 8.1 Three typical micromechanics models
\[ H_{2e} = \left(1 + \frac{2\pi\varepsilon k_M}{k_{11M}}\right) \left(H_{2e} + \hat{H}_{2p}\right) \quad (8.2.52) \]

Hence

\[ p_{22}^{DS} = \frac{2\pi\varepsilon k_M / k_{11M}}{1 + 2\pi\varepsilon k_M / k_{11M}} \quad (8.2.53) \]

It can be seen from Eq (8.2.53) that this procedure obtains the same results as Eq (8.2.49).

(iv) Differential method

As was pointed out in Chapter 2, the essence of the differential scheme is the construction of the final cracked medium from the intact material through successive replacement of an incremental area of the current cracked material with that of the cracks [47]. The result obtained below is along the lines given in [44] in the study of the overall moduli of isotropic elastic solids with a penny-shaped crack.

\[ \lim_{v_1 \to 0} \left( v_2 \frac{dK^{DS}}{dv_2} \right) = -k_i^{DS} p_i^{DS}, \quad p_{22}^{DS} = \frac{2\pi\varepsilon k_M}{k_{11M}} \quad (8.2.54) \]

Assume that the cracks are obtained by flattening elliptical voids which have the axes \(a\) and \(ap\) where \(p\) can be made infinitely small. Then the area fraction of the voids is

\[ v_2 = \pi p \sum a^2 / A = \pi \rho_p \quad (8.2.55) \]

Inserting Eq (8.2.55) into Eq (8.2.54) and noting that \(dv_2 = \pi p d\varepsilon\), we have

\[ \frac{dK^{DS}}{d\varepsilon} = -k_i^{DS} p_i^{DS} \quad (8.2.56) \]

with initial condition

\[ k_i^{DS} \big|_{\varepsilon=0} = k_i \quad (8.2.57) \]

where superscript ‘DS’ stands for the quantity associated with the differential scheme. Eq (8.2.56) represents a set of 2x2 coupled nonlinear ordinary differential equations, which can be solved using certain numerical methods, such as the well-known fourth order Runge-Kutta integration scheme.

(v) Generalized self-consistent method

The generalized self-consistent method considered here is based on the effective cracked medium model shown in Fig. 8.2 [48], a crack of length 2\(a\) embedded
in an elliptical matrix material, which in turn is embedded in a material with the as yet unknown effective property of a microcracked solid. The major axis of the elliptical matrix is chosen to be aligned along the crack line, and the area of the surrounding matrix is chosen so as to preserve the corresponding crack density in the matrix. Based on this understanding, the major and minor axes of the ellipse in Fig. 8.2 are assumed to be [48]:

\[ a^* = a + \delta, \quad b^* = \delta \]  

(8.2.58)

where \( \delta \) is determined by

\[ \delta = \frac{a^2}{\pi(a + \delta)} \frac{N a^3}{A} \]  

(8.2.59)

Since it is impossible to find the analytical temperature field for the effective cracked medium model, the approach presented in [48] is used to calculate \( \Delta T \). The method is based on the minimum potential principle of the following functional:

\[ J(T) = \int_{S^H} k_0 T \hat{T} ds + \int_{S^EM} k_0 T \hat{T} ds - \int_{\partial E^M} \Delta T h_{n} n \, dc \]  

(8.2.60)

where \( S^H \) and \( S^{EM} \) are, respectively, regions occupied by the matrix and effective medium, and \( T \) is a kinematic admissible temperature field. Among all the kinematic admissible temperature fields, the exact temperature field gives the minimum potential energy.

![Fig. 8.2 Effective cracked medium model for generalized self-consistent method](image)

Let \( T^H \) be the temperature field for an infinite matrix medium containing a crack of length \( 2a \) and subjected to the far field \( h_{2a} \), and let \( T^{EM} \) be the temperature field for an infinite effective medium having the as yet unknown material properties of a cracked matrix, where inside the medium, there is a crack of length
2a and it is subjected to the far field $h_{2a}$. These temperature fields have been given in [16, 45] as:

$$\Delta T^I = \frac{4q_{2a}}{k^I} - (a^2 - x_2^2)^{1/2} \quad (I=\text{M, EM}) \quad (8.2.61)$$

$$T^I_{(l,1)} = \frac{2q_{2a}}{k^I} \text{Re}[(a^2 - z_1^2)^{1/2} + i\xi_1] = t^I_{(l,1)} q_{2a}, \quad (x_2 > 0, I=\text{M, EM}) \quad (8.2.62)$$

$$T^I_{(l,2)} = -\frac{2q_{2a}}{k^I} \text{Re}[(a^2 - z_2^2)^{1/2} + i\xi_2] = t^I_{(l,2)} q_{2a}, \quad (x_2 < 0, I=\text{M, EM}) \quad (8.2.63)$$

where superscripts ‘M’ and ‘EM’ represent the quantity associated with the solution in an infinite matrix medium and an infinite effective medium, respectively.

The approximate temperature field $T$ is assumed to be the linear superimposition of the above two solutions:

$$T = q_{2a}(\xi^M T^M + \xi^{EM} T^{EM}) \quad (8.2.64)$$

where $T^I = t^I_{(l,1)}$, when $x_2 > 0$, otherwise $T^I = t^I_{(l,2)}$, and $\xi^M$ and $\xi^{EM}$ are the constants to be determined by the principle of minimum potential energy. To determine $\xi^M$ and $\xi^{EM}$, substituting Eq (8.2.64) into Eq (8.2.60) and the vanishing variation of Eq (8.2.60) with respect to $\xi^M$ and $\xi^{EM}$ yields

$$\begin{bmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{bmatrix} \begin{bmatrix} \xi^M \\ \xi^{EM} \end{bmatrix} = a^2 \pi \begin{bmatrix} 1/k^M \\ 1/k^{EM} \end{bmatrix} \quad (8.2.65)$$

where

$$I_{11} = \int_{x_1} k^M T^M_{i,j} ds + \int_{x_1} k^{EM} T^{EM}_{i,j} ds \quad (8.2.66)$$

$$I_{12} = \int_{x_1} k^M T^M_{i,j} ds + \int_{x_1} k^{EM} T^{EM}_{i,j} ds \quad (8.2.67)$$

$$I_{21} = \int_{x_1} k^M T^M_{i,j} ds + \int_{x_1} k^{EM} T^{EM}_{i,j} ds \quad (8.2.68)$$

Thus, $\xi^M$ and $\xi^{EM}$ can be determined by solving Eq (8.2.65), and then substituting the solution of $\xi^M$ and $\xi^{EM}$ into Eq (8.2.63) and subsequently into Eq (8.2.34) for determining $\Delta T$. It can be seen from Eqs (8.2.37) and (8.2.39) that

$$\mathbf{P}^\text{GSC} = \xi^M \mathbf{P}^\text{DIL} + \xi^{EM} \mathbf{P}^\text{SC} \quad (8.2.69)$$

where superscript ‘GSC’ denotes the quantity associated with generalized self-consistent method.
8.2.3 Effective electroelastic constants

To obtain the relations between the effective electroelastic moduli of a cracked medium, the following auxiliary problem is considered:

\[ \mathbf{t}(s) = \mathbf{n} \mathbf{P} \quad \text{or} \quad \mathbf{U}(s) = \mathbf{Z} \mathbf{x} \]  
(8.2.71)

and

\[ T(s) = 0 \]  
(8.2.72)

where \( \mathbf{x} = [x_1, x_2]^T = [x, y]^T \) is a position vector. When the boundary conditions (8.2.71) and (8.2.72) are applied, it follows from the energy theorem that [43]

\[ \bar{\mathbf{P}} = \mathbf{P}_1 \quad \text{or} \quad \bar{\mathbf{Z}} = \mathbf{Z}_1 \]  
(8.2.73)

and

\[ \bar{T} = 0 \]  
(8.2.74)

In the case of a cracked body, the average stress \( \bar{\mathbf{P}} \) and strain \( \bar{\mathbf{Z}} \) defined on the basis of the integral average are [16]:

\[ \bar{\mathbf{P}} = \mathbf{P}_1, \quad \bar{\mathbf{Z}} = \mathbf{Z}_1 + \bar{\mathbf{Z}} \]  
(8.2.75)

where \( \bar{\mathbf{Z}} \) can be calculated through use of Eq (8.2.25), i.e.,

\[ \bar{\mathbf{Z}}_c = \lim_{v \to 0} \left[ (\mathbf{Z}_c)_v \right] = \frac{1}{2\Omega} \int \left[ (1 + H(i - 3))\Delta U_n + \Delta U_n \right] dl = \mathbf{X}_y \]  
(8.2.76)

with \( \Delta U_n \) being the jump of generalized displacement field across the crack faces. Thus, Eq (8.2.75) can be further written as

\[ \bar{\mathbf{E}}^T \bar{\mathbf{Z}}_c = \mathbf{E}_1 \mathbf{Z}_1 - \mathbf{E}_1 \bar{\mathbf{Z}}_c \]  
(8.2.77)

\[ \bar{\mathbf{F}}^T \bar{\mathbf{P}}_c = \mathbf{F}_1 \mathbf{P}_1 + \bar{\mathbf{P}}_c \]  
(8.2.78)

It can be seen from the discussion above that the estimation of \( \bar{\mathbf{Z}}_c \) is the key to predicting the effective electroelastic moduli, while the estimation of \( \bar{\mathbf{P}}_c \) requires the solution of \( \Delta U_n \). For a piezoelectric sheet containing a crack of length \( 2a \) and subjected to a set of far fields \( \mathbf{P}_1 \), the solution has been given in Eq (3.7.115). When there is no applied temperature load, Eq (3.7.115) becomes:

\[ \Delta \mathbf{U}(x, 0) = (a^2 - x^2)^{1/2} \mathbf{C} \mathbf{P}_1 \quad |x| < a \]  
(8.2.79)
where \( \Pi_e = [\sigma_{1(e)}, \sigma_{2(e)}, \sigma_{3(e)}] \) are the applied far fields, and the matrix \( C \) is defined by Eq (3.7.100), which depends on the material constants, i.e. \( C_y = C_y(E) \).

Substituting Eq (8.2.79) into Eq (8.2.76) yields the expression of \( \bar{Z}_e \) as:

\[
\bar{Z}_{ij} = \frac{n_0}{4} (1 + H(i - 3)) C_i n_j + C_{ij} n_i \Pi_e \tag{8.2.80}
\]

where

\[
C_i = \{C_{1i}, C_{2i}, C_{3i}\} \tag{8.2.81}
\]

When all cracks are in the horizontal direction, noting that \( n_1 = 0, n_2 = 1 \) and

\[
\bar{Z}_e = \text{PF}_M \Pi_e = \{\bar{Z}_{22e}, 2\bar{Z}_{12e}, \bar{Z}_{32e}\}^T \tag{8.2.82}
\]

we have

\[
P = \frac{n_0}{2} R_s(E) E_M \tag{8.2.83}
\]

\[
R_s(E) = \begin{bmatrix}
C_{11}(E) & C_{12}(E) & C_{13}(E) \\
C_{12}(E) & C_{22}(E) & C_{23}(E) \\
C_{13}(E) & C_{23}(E) & C_{33}(E)
\end{bmatrix}, \quad E_M = \begin{bmatrix}
c_{33M} & 0 & e_{33M} \\
0 & c_{44M} & 0 \\
e_{33M} & 0 & -\kappa_{33M}
\end{bmatrix} \tag{8.2.84}
\]

where \( C_y(E) \) are functions of as yet unknown material constants. In the following, the results of Eqs (8.2.76)-(8.2.84) are used to establish five micromechanics approximation theories for estimating the effective electroelastic moduli.

(i) Dilute method

For the dilute method, we have \( C_y = C_{yM} = C_y(E_M) \). The concentration factor \( P_{DIL} \) is thus given by

\[
P_{DIL} = \frac{n_0}{2} R_s(E_M) E_M \tag{8.2.85}
\]

(ii) Self-consistent method

Self-consistent theory gives results with the same form as Eq (8.2.85) except that \( R_s(E_M) \) in Eq (8.2.85) is replaced by \( R_s(E') \):

\[
P_{SC} = \frac{n_0}{2} R_s(E') E_M \tag{8.2.86}
\]

(iii) Mori-Tanaka method
For the Mori-Tanaka theory, we have the same form of $P^{MT}$ as in Eq (8.2.48), i.e.,

$$P^{MT} = P^{DL} (I + P^{DL})^{-1} = \frac{\pi E}{2} \mathbf{R}_c (E_M) E_m [I + \frac{\pi E}{2} \mathbf{R}_c (E_M) E_m]^{-1}$$  \hspace{1cm} (8.2.87)

(vi) Differential method

Similar to the formulation of differential theory in Subsection 8.2.2, we have

$$\varepsilon \frac{dE^{DS}}{dc} = -E^{DS} P^{DS}$$  \hspace{1cm} (8.2.88)

with the initial condition

$$E^{DS} |_{c=0} = E_M$$  \hspace{1cm} (8.2.89)

Generally, Eq (8.2.88) represents a set of 3x3 coupled nonlinear ordinary differential equations, which can also be solved with the well-known fourth order Runge-Kutta integration scheme.

(v) Generalized self-consistent method

Similar to the treatment in Subsection 8.2.2.5, the generalized self-consistent method here is also based on the effective cracked medium model in Fig. 8.2. The energy functional corresponding to the electroelastic problem can be defined as

$$J(U) = \frac{1}{2} \int_{S^0} Z' E_{a} Z ds + \frac{1}{2} \int_{S^0} Z' E' Z ds - \int_{a}^{c} \Delta U' \mathbf{\Pi}_e dc$$  \hspace{1cm} (8.2.90)

where

$$Z = \begin{bmatrix} Z_{22} \\ 2Z_{12} \\ Z_{32} \end{bmatrix} = \begin{bmatrix} U_{2,2} \\ U_{2,1} + U_{1,2} \\ U_{3,2} \end{bmatrix} = Z_e \mathbf{\Pi}_e$$  \hspace{1cm} (8.2.91)

and $Z_e$ can be evaluated by substituting Eq (3.7.113) or Eq (3.7.114) into Eq (8.2.91). For a piezoelectric sheet containing a crack of length $2a$ and subjected to a set of far fields $\mathbf{\Pi}_e$, the solution $U$ has been given in Eqs (3.7.113) and (3.7.114). When there is no applied temperature load, Eqs (3.7.113) and (3.7.114) become:

$$U_{(1)} = \text{Re}[\bar{A}F(\bar{z})B^{-1}] \mathbf{\Pi}_e, \quad x_2 > 0,$$  \hspace{1cm} (8.2.92)

$$U_{(2)} = \text{Re}[A F(z) B^{-1}] \mathbf{\Pi}_e, \quad x_2 < 0.$$  \hspace{1cm} (8.2.93)
If we denote \( \mathbf{U}_{(e)} = \text{Re}[\mathbf{A}\mathbf{F}(\mathbf{z})\mathbf{B}^{-1}] \) and \( \mathbf{U}_{(s)} = \text{Re}[\mathbf{A}\mathbf{F}(\mathbf{z})\mathbf{B}^{-1}] \), Eqs (8.2.92) and (8.2.93) can be written in one equation as

\[
\mathbf{U} = \begin{cases} 
\mathbf{U}_{(e)} \Pi_x & x_2 > 0 \\
\mathbf{U}_{(s)} \Pi_x & x_2 < 0 
\end{cases} 
\]

(8.2.94)

Hence, with the generalized self-consistent method, \( \mathbf{U} \) and \( \mathbf{Z} \) can be assumed in the form:

\[
\mathbf{U} = (\xi^M \mathbf{U}_M^M + \xi^EM \mathbf{U}_EM^M) \Pi_x, \quad \mathbf{Z} = (\xi^M \mathbf{Z}_M^M + \xi^EM \mathbf{Z}_EM^M) \Pi_x
\]

(8.2.95)

where \( \xi^M \) and \( \xi^EM \) are unknown constants which can be determined by taking the vanishing variation of the functional (8.2.90) with respect to \( \xi^M \) and \( \xi^EM \). To find the solution \( \xi^M \) and \( \xi^EM \), we consider first the cracked body subjected to a set of far fields \( \Pi_x = \{\sigma_{31e}, 0\}^T \). Substituting Eq (8.2.79) into Eqs (8.2.97) and (8.2.95), we obtain the expression of \( \mathbf{U} \) and \( \mathbf{Z} \) as

\[
\Delta \mathbf{U} = \left[ \begin{array}{c} \mathbf{C}_{11M} \xi^M + \mathbf{C}_{11} \xi^EM \\
\mathbf{C}_{21M} \xi^M + \mathbf{C}_{21} \xi^EM \\
\mathbf{C}_{31M} \xi^M + \mathbf{C}_{31} \xi^EM 
\end{array} \right] \sigma_{31e} = \left( \mathbf{C}_{11} \xi^M + \mathbf{C}_{11} \xi^EM \right) \sigma_{31e}
\]

(8.2.96)

\[
\mathbf{U} = \left[ \begin{array}{c} (\mathbf{U}_R^M)_{11} \xi^M + (\mathbf{U}_R^EM)_{11} \xi^EM \\
(\mathbf{U}_R^M)_{21} \xi^M + (\mathbf{U}_R^EM)_{21} \xi^EM \\
(\mathbf{U}_R^M)_{31} \xi^M + (\mathbf{U}_R^EM)_{31} \xi^EM 
\end{array} \right] \sigma_{31e} = \left( \mathbf{U}_R^M \xi^M + \mathbf{U}_R^EM \xi^EM \right) \sigma_{31e}
\]

(8.2.97)

\[
\mathbf{Z} = \left[ \begin{array}{c} (\mathbf{Z}_R^M)_{11} \xi^M \sigma_{31e} + (\mathbf{Z}_R^EM)_{11} \xi^EM \sigma_{31e} \\
(\mathbf{Z}_R^M)_{21} \xi^M \sigma_{31e} + (\mathbf{Z}_R^EM)_{21} \xi^EM \sigma_{31e} \\
(\mathbf{Z}_R^M)_{31} \xi^M \sigma_{31e} + (\mathbf{Z}_R^EM)_{31} \xi^EM \sigma_{31e} 
\end{array} \right] \xi^M \sigma_{31e} = \left( \mathbf{Z}_R^M \xi^M + \mathbf{Z}_R^EM \xi^EM \right) \sigma_{31e}
\]

(8.2.98)

Then, substituting Eq (8.2.96)-(8.2.98) into Eq (8.2.90) and taking the vanishing variation of Eq (8.2.90) with respect to \( \xi^M \) and \( \xi^EM \) yields

\[
\begin{bmatrix} \Phi_{11} \xi^M \\
\Phi_{12} \xi^M \\
\Phi_{21} \xi^EM \\
\Phi_{22} \xi^EM \end{bmatrix} = \frac{\pi a^2}{2} \begin{bmatrix} \mathbf{C}_{11M} \\
\mathbf{C}_{11} 
\end{bmatrix}
\]

(8.2.99)

where
\[ \Phi^{(1)}_{11} = \int_{S^M} (u^M_{x(1)})^T E^M u^M_{x(1)} ds + \int_{S^{EM}} (u^M_{x(1)})^T E^{EM} u^M_{x(1)} ds \]  
(8.2.100)

\[ \Phi^{(1)}_{12} = \int_{S^M} (u^M_{x(1)})^T E^M u^{EM}_{x(1)} ds + \int_{S^{EM}} (u^{EM}_{x(1)})^T E^{EM} u^{EM}_{x(1)} ds \]  
(8.2.101)

\[ \Phi^{(1)}_{22} = \int_{S^M} (u^{EM}_{x(1)})^T E^M u^{EM}_{x(1)} ds + \int_{S^{EM}} (u^{EM}_{x(1)})^T E^{EM} u^{EM}_{x(1)} ds \]  
(8.2.102)

Solve Eq (8.2.99) for \( \xi^M \) and \( \xi^{EM} \) and denote the solution as \( (1) \) \( \xi^M \) and \( (1) \) \( \xi^{EM} \). Finally, substituting the solution \( (1) \) \( \xi^M \) and \( (1) \) \( \xi^{EM} \) into Eq (8.2.96) and subsequently into Eq (8.2.82) yields three equations for the three components of \( \mathbf{P}^{GSC} \):

\[ p_{GSC} f_{3iM} + p_{GSC} f_{3iM} = \frac{\bar{\rho} \bar{\varepsilon}}{2} (C_{i1} \xi^M_{n(1)} + C^* \xi^{EM}_{n(1)}) \]
(8.2.103)

Similarly, assume \( \mathbf{I}_n = [0 \sigma_{s3} \sigma_{s3}]^T \) and \( \mathbf{I}_n = [0 0 D_{s3}]^T \), and, using the procedure described above, we can finally obtain the following equations for the remaining six components of \( \mathbf{P}^{GSC} \):

\[ p_{GSC} f_{5iM} = \frac{\bar{\rho} \bar{\varepsilon}}{2} (C_{i1} \xi^M_{n(2)} + C^* \xi^{EM}_{n(2)}) \]  
(8.2.104)

Thus, the concentration factor \( \mathbf{P}^{GSC} \) can be determined by solving the nine equations above.
8.2.4 Effective thermal expansion and pyroelectric constants

To ascertain the relations between the thermal and electroelastic moduli of a cracked medium, similar to the treatment in Subsection 8.2.3, an auxiliary remote uniform temperature problem is considered in which the following boundary conditions are prescribed:

$$T(s) = T_\infty$$  \hspace{1cm} \text{(8.2.106)}

and

$$t(s) = 0 \quad \text{or} \quad U(s) = 0$$  \hspace{1cm} \text{(8.2.107)}

When the boundary conditions (8.2.106) and (8.2.107) exist, it follows from the energy theorem that [43]:

$$\bar{\Pi} = 0, \quad \bar{T} = T_\infty, \quad \bar{Z} = a'T_\infty, \quad Z_{\mu} = F_{\mu}\Pi + a_\mu T$$  \hspace{1cm} \text{(8.2.108)}

For the boundary conditions (8.2.106) and (8.2.107), the corresponding fields are defined as:

$$\bar{Z} = 0, \quad \bar{T} = T_\infty, \quad \bar{\Pi} = -\Gamma'T_\infty, \quad \Pi_{\mu} = E_{\mu}Z - \Gamma_{\mu}T$$  \hspace{1cm} \text{(8.2.109)}

where

$$a = [a_{11}, a_{33}, \gamma_3]^T, \quad \Gamma = [\lambda_{11}, \lambda_{33}, \rho_3]^T$$  \hspace{1cm} \text{(8.2.110)}

Making use of Eqs (8.2.12) and (8.2.75), we have

$$a'T_\infty = a_\mu T_\infty + \bar{Z}_\gamma, \quad \Gamma'T_\infty = \Gamma_{\gamma}T_\infty + E_{\gamma}\bar{Z}_\gamma$$  \hspace{1cm} \text{(8.2.111)}

where $\bar{Z}_\gamma$ is defined by Eq (8.2.76) in which $\Delta U$ is given by [see Eq (3.7.115)]

$$\Delta U(x, 0) = -b(a - x_0^2)^{1/2}T_\infty$$  \hspace{1cm} \text{(8.2.112)}

with $b$ being evaluated from Eq (3.7.101). Substituting Eq (8.2.112) into Eq (8.2.76) yields the explicit expression of $\bar{Z}_\gamma$ as:

$$\bar{Z}_\gamma = \begin{bmatrix} Z_{11} \\ Z_{22} \\ Z_{12} \end{bmatrix} = -\frac{\pi\epsilon}{2} \begin{bmatrix} 0 \\ b_2 \\ b_3 \end{bmatrix} T_\infty$$  \hspace{1cm} \text{(8.2.113)}

With the dilute and self-consistent methods, the substitution of Eq (8.2.113) into Eq (8.2.111) yields

$$a^{DL} = \begin{bmatrix} a_{11}^{DL} \\ a_{22}^{DL} \\ \gamma_3^{DL} \end{bmatrix} = \begin{bmatrix} a_{11M} \\ a_{22M} \\ \gamma_{3M} \end{bmatrix} - \frac{\pi\epsilon}{2} \begin{bmatrix} 0 \\ b_{2M} \\ b_{3M} \end{bmatrix}$$  \hspace{1cm} \text{(8.2.114)}
\[ a^{\text{SC}} = \begin{bmatrix} a_{11}^{\text{SC}} \\ a_{22}^{\text{SC}} \\ \frac{\pi c}{2} b_{2M} \\ \gamma_{3M}^{\text{SC}} \\ b_{3M}^{\text{SC}} \end{bmatrix} = \begin{bmatrix} a_{11M} \\ a_{22M} \\ \frac{\pi c}{2} b_{2M} \\ \gamma_{3M} \\ b_{3M} \end{bmatrix} \]  \hspace{1cm} (8.2.115)

For the concentration factors \( P \) and \( Q \), it can be shown that [16]:
\[ \Gamma' = (I - P)\Gamma_M, \quad a' = (I + Q)a_M \]  \hspace{1cm} (8.2.116)

Comparing Eq (8.2.114) with (8.2.116), we see that
\[ Q^{\text{DL}} = -\frac{\pi c}{2} \text{diag} \begin{bmatrix} 0 & b_{2M} \\ \alpha_{33M} & b_{3M} \end{bmatrix} \]  \hspace{1cm} (8.2.117)

which yields \( Q^{\text{MT}} \) of Mori-Tanaka theory:
\[ Q^{\text{MT}} = Q^{\text{DL}} (I + Q^{\text{DL}})^{-1} \]  \hspace{1cm} (8.2.118)

### 8.3 Micromechanics model of thermopiezoelectricity with microvoids

In this section, the effective electroelastic behaviour of void-weakened 2D material is studied via the dilute, self-consistent, Mori-Tanaka, and differential micromechanics theories. For simplicity, all holes are assumed to have the same size and orientation. First the results of perturbed heat intensity, strain, and electric field due to the presence of voids are presented for two-dimensional piezoelectric plates with voids of various shapes, and then the above four micromechanics models can be established based on the perturbation results. These models are applicable to a wide range of holes such as ellipse, circle, crack, triangle, square and pentagon.

In the case of voids, Eqs (8.2.21)-(8.2.24) are still applicable. It can be seen from Eq (8.2.24) that the estimation of temperature, elastic displacement, electric potential, and their integration along the hole boundary is the key to predicting the effective material properties of void-weakened piezoelectric plate. To this end, consider an infinite sheet containing a hole of any one of various shapes, whose contour is described by [49]
\[ x_1 = a(\cos \psi + \eta \cos mp), \quad x_2 = a(\sin \psi - \eta \sin mp) \]  \hspace{1cm} (8.3.1)

where \( 0 < c \leq 1 \), and \( m \) is an integer. By appropriate selection of the parameters \( c \), \( m \), and \( \eta \), we can obtain various shapes of voids, such as ellipse, square, and the like.
8.3.1 Effective conductivity

When a set of far-field $\mathbf{h}_\infty = \{h_{1\infty}, h_{2\infty}\}^T$ is applied on the voided infinite sheet above, the temperature change $T$ at a point on the void boundary has been given in [49] as

$$T = -\frac{a}{k}[c q_{1\infty} \cos \psi + q_{2\infty} \sin \psi - \eta(q_{1\infty} \cos m\psi - q_{2\infty} \sin m\psi)] \quad (8.3.2)$$

$$k^\star = \mathbf{k}_M (\mathbf{I} - \mathbf{A}_{T_0} \mathbf{v}_2), \quad \mathbf{p}^\star = \mathbf{p}_M (\mathbf{I} + \mathbf{B}_{T_0} \mathbf{v}_2) \quad (8.3.3)$$

where $\mathbf{A}_{T_0}$ and $\mathbf{B}_{T_0}$ are defined by:

$$\mathbf{W}_2 = \mathbf{p}_{M} \mathbf{B}_{T_0} \mathbf{h}_s = \mathbf{A}_{T_0} \mathbf{p}_{M} \mathbf{h}_u = \frac{1}{\Omega} \int_{\Omega_\infty} T n ds \quad (8.3.4)$$

Substituting Eq (8.3.3) into Eq (8.3.4) and integrating it along the contour of the void yields

$$\mathbf{W}_2 = \mathbf{R}_T \mathbf{h}_u \quad (8.3.5)$$

where $\mathbf{R}_T$ is a $2 \times 2$ diagonal matrix whose components are

$$R_{12} = R_{21} = 0, \quad R_{11} = \frac{c^2 + m\eta^2}{k(c - m\eta)}, \quad R_{22} = \frac{1 + m\eta^2}{k(c - m\eta)} \quad (8.3.6)$$

Thus, from Eqs (8.3.3)-(8.3.5), we find

$$\mathbf{A}_{T_0} = \mathbf{A}_{T_0} (\mathbf{k}_M, \mathbf{k}^\star) = \mathbf{R}_T \mathbf{k}_M, \quad \mathbf{B}_{T_0} = \mathbf{B}_{T_0} (\mathbf{k}_M, \mathbf{k}^\star) = \mathbf{k}_M \mathbf{R}_T \quad (8.3.7)$$

8.3.2 Effective electroelastic constants

Consider a piezoelectric plate containing a hole whose contour is defined by Eq (8.3.1) and subjected a set of far fields $\mathbf{H}_\infty$. The elastic displacement and electric potential at a point of the hole boundary has been given in [15]:

$$\mathbf{U} = \psi \mathbf{Z}_{1\infty} + \chi \mathbf{Z}_{2\infty} + [\alpha \mathbf{L}^{-1} \mathbf{cos} \psi - \alpha \mathbf{L}^{-1} \mathbf{m} \sin \psi \\
+ \alpha \mathbf{L}^{-1} \mathbf{n} \sin m\psi] \mathbf{t}_{1\infty} - [\alpha \mathbf{S}^{-1} \mathbf{L}^T \mathbf{cos} \psi + \alpha \mathbf{L}^{-1} \mathbf{S}^{-1} \mathbf{m} \sin m\psi \\
- \alpha (\mathbf{H} + \mathbf{S}^{-1} \mathbf{S}^T) (\mathbf{sin} \psi + \eta \mathbf{n} \mathbf{m} \sin m\psi)] \mathbf{t}_{2\infty}$$

(8.3.8)

where

$$\mathbf{Z}_{1\infty} = [\epsilon_{1\infty} \epsilon_{2\infty} - E_{1\infty}]^T, \quad \mathbf{Z}_{2\infty} = [\epsilon_{3\infty} \epsilon_{3\infty} - E_{3\infty}]^T, \quad \mathbf{t}_{1\infty} = [\sigma_{1\infty} \sigma_{2\infty} D_{1\infty}]^T, \quad \mathbf{t}_{2\infty} = [\sigma_{3\infty} \sigma_{3\infty} D_{3\infty}]^T \quad (8.3.9)$$
and $Z_e = E_{m} \Pi_{e}$, $L$, $S$ and $H$ are the well-known real matrices in the Stroh formalism, which is defined by Eq (3.3.49), while $Z_e$ and $\Pi_{e}$ are
\begin{equation}
Z_e = [\varepsilon \kappa e \kappa e \kappa e \varepsilon e \kappa e] - [E_{\kappa e} - E_{\kappa e}]^T, \quad \Pi_{e} = [\sigma_{1e} \sigma_{1e} \sigma_{3e} \sigma_{1e} \sigma_{3e} \sigma_{1e} \sigma_{3e} \sigma_{1e} \sigma_{3e} \sigma_{1e} \sigma_{3e}]^T \quad (8.3.10)
\end{equation}

By substituting Eq (8.3.8) into Eq (8.2.24) and integrating it along the whole contour of the hole, we obtain
\begin{equation}
\bar{Z}_2 = R \Pi_{e} \quad (8.3.11)
\end{equation}

where $R$ is a $5 \times 5$ symmetric matrix whose components are
\begin{align*}
R_{11} &= f_{11} (c - m\eta^2) + (L^{-1})_{11} (c^2 + m\eta^2) \\
R_{12} &= (c - m\eta^2) [f_{13} - (L^{-1}S^T)_{12}] \\
R_{13} &= (c^2 + m\eta^2) (L^{-1})_{12} - (c - m\eta^2) (L^{-1}S^T)_{11} \\
R_{14} &= (c^2 + m\eta^2) (L^{-1})_{13} \\
R_{15} &= (c - m\eta^2) [p_{11} - (L^{-1}S^T)_{13}] \\
R_{22} &= (c - m\eta^2) f_{31} + (1 + m\eta^2) [(H)_{22} + (SL^{-1}S^T)_{22}] \\
R_{23} &= (-c + m\eta^2) (SL^{-1})_{22} + (1 + m\eta^2) [(H)_{21} + (SL^{-1}S^T)_{21}] \\
R_{24} &= (-c + m\eta^2) (SL^{-1})_{23} \\
R_{25} &= (c - m\eta^2) [p_{13} + (1 + m\eta^2) [(H)_{23} + (SL^{-1}S^T)_{23}] \\
R_{33} &= (c - m\eta^2) [f_{44} - 2(SL^{-1})_{12} + (c^2 + m\eta^2) (L^{-1})_{22} + (1 + m\eta^2) [(H)_{31} + (SL^{-1}S^T)_{31}] \\
R_{34} &= (c - m\eta^2) [p_{13} - (SL^{-1})_{32} + (c^2 + m\eta^2) (L^{-1})_{23} \\
R_{35} &= (1 + m\eta^2) [(SL^{-1}S^T)_{23} + (H)_{31}] - (c - m\eta^2) (L^{-1}S^T)_{23} \\
R_{44} &= (c^2 + m\eta^2) (L^{-1})_{33} + (c - m\eta^2) p_{13} \\
R_{45} &= (m\eta^2 - c) (L^{-1}S^T)_{33} \\
R_{33} &= (c - m\eta^2) p_{13} + (1 + m\eta^2) [(SL^{-1}S^T)_{33} + (H)_{33}]
\end{align*}

Thus, from Eqs (8.2.20), (8.2.23) and (8.3.11), we have:
\begin{equation}
A_0 = A_0 (E_{m}, E') = RE_{m}, \quad B_0 = B_0 (E_{m}, E') = E_{m} R \quad (8.3.12)
\end{equation}

In the following, the results (8.3.11) and (8.3.12) are used to establish various micromechanics models for the effective thermoelectroelastic moduli.
8.3.3 Effective concentration factors based on various micromechanics models

8.3.3.1 Effective temperature field

The results (8.3.5) and (8.3.7) can be used to establish micromechanics models for effective conductivity. First, we consider the dilute method. Since the interaction among voids is ignored in the dilute method, noting Eq (8.3.7), \( A_{T_0} \) and \( B_{T_0} \) can be written as:

\[
A_{T_0}^{DIL} = R_f (k_M) k_M, \quad B_{T_0}^{DIL} = k_M R_f (k_M)
\]  
(8.3.13)

Substituting Eq (8.3.13) into Eq (8.3.3) yields

\[
k^{DIL} = k_M [I - v_2 R_f (k_M)] \quad \rho^{DIL} = \rho_M [I + v_2 R_f (k_M)]
\]  
(8.3.14)

\[
A_{T_0}^{SC} = R_f (k^*) k_M, \quad B_{T_0}^{SC} = k_M R_f (k^*)
\]  
(8.3.15)

\[
k^{SC} = k_M [I - v_2 R_f (k^*)] \quad \rho^{SC} = \rho_M [I + v_2 R_f (k^*)]
\]  
(8.3.16)

In the Mori-Tanaka method, we assume that the average perturbed heat intensity \( \tilde{W}_2 \) is related to the average heat intensity of the matrix, \( \tilde{W}_1 \), by [15]:

\[
\tilde{W}_2 = R_f (k_M) k_M \tilde{W}_1 = A_{T_0}^{DIL} \tilde{W}_1
\]  
(8.3.17)

Multiplying the both sides of Eq (8.3.17) by \( v_1 \) and then substituting it into Eq (8.2.46) yields

\[
\tilde{W}_2 = (v_1 I + v_2 A_{T_0}^{DIL})^{-1} A_{T_0}^{DIL} \tilde{W}_1 = A_{T_0}^{MT} \tilde{W}_1
\]  
(8.3.18)

Noting that \( A_{T_0} \) is symmetric, \( A_{T_0}^{MT} \) can be written as:

\[
A_{T_0}^{MT} = (v_1 I + v_2 A_{T_0}^{DIL})^{-1} A_{T_0}^{DIL} = A_{T_0}^{DIL} (v_1 I + v_2 A_{T_0}^{DIL})^{-1}
\]  
(8.3.19)

Similarly, the concentration factor \( B_{T_0}^{MT} \) can be obtained as

\[
B_{T_0}^{MT} = B_{T_0}^{DIL} (v_1 I + v_2 B_{T_0}^{DIL})^{-1}
\]  
(8.3.20)

With regard to the differential method, similar to the treatment in Subsection 8.2.2, we have

\[
\frac{d \kappa^{DS}}{dv_2} = - \frac{k^{DS} A_{T_0}^{DS}}{1 - v_2}, \quad \kappa^{DS} = R_f (k^{DS}) k_M
\]  
(8.3.21)

Subjected to the initial condition
\[ k^{DS} \bigg|_{v_2 \to 0} = k_M \]  

Eq (8.3.21) represents a 2x2 coupled nonlinear differential equations which has a similar structure to that of Eq (8.2.56).

### 8.3.3.2 Effective electroelastic moduli

Making use of Eqs (8.3.11) and (8.3.12), the concentration factors and effective electroelastic moduli corresponding to the following four micromechanics theories can be obtained and listed:

1. **Dilute method**
   
   \[ A_0^{DL} = R(E_M)E_M, \quad B_0^{DL} = E_M R(E_M) \]  
   \[ E^{DL} = E_M \{ I - v_2 R(E_M)E_M \}, \quad F^{DL} = F_M + v_2 R(E_M) \]  

2. **Self-consistent method**
   
   \[ A_0^{SC} = R(E^*)E_M, \quad B_0^{SC} = E_M R(E^*) \]  
   \[ E^{SC} = E_M \{ I - v_2 R(E^*)E_M \}, \quad F^{SC} = F_M + v_2 R(E^*) \]  

3. **Mori-Tanaka method**
   
   \[ A_0^{MT} = A_0^{DL} (v_1 I + v_2 A_0^{DL})^{-1}, \quad B_0^{MT} = B_0^{DL} (v_1 I + v_2 B_0^{DL})^{-1}, \]  
   \[ E^{MT} = E_M \{ I - v_2 R(E_M)E_M \} [I + v_2 E_M R(E_M)]^{-1}, \quad F^{MT} = F_M \{ I + v_2 E_M R(E_M) \} [I + v_2 E_M R(E_M)]^{-1} \]  

4. **Differential scheme**
   
   \[ \frac{dE^{DS}}{dv_2} = - \frac{E^{DS} A_0^{DS}}{1 - v_2}, \quad A_0^{DS} = R(E^{DS})E_M \]  
   \[ E^{DS} \bigg|_{v_2 \to 0} = E_M \]
8.4 Micromechanics model of piezoelectricity with inclusions

8.4.1 Eshelby’s tensors for a composite with an ellipsoidal inclusion

For problems of piezoelectricity with inclusions, Eqs (8.2.15)-(8.2.20) can still be used to predict effective electroelastic properties. Evaluation of $\mathbf{Z}_2$ and the related concentration factor is the key to predicting the effective electroelastic properties. In this section, approaches presented in [4, 21, 50] are described to show how the micromechanics models can be derived. To this end, consider a piezoelectric composite consisting of an infinite domain $D$ containing an ellipsoidal inclusion $\Omega$ defined by

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} \leq 1 \quad \text{(in } \Omega)$$  \hspace{1cm} (8.4.1)

where $a_1$, $a_2$, and $a_3$ are the semi-axes of the ellipsoid with the $a_i$ principle axis coincident with the $x_i$ axis. The assumption that the shape of the inclusion is ellipsoidal allows treatment of composite reinforcement geometries ranging from thin flake to continuous fibre reinforcement. Suppose that the inclusion $\Omega$ has electroelastic moduli $E_i$, while the matrix, $D-\Omega$, has electroelastic moduli $E_M$. The composite is subjected to a set of far fields $Z_\infty$. Using the equivalent inclusion method for piezoelectric composites [4], the generalized stress in the representative inclusion can be written as:

$$\mathbf{\Pi}_i = E_i Z_i = E_i (Z_{in} + Z) = E_{eq}(Z_{in} + Z - Z_*)$$  \hspace{1cm} (8.4.2)

where $Z$ represents the perturbation of the generalized strain in the inclusion with respect to the generalized strain in the matrix and $Z_*$ is the fictitious eigenfield required to ensure that the equivalency of Eq (8.4.2) holds. In Eq (8.4.2), $Z$ and $Z_*$ are related through [4]:

$$Z = \mathbf{S}Z_*$$  \hspace{1cm} (8.4.3)

where $\mathbf{S}$ is the coupled electroelastic analog of Eshelby’s tensor whose components can be expressed in terms of surface integrals over the unit sphere as [4]:

$$\mathbf{S}_{\text{Mlab}} = \begin{cases} 
\frac{a_i a_j a_k}{8\pi} E_{\text{lab}} \int_{|z|=1} \frac{1}{\zeta^2} [G_{m|\text{in}}(z) + G_{\text{in|m}}(z)] ds(z), & M \leq 3 \\
\frac{a_i a_j a_k}{4\pi} E_{\text{lab}} \int_{|z|=1} \frac{1}{\zeta^2} G_{m|\text{in}}(z) ds(z), & M = 4
\end{cases}$$  \hspace{1cm} (8.4.4)

where $|z| = 1$ is the surface of the unit sphere $\zeta = [a_1^2 z_1^2 + a_2^2 z_2^2 + a_3^2 z_3^2]^{1/2}$, and
\[
G_{M|\text{in}}(z) = \xi_n z_n K_{M|\text{in}}^z(z), \quad K_{M}(z) = E_{\epsilon|\text{in}} z_n \xi_n
\] (8.4.5)

To perform the integration in Eq (8.4.4), the unit sphere is parameterized as

\[
z_1 = \frac{\xi_1}{a_1} = \left(1 - \xi_1^2\right)^{1/2} \cos \theta, \quad z_2 = \frac{\xi_2}{a_2} = \left(1 - \xi_2^2\right)^{1/2} \sin \theta, \quad z_3 = \frac{\xi_3}{a_3}
\] (8.4.6)

In general, for an anisotropic medium the integrals in Eq (8.4.4) cannot be evaluated analytically. In this case, the integration is easily performed by Gaussian quadrature.

It should be mentioned that Huang [50] presented an equivalent formulation to Eq (8.4.4) as below:

\[
S_{\text{M|\text{in}}} = \begin{cases}
\frac{1}{8\pi} E_{\epsilon|\text{in}} \left( \vec{G}_{\text{M|\text{in}}} + \vec{G}_{\text{M|\text{in}}} \right) & M \leq 3 \\
\frac{1}{4\pi} E_{\epsilon|\text{in}} \vec{G}_{\text{M|\text{in}}} & M = 4
\end{cases}
\] (8.4.7)

where

\[
\vec{G}_{\text{M|\text{in}}} = a_1 a_2 a_3 \int_{\mathbb{R}^3} \frac{1}{|\xi|^3} N_{M|\text{in}} \left( \xi \bar{\xi} \xi_n \right) D^{-1}(\xi)d\xi
\] (8.4.8)

with \(\xi_n\) being defined in Eq (8.4.6), and \(N_{M|\text{in}}(\xi)\) and \(D(\xi)\) being the cofactor and the determinant of the \(4\times4\) matrix \(E_{\epsilon|\text{in}} \xi_n \xi_n\), respectively [50]. The evaluation of \(N_{M|\text{in}}(\xi)\) and \(D(\xi)\) has been substantively discussed in [50] and we will not repeat it here as it is tedious and algebraic.

Note that \(S\) is a fourth order tensor and it is useful to use the generalized Voight two-index notation. With the two-index notation, the electroelastic Eshelby’s tensor \(S_{\text{M|\text{in}}}\) for an ellipsoidal inclusion in transversely isotropic piezoelectric materials can be expressed in the following form [21]:

\[
[S_{\text{M|\text{in}}} = \begin{bmatrix}
S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\
S_{21} & S_{22} & S_{23} & 0 & 0 & 0 \\
S_{31} & S_{32} & S_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & S_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & S_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & S_{66}
\end{bmatrix}, \quad [S_{\text{M|\text{in}}} = \begin{bmatrix}
0 & 0 & S_{19} \\
0 & 0 & S_{29} \\
0 & 0 & S_{39} \\
0 & S_{49} & 0 \\
S_{57} & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
(8.4.9)

\[
[S_{\text{in}} = \begin{bmatrix}
0 & 0 & 0 & 0 & S_{55} & 0 \\
0 & 0 & 0 & S_{64} & 0 & 0 \\
S_{51} & S_{52} & S_{53} & 0 & 0 & 0
\end{bmatrix}, \quad [S_{\text{in}} = \begin{bmatrix}
S_{77} & 0 & 0 \\
0 & S_{88} & 0 \\
0 & 0 & S_{99}
\end{bmatrix}
\]
(8.4.10)
where

\[S_{11} = S_{111}, \, S_{12} = S_{112}, \, S_{13} = S_{133}, \, S_{19} = S_{143}, \, S_{21} = S_{2211}, \, S_{22} = S_{2222}, \]

\[S_{23} = S_{2323}, \, S_{29} = S_{2329}, \, S_{31} = S_{3311}, \, S_{32} = S_{3322}, \, S_{33} = S_{3333}, \, S_{39} = S_{3343},\]

\[S_{44} = S_{2323}, \, S_{48} = S_{2323}, \, S_{57} = S_{2341}, \, S_{77} = S_{4411}, \] (8.4.11)

\[S_{55} = S_{3311}, \, S_{56} = S_{3322}, \, S_{66} = S_{3333}, \, S_{1221} = S_{2112} = S_{2121}, \]

In particular, the above electroelastic Eshelby tensors for an elliptic cylinder, a circular cylinder, and a penny-shaped inclusion in transversely isotropic piezoelectric solids have been obtained by Huang as [19]:

(i) Elliptic cylinder \((a_1/a_2 = a, \, a_3 \to \infty)\):

\[S_{11} = \frac{a}{2(1+a)^2} \left[ \frac{2c_{11} + c_{12}}{c_{11}} + \frac{2+a}{a} \right], \quad S_{12} = \frac{a}{2(1+a)^2} \left\{ \frac{(2+a)c_{12}}{ac_{11}} - 1 \right\},\]

\[S_{13} = \frac{c_{13}}{(1+a)c_{11}}, \quad S_{19} = \frac{e_{31}}{(1+a)c_{11}}, \quad S_{21} = \frac{a}{2(1+a)^2} \left\{ \frac{(1+2a)c_{12}}{ac_{11}} - 1 \right\},\]

\[S_{22} = \frac{a}{2(1+a)^2} \left[ 2a + \frac{3c_{11} + c_{12}}{c_{11}} \right], \quad S_{23} = aS_{13}, \quad S_{29} = aS_{19},\] (8.4.12)

\[S_{44} = \frac{a}{2(1+a)}, \quad S_{55} = \frac{1}{2(1+a)}, \quad S_{66} = \frac{a}{2(1+a)^2} \left( \frac{a^2 + a + 1}{a} - \frac{c_{12}}{c_{11}} \right),\]

\[S_{77} = \frac{1}{(1+a)}, \quad S_{88} = \frac{a}{1+a} .\]

(ii) Circular cylinder \((a_1 = a_2, \, a_3 \to \infty)\):

\[S_{11} = S_{22} = \frac{5c_{11} + c_{12}}{8c_{11}}, \quad S_{12} = S_{21} = \frac{3c_{11} - c_{13}}{8c_{11}}, \quad S_{23} = S_{32} = \frac{c_{13}}{2c_{11}}, \]

\[S_{29} = S_{19} = \frac{e_{31}}{2c_{11}}, \quad S_{44} = S_{55} = \frac{1}{4}, \quad S_{66} = \frac{3c_{11} - c_{12}}{8c_{11}}, \quad S_{77} = S_{88} = \frac{1}{2},\] (8.4.13)

(iii) Penny-shaped inclusion \((a_1 = a_2 \gg a_3, \, a_3 \to 0)\)

\[S_{44} = S_{55} = \frac{1}{2}, \quad S_{57} = S_{75} = S_{48} = S_{84} = \frac{c_{13}}{2c_{11}}, \quad S_{33} = S_{99} = 1,\]

\[S_{31} = S_{32} = \frac{c_{11}^3K_{13} + e_{11}^2e_{33}}{c_{11}^3K_{33} + e_{33}^2}, \quad S_{91} = S_{92} = \frac{c_{11}e_{33} - c_{13}e_{11}^2}{c_{11}^3K_{33} + e_{33}^2}.\] (8.4.14)
8.4.2 Effective elastoelectric moduli

Substituting Eq (8.4.3) into Eq (8.4.2) yields the generalized strain in the inclusion $Z_2$ as

$$Z_2 = Z_o + SZ_a$$  \hspace{1cm} (8.4.15)

Making use of Eqs (8.4.2) and (8.4.15), $Z_2$ can be further written as

$$Z_2 = [I + SE_{ij} (E_i - E_{ij})]^{-1} Z_o$$  \hspace{1cm} (8.4.16)

By comparing Eq (8.4.16) with Eq (8.2.20), we observe

$$A_2 = [I + SE_{ij} (E_i - E_{ij})]^{-1}$$  \hspace{1cm} (8.4.17)

Similarly, the concentration factor $B_2$ can also be obtained as:

$$B_2 = [I + F_{ij} (I - S)(F_i - F_{ij})]^{-1}$$  \hspace{1cm} (8.4.18)

Eqs (8.4.17) and (8.4.18) provide the results of the concentration factors $A_2$ and $B_2$ by ignoring interaction among inclusions. Therefore, they represent concentration factors $A_2^{MT}$ and $B_2^{MT}$.

For the self-consistent method, noting that each inclusion is assumed to be embedded in an infinite piezoelectric medium, Eqs (8.4.17) and (8.4.18) become:

$$A_2^{SC} = [I + S E^{* -1} (E_i - E^*)]^{-1}$$  \hspace{1cm} (8.4.19)

$$B_2^{SC} = [I + F^{* -1} (I - S^*)(F_i - F^*)]^{-1}$$  \hspace{1cm} (8.4.20)

With regard to Mori-Tanaka method, it can be shown that [4]:

$$A_2^{MT} = A_2^{DDL} (v_1 I + v_2 A_2^{DDL})^{-1}, \quad B_2^{MT} = B_2^{DDL} (v_1 I + v_2 B_2^{DDL})^{-1}$$  \hspace{1cm} (8.4.21)

Finally, we discuss the differential scheme. Following Mclaughlin [47], the removal of a volume increment $dV$ of the instantaneous configuration (thus a removal of $v_2 dV$ of the reinforcing phase) leads to

$$dv_2 = \frac{dV}{V (1-v_2)}$$  \hspace{1cm} (8.4.22)

where $V$ is the volume of the composite. Denoting $E^*(v_2 + dv_2)$ as the effective electroelastic moduli at a reinforcement volume fraction of $(v_2 + dv_2)$, use of Eqs (8.2.18) and (8.4.22) leads to [4]

$$\frac{dE}{dv_2} = \frac{1}{1-v_2} (E_i - E^*) A_2^{DDL}$$  \hspace{1cm} (8.4.23)

where
As in the self-consistent scheme, $S^{\text{off}}$ is a function of $E^*$ of the composite material at a reinforcement volume fraction of $(\nu_2 + d\nu_2)$. Formally, Eq (8.4.23) represents a set of $9 \times 9 = 81$ coupled nonlinear ordinary differential equations which

\[ E^* (\nu_2 = 0) = E_m \]  

(8.4.25)

### 8.4.3 Effective thermal expansion and pyroelectric coefficients

As mentioned in Subsection 8.2.1, evaluation of the effective thermal expansion and pyroelectric coefficients requires information about the effective electroelastic moduli. To obtain the relationships between thermal and coupled electroelastic effects, Dunn [5] considered the following two auxiliary problems:

(i) **Applied uniform electroelastic far-fields**

Consider a two-phase composite subjected to the boundary conditions (8.2.71) and (8.2.72). For the boundary conditions (8.2.71) and (8.2.72), the volume average fields and the phase and overall equations follow from the energy theorem that [43]:

\[ \Pi \equiv \Pi_0, \quad \Pi = 0, \quad \Pi_E = E_0, \quad \Pi_{EM} = E_m \Pi_{EM}, \quad \Pi = F \Pi_0. \]  

(8.4.26)

To distinguish the fields induced by different loading conditions, the left-superscript ‘$\Pi$’ is used to represent fields associated with the applied far-field $\Pi_0$. With the boundary conditions (8.2.71) and (8.2.72), the average electroelastic concentration factors for each phase are defined in a way similar to that in Eq (8.2.20) as:

\[ \Pi_1 = B_1 \Pi, \quad \Pi_2 = B_2 \Pi_0 \]  

(8.4.27)

Likewise, for the boundary conditions (8.2.71) and (8.2.72), the corresponding fields are:

\[ Z = Z_0, \quad Z = 0, \quad Z_E = E_0, \quad Z_{EM} = E_m Z_m, \quad Z = E Z_0 \]  

(8.4.28)

where the left-superscript ‘$Z$’ denotes fields induced by loading conditions (8.2.71) and (8.2.72). The average electroelastic concentration factors for each phase are defined as

\[ Z_1 = A_1 Z_0, \quad Z_2 = A_2 Z_0 \]  

(8.4.29)

(ii) **Applied uniform temperature change**
Consider again the two-phase composite, but subjected to the boundary conditions (8.2.106) and (8.2.107). When the boundary conditions (8.2.106) and (8.2.107) are applied, the volume average fields and the phase and overall equations are as follows

\[
\begin{align*}
\n^T \mathbf{\bar{F}} &= T_e, \\
\n^T \mathbf{\bar{Z}} &= a^T T_e, \\
\n^T \mathbf{Z}_j &= \mathbf{F}_j, \quad \mathbf{\Pi}_j + a_j T_e, \\
\n^T \mathbf{Z}_M &= \mathbf{F}_M, \quad \mathbf{\Pi}_M + a_M T
\end{align*}
\]

(8.4.30)

where the left-superscript ‘T’ denotes fields induced by loading conditions (8.2.106) and (8.2.107). Those for the boundary conditions (8.2.106) and (8.2.107) are

\[
\begin{align*}
\n^T \mathbf{\bar{Z}} &= 0, \\
\n^T \mathbf{\bar{F}} &= T_e, \\
\n^T \mathbf{\bar{Z}} &= -\mathbf{\Gamma}^T T_e, \\
\n^T \mathbf{\Pi}_j &= \mathbf{E}_j, \quad \mathbf{Z}_j - \mathbf{\Gamma}^T T_e, \\
\n^T \mathbf{\Pi}_M &= \mathbf{E}_M, \quad \mathbf{Z}_M - \mathbf{\Gamma}^T T_M
\end{align*}
\]

(8.4.31)

With the boundary conditions (8.2.106) and (8.2.107), the average thermal concentration factors for each phase are defined as:

\[
\begin{align*}
\n^T \mathbf{\bar{Z}}_1 &= \mathbf{B}_1 T_e, \\
\n^T \mathbf{\bar{Z}}_2 &= \mathbf{B}_2 T_e, \\
\n^T \mathbf{\bar{Z}}_3 &= \mathbf{V}_1 T_e, \\
\n^T \mathbf{\bar{Z}}_4 &= \mathbf{V}_2 T_e
\end{align*}
\]

(8.4.32)

It is then necessary to establish relationships between effective thermal property and electroelastic property based on the above results. Based on the theorem of average strain energy [43]

\[
\int_{\Omega} \mathbf{\Pi} \cdot \mathbf{Z} d\Omega = \mathbf{\bar{F}} \cdot \mathbf{\bar{Z}} d\Omega
\]

(8.4.33)

and considering the electroelastic fields due to the boundary conditions (8.2.71) and (8.2.72), substituting Eq (8.4.26) into Eq (8.4.33) and then using Eqs (8.4.26) and (8.4.30), we obtain

\[
\int_{\Omega} \mathbf{\Pi} \cdot \mathbf{Z} d\Omega = \int_{\Omega_1} \mathbf{\Pi} \cdot \mathbf{Z} d\Omega + \int_{\Omega_2} \mathbf{\Pi} \cdot \mathbf{Z} d\Omega = 0
\]

(8.4.34)

Similarly, substituting Eq (8.4.30) into Eq (8.4.33) and then using Eqs (8.4.26), (8.4.30), and (8.4.34) leads to

\[
\int_{\Omega} \mathbf{\Pi} \cdot \mathbf{Z} d\Omega = \int_{\Omega_1} \mathbf{\Pi} \cdot \mathbf{Z} d\Omega + \int_{\Omega_2} \mathbf{\Pi} \cdot \mathbf{Z} d\Omega = 0
\]

(8.4.35)

Similar manipulation with the quantities associated with the loading conditions (8.2.71) and (8.2.72) yields

\[
\int_{\Omega} \mathbf{\Pi} \cdot \mathbf{Z} d\Omega = \int_{\Omega} \mathbf{\Pi} \cdot \mathbf{Z} d\Omega = 0
\]

(8.4.36)

\[
\int_{\Omega} \mathbf{\Pi} \cdot \mathbf{Z} d\Omega = \int_{\Omega} \mathbf{\Pi} \cdot \mathbf{Z} d\Omega = 0
\]

(8.4.37)
Substituting Eqs (8.4.26), (8.4.28), (8.4.27), and (8.4.29) into Eq (8.2.17) leads to the relations

\[ v_1 A_1 + v_2 A_2 = I, \quad v_1 B_1 + v_2 B_2 = I \]  

(8.4.38)

Enforcing the quantity on the right-hand side of Eq (8.4.35) to the results of the middle integrals of Eq (8.4.35), we obtain

\[ \alpha' = v_1 B_1 a_1 + v_2 B_2 a_2 \]  

(8.4.39)

Similar manipulation for Eq (8.4.37) yields

\[ \Gamma' = v_1 A_1 \Gamma_1 + v_2 A_2 \Gamma_2 \]  

(8.4.40)

To express effective thermal property in terms of effective electroelastic property, we need to find relationships between the concentration factors \( A_i \) and \( B_i \) appearing in Eqs (8.4.39) and (8.4.40) and the effective electroelastic moduli of the composite. To this end, substituting Eq (8.4.26) into Eq (8.2.17) and making use of Eq (8.4.27) yields

\[ F' = v_1 F_1 B_1 + v_2 F_2 B_2 \]  

(8.4.41)

which is similar to the expression in Eq (8.4.39). Substituting Eq (8.4.28) into Eq (8.2.17) and making use of Eq (8.4.29) yields

\[ E' = v_1 E_1 A_1 + v_2 E_2 A_2 \]  

(8.4.42)

Inserting Eq (8.4.38) into Eq (8.4.41) to eliminate \( B_2 \) in favor of \( B_1 \) then yields

\[ v_1 B_1 = (F' - F_j)(F' - F_j)^{-1} \]  

(8.4.43)

Similarly, we have

\[ v_1 A_1 = (E' - E_j)(E' - E_j)^{-1} \]  

(8.4.44)

Finally, substituting Eqs (8.4.43) and (8.4.44) into Eqs (8.4.39) and (8.4.40) we obtain

\[ \alpha' = a_j + (F' - F_j)(F' - F_j)^{-1}(a_j - a_j) \]  

(8.4.45)

\[ \Gamma' = \Gamma_j + (E' - E_j)(E' - E_j)^{-1}(\Gamma_j - \Gamma_j) \]  

(8.4.46)

It can be seen from Eqs (8.4.45) and (8.4.46) that \( \alpha' \) and \( \Gamma' \) can be easily evaluated when the effective electroelastic moduli \( F' \) and \( E' \) are obtained in a manner such as the results presented in Subsection 8.4.2. In the following, Eqs (8.4.39), (8.4.40), (8.4.45), and (8.4.46) are combined with the results of micromechanics theories obtained in Subsection 8.4.2 to obtain \( \alpha' \) and \( \Gamma' \) of the composite.

From Eqs (8.4.38)-(8.4.40), it is easy to prove that
\[ \alpha' = a_M + v_2 B_2 (a_j - a_M), \quad \Gamma' = \Gamma_M + v_2 A_2 (\Gamma_j - \Gamma_M) \]  
(8.4.47)

Substituting Eqs (8.4.17) and (8.4.18) into Eq (8.4.47), the dilute method yields the effective thermal expansion and pyroelectric coefficients as

\[ \alpha' = a_M + v_2 [I + F_m' (I - S)(F_j - F_m)]^{-1} (a_j - a_M) \]  
(8.4.48)

\[ \Gamma' = \Gamma_M + v_2 [I + SE_m' (E_j - E_m)]^{-1} (\Gamma_j - \Gamma_M) \]  
(8.4.49)

The substitution of Eqs (8.4.19) and (8.4.20) into Eq (8.4.47) yields the expressions of \( \alpha' \) and \( \Gamma' \) for the self-consistent method as

\[ \alpha' = a_M + v_2 [I + S E^{-1} (E_j - E_m)]^{-1} (a_j - a_M) \]  
(8.4.49)

\[ \Gamma' = \Gamma_M + v_2 [I + S E^{-1} (E_j - E_m)]^{-1} (\Gamma_j - \Gamma_M) \]  
(8.4.50)

With the Mori-Tanaka method, the insertion of Eq (8.4.21) into Eq (8.4.47) yields

\[ \alpha' = a_M + v_2 B^{\text{diff}}_2 (v_1 I + v_2 B^{\text{diff}}_2)^{-1} (a_j - a_M) \]  
(8.4.51)

\[ \Gamma' = \Gamma_M + v_2 A^{\text{diff}}_2 (v_1 I + v_2 A^{\text{diff}}_2)^{-1} (\Gamma_j - \Gamma_M) \]  
(8.4.52)

For the differential method, \( E' \) can be evaluated from Eqs (8.4.23)-(8.4.25), while \( F' \) is determined from the following equations [5]:

\[ \frac{dF}{dv_2} = \frac{1}{1 - v_2} (F_j - F') B^{\text{diff}}_2 \]  
(8.4.53)

subjected to the initial conditions

\[ F'(v_2 = 0) = F_m \]  
(8.4.54)

where

\[ B^{\text{diff}}_2 = [I + F^{-1} (I - S^{\text{diff}})(E_j - E')]^{-1} \]  
(8.4.55)

Substituting Eqs (8.4.24) and (8.4.55) into Eq (8.4.47) yields

\[ \alpha' = a_M + v_2 [I + F^{-1} (I - S^{\text{diff}})(E_j - E')]^{-1} (a_j - a_M) \]  
(8.4.56)

\[ \Gamma' = \Gamma_M + v_2 [I + S^{\text{diff}} E^{-1} (E_j - E')]^{-1} (\Gamma_j - \Gamma_M) \]  
(8.4.57)
8.5 Micromechanics-boundary element mixed approach

It is noted that common to each of the micromechanics theories described in this chapter is the use of the well-known stress and strain concentration factors obtained through an analytical solution of a single crack, void, or inclusion embedded in an infinite medium. However, for a problem with complexity in the aspects of geometry and mechanical deformation, a combination of these micromechanics approaches and numerical methods such as finite element method and boundary element method (BEM) presents a powerful computational tool for estimating effective material properties. It is also noted from Section 8.2 that estimation of the integral (8.2.24), which contains unknown variables on the boundary only, is the key to predicting the concentration factor $A_2$ (or $B_2$). Therefore, BEM is very suitable for performing this type of calculation. In this section, a micromechanics-BE mixed algorithm is presented for analyzing the effective behaviour of piezoelectric composites. The algorithm is based on two typical micromechanics models (self-consistent and Mori-Tanaka methods) and a two-phase BE formulation. An iteration scheme is designated for the self-consistent-BE mixed method.

8.5.1 Two-phase BE formulation

In this subsection, a two-phase BE model is introduced for generalized displacements and generalized stresses on the boundary of the subdomain of each phase [34]. The two subdomains are separated by the interfaces between inclusion and matrix (see Fig. 8.1). Each subdomain can be separately modelled by direct BEM. Global assembly of the BE subdomains is then performed by enforcing continuity of the generalized displacements and generalized stresses at the subdomain interface.

In a two-dimensional piezoelectric composite, the BE formulation takes the form [51]

$$
e^{(\alpha)}(\xi)|U^{(\alpha)}(\xi) = \int_{S^{(\alpha)}} [U_{ij}^{(\alpha)}(x, \xi)T_{ij}^{(\alpha)}(x) - T_{ij}^{(\alpha)}(x, \xi)U_{ij}^{(\alpha)}(x)]dS(x)$$

(8.5.1)

where the superscript $(\alpha)$ stands for the quantity associated with the $\alpha$th phase ($\alpha=1$ being matrix and $\alpha=2$ being inclusion), $T_i = \sigma_{ij}n_j$ ($i=1,2$), $T_i = D_{ij}n_j$ and

$$S^{(\alpha)} = \begin{cases} S + \Gamma & \alpha = 1 \\ S & \alpha = 2 \end{cases}, \quad c^{(\alpha)}(\xi) = \begin{cases} 1 & \text{if } \xi \in \Omega^{(\alpha)} \\ 0.5 & \text{if } \xi \in S^{(\alpha)} (S^{(\alpha)} \text{ smooth}) \\ 0 & \text{if } \xi \notin \Omega^{(\alpha)} \cup S^{(\alpha)} \end{cases}$$

(8.5.2)

$$[U']_{ij} = \begin{bmatrix} u'_{11} & u'_{12} & -\phi'_{1} \\ u'_{21} & u'_{22} & -\phi'_{2} \\ u'_{31} & u'_{32} & -\phi'_{3} \end{bmatrix}, \quad [T']_{ij} = \begin{bmatrix} t'_{11} & t'_{12} & -\omega'_{1} \\ t'_{21} & t'_{22} & -\omega'_{2} \\ t'_{31} & t'_{32} & -\omega'_{3} \end{bmatrix}$$

(8.5.3)
in which \( \Gamma \) and \( S \) are the boundaries of the representative area element (RAE) and inclusions, respectively (see Fig. 8.1), \( \mathbf{u}^{i}_{j} \) and \( \mathbf{t}^{i}_{j} \) (\( i, j = 1, 2 \)) denote, respectively, the displacement and traction component in the \( i \)th direction at a field point \( \mathbf{x} \) due to a unit point force acting in the \( j \)th direction at source point \( \mathbf{x} \). \( \mathbf{u}^{i}_{j} \) and \( \mathbf{t}^{i}_{j} \) (\( i = 1, 2 \)) represent the \( i \)th displacement and traction at \( \mathbf{x} \) due to a unit electric charge at \( \mathbf{x} \). \( \phi^{i} \) and \( \omega^{i} \) (\( i = 1, 2 \)) stand for the electric potential and surface charge at \( \mathbf{x} \) due to a unit point force acting in the \( i \)th direction at \( \mathbf{x} \). These fundamental solutions are well documented in the literature and can be found in [51].

To obtain a weak solution of Eq (8.5.1) as in conventional BEM, the boundary \( S^{(i)} \) is divided into a series of boundary elements. After performing discretization using various kinds of boundary element (e.g., constant element, linear element, higher-order element) and collecting the unknown terms to the left-hand side and the known terms to the right-hand side, as well as using continuity conditions at the interface \( S \) (Fig. 1b), the boundary integral equation (8.5.1) becomes a set of linear algebraic equations:

\[
\mathbf{A} \mathbf{Y} = \mathbf{P} \tag{8.5.4}
\]

where \( \mathbf{Y} \) and \( \mathbf{P} \) are the total unknown and known vectors, respectively, and \( \mathbf{A} \) is a known coefficient matrix.

When the inclusion in Fig. 8.1a becomes a hole, the boundary integral equation (8.5.1) still holds true if we take \( \alpha = 1 \) only. In this case the interfacial continuity condition is replaced by the hole boundary condition: \( T_{ij} = 0 \) along the boundary \( S \) (Fig. 8.1b).

**8.5.2 Algorithms for self-consistent and Mori-Tanaka approaches**

*(i) Self-consistent-BEM approach*

As stated in Subsection 8.2.2, in the self-consistent method, for each inclusion (or hole), the effect of inclusion (or hole) interaction is taken into account approximately by embedding each inclusion (or hole) in the effective medium whose properties are unknown. In this case, the material constants appearing in the boundary element formulation (8.5.1) are unknown. Consequently a set of initial trial values of the effective properties is needed and an iteration algorithm is required. In detail, the algorithm is:

(a) Assume initial values of material constants \( \mathbf{E}_{(i)}^{(0)} \).

(b) Solve Eq (8.5.1) for \( \mathbf{U}_{(i)}^{(0)} \) using the values of \( \mathbf{E}_{(i)}^{(0)} \), where the subscript “(i)” stands for the variable associated with the \( i \)th iterative cycle.

(c) Calculate \( \mathbf{A}_{(i)}^{(0)} \) in Eq (8.2.20) by way of Eq (8.2.24) and using the current values of \( \mathbf{U}_{(i)}^{(0)} \), and then determine \( \mathbf{E}_{(i)}^{(0)} \) by way of Eq (8.2.18).
(d) If $\varepsilon_{(i)} = \left\| E_{(i)} - E_{(i-1)} \right\| \leq \varepsilon$, where $\varepsilon$ is a convergent tolerance, terminate the iteration; otherwise take $E'_{(i)}$ as the initial value and go to step (b).

(ii) Mori-Tanaka-BEM approach

With the Mori-Tanaka method, the concentration matrix $A_{2i}^{MT}$ is given by the solution for a single inclusion (or void) embedded in an intact solid subjected to an applied strain field equal to the as yet unknown average field in the composite, which means that the introduction of inclusions in the composite results in a value of $Z_2$ given by

$$Z_2 = A_{2i}^{MT} \tilde{Z}_i$$

(8.5.5)

where $A_{2i}^{MT}$ is the concentration matrix associated with the dilute model, which can be calculated by way of Eqs (8.2.20), (8.2.24) and (8.5.1). Then, Eq (8.4.21) is used to calculate $A_{2i}^{MT}$. It can be seen from Eq. (8.4.21) that the Mori-Tanaka approach provides explicit expressions for effective constants of piezoelectric composites. Therefore, no iteration is required with the Mori-Tanaka-BE method.

References


