Advanced topics for further programming development

9.1 Introduction

In the preceding chapters we have attempted to show how HT-FE formulations are formed and programmed for potential analysis and linear elastic applications. In particular, we have discussed the theoretical and practical implementations of HT-FEM for solving potential and linear elastic problems with or without non-homogeneous right-hand sources (or body forces) in two-dimensional space. Further, circular hole elements were developed using special purpose Trefftz functions in Chapter 8. The aim of this chapter is to present some advanced topics with which the computer programming given in the previous chapters can be further improved. These topics include construction of Trefftz elements, dimensionless transformation, stress evaluation and smooth treatment, sparse matrix generation and a new idea for developing hybrid elements using fundamental solutions as intra-element trial functions.

9.2 Construction of Trefftz elements

In this book, only conventional four-node linear quadrilateral elements and eight-node quadratic quadrilateral elements are involved in our numerical computation. It should, however, be mentioned that one of the advantages of T-elements is the possibility of constructing arbitrary shaped elements to fulfill the specific purposes of users. For example, we can easily construct triangular elements, pentagonal elements, and so on, by adjusting the relevant frame fields. On the other hand, an increase in elemental sides will create more element nodes and thus more nodal DOF, which requires more terms of Trefftz functions to satisfy the rank requirements (5.44) and (6.61); thus, a higher order of $\mathbf{H}$ matrix may be formed at element level and its inverse operation becomes difficult. In this case, dimensionless transformation can be introduced to overcome this obstacle. The dimensionless procedure is described in the next section.

Besides the regular and special purpose elements described in the previous chap-
ters, the HT p-element model based on the p-extension concept in conventional FEM [1] can also apply to HT elements. Unlike in the h-version of FEM, the purpose in using the p-version FEM is to increase the degree of the interpolation functions while keeping elements and subdomains unchanged. The idea of providing the HT elements with p-method capabilities goes back to 1982 [2], but the first practical implementation for thin plate bending was reported only in 1987 [3]. Since that time, applications of HT p-element methods have extended to orthotropic plate bending [4], thick plate bending [5], plane elasticity [6], and so on. The results obtained have been so convincing that this kind of new HT p-element has been widely used for engineering analysis, due to its high accuracy in modelling field distribution and ease of use [7, 8]. For illustration, we consider a typical triangular p-version element in potential problems. It is constructed by adding an optional number of hierarchical side mode DOF, say \( M \), associated with mid-side nodes (see Figure 9.1) to the frame field of a regular element, to achieve higher-order variations and the desired level of precision. It should be noted that the p-element model differs from the standard one described earlier only in the definition of the frame field of nodes. For example, consider the side 1-C-2 of a particular element shown in Figure 9.1. The frame function is now defined in the form [6, 9]

\[
\tilde{u}_{12} = \tilde{N}_1 u_1 + \tilde{N}_2 u_2 + \sum_{i=1}^{M} \xi^{i-1} (1 - \xi^2) u_{C_i} \quad (9.1)
\]

In contrast, the conventional interpolation expression is

\[
\tilde{u}_{12} = \tilde{N}_1 u_1 + \tilde{N}_2 u_2 \quad (9.2)
\]

which produces linear variation along the side 1-C-2. \( \tilde{N}_1 \) and \( \tilde{N}_2 \) are linear shape functions defined in Figure 9.1, \( \xi \) represents the non-dimensional variable, and \( u_{C_i} \) denotes the hierarchical DOF associated with the mid-side node C.

### 9.3 Dimensionless transformation

From the discussion in the previous chapters, we find that homogeneous Trefftz solutions are usually related to the different order quantities of the distance variable \( r \). The order of \( r \) increases along with an increase in the term number \( m \) of Trefftz functions, which may affect the properties of the element flexibility matrix \( \mathbf{H}_e \) defined as

\[
\mathbf{H}_e = \int_{\Gamma_e} \mathbf{Q}_e^T \mathbf{N}_e \, d\Gamma \quad (9.3)
\]

*Because the maximum dimension of finite elements is usually denoted by \( h \), researchers refer to this conventional mesh refinement as the h-version of FEM.*
and increase the numerical difficulty of the inverse computation of $H_e$. Additionally, the geometrical size of the HT element also influences the numerical accuracy of the value of $H_e$. For a large HT element in terms of geometry, the large distance between the centre and the element boundary may induce greater quantitative difference among the components of $N_e$ due to the different orders of $r$. This phenomenon makes it difficult to evaluate the inverse of the matrix $H_e$. Moreover, for special circular hole elements, the radius of the hole is another important factor affecting numerical accuracy. Too small a size of circular hole also causes numerical instability involving the computation of $H_e$ and its inverse.

To ensure good numerical conditioning of the matrix $H_e$ and to prevent overflow or underflow in evaluating the inverse of $H_e$, the introduction of a local non-dimensional coordinates system is usually suggested [6, 9].

### 9.3.1 Dimensionless transformation in regular HT element for plane potential problems

To show the generalisation of the procedure of dimensionless transformation, a typical T-element as shown in Figure 9.2 is considered. In Figure 9.2, $(X_1, X_2)$ and $(x_1, x_2)$ represent the global coordinates system and the local coordinates system whose origin is at the centre of the element.

The dimensionless transformation can be achieved by introducing a local coordi-
FIGURE 9.2
Dimensionless transformation in regular HT element

The coordinates system \((x_1, x_2)\) centred at the centroid \((X_{1c}, X_{2c})\) of the element. The relationship between \((X_1, X_2)\) and \((x_1, x_2)\) is defined as

\[
\begin{align*}
x_1 &= X_1 - X_{1c} \\
x_2 &= X_2 - X_{2c}
\end{align*}
\]  

(9.4)

where the central coordinates \((X_{1c}, X_{2c})\) measured in the global coordinates system \((X_1, X_2)\) can be determined by

\[
\begin{align*}
X_{1c} &= \frac{1}{n} \sum_{i=1}^{n} X_{1i} \\
X_{2c} &= \frac{1}{n} \sum_{i=1}^{n} X_{2i}
\end{align*}
\]  

(9.5)

in which \(n\) is the number of nodes of the element under consideration. Eq. (9.5) was used in previous chapters to determine the origin of the local coordinate system to an element.

Then, a dimensionless coordinate system \((\xi, \eta)\) is employed:

\[
\begin{align*}
\xi &= \frac{x_1}{a_e} \\
\eta &= \frac{x_2}{a_e}
\end{align*}
\]  

(9.6)

where \(a_e\) is usually taken as the element average distance between the element centroid and its nodes measured in the local coordinates system \((x_1, x_2)\)

\[
a_e = \frac{\sum_{i=1}^{n} \sqrt{x_{1i}^2 + x_{2i}^2}}{n}
\]  

(9.7)
which is used to guarantee the distance between the element boundary points and the
centroid to be close to 1.

It should be noted that the previously defined matrices and vectors such as \( H_e \), \( G_e \) and \( K_e \) associated with the element are constructed in the local Cartesian coordi-
nate system \((x_1, x_2)\). The process of dimensionless transformation from \((x_1, x_2)\) to
\((\xi, \eta)\) will cause some changes in the expressions of these matrices. For illustration,
consider the Trefftz functions used in Chapter 5:

\[
N_{2k-1} = r^k \cos (k \theta), \quad N_{2k} = r^k \sin (k \theta) \quad (k = 1, 2, 3, \ldots)
\]  

(9.8)

where

\[
r = \sqrt{x_1^2 + x_2^2} \quad \text{and} \quad \theta = \arctan \frac{x_2}{x_1}
\]  

(9.9)

When the dimensionless coordinates \((\xi, \eta)\) are employed, we have

\[
r\big|_{(x_1, x_2)} = a_e \bar{r}\big|_{(\xi, \eta)}, \quad \theta\big|_{(x_1, x_2)} = \bar{\theta}\big|_{(\xi, \eta)}
\]  

(9.10)

where

\[
\bar{r} = \sqrt{\xi^2 + \eta^2} \quad \text{and} \quad \bar{\theta} = \arctan \frac{\eta}{\xi}
\]  

(9.11)

Substituting Eq. (9.10) into Eq. (9.8) leads to the following transformation rela-
tion on Trefftz functions:

\[
N_{2k-1}\big|_{(x_1, x_2)} = a_e^k N_{2k-1}\big|_{(\xi, \eta)}
\]

\[
N_{2k}\big|_{(x_1, x_2)} = a_e^k N_{2k}\big|_{(\xi, \eta)}
\]  

(9.12)

As a result, the interpolation vector \( N_e(\mathbf{x}) \) defined in Eq. (5.8) becomes

\[
N_e\big|_{(x_1, x_2)} = N_e\big|_{(\xi, \eta)} a
\]  

(9.13)

where the diagonal matrix \( a \) denotes the dimensionless transformation matrix:

\[
a = \begin{bmatrix}
a_e & 0 & 0 & 0 & 0 & 0 \\
0 & a_e & 0 & 0 & 0 & 0 \\
0 & 0 & a_e^2 & 0 & 0 & 0 \\
0 & 0 & 0 & a_e^2 & 0 & 0 \\
0 & 0 & 0 & 0 & \ddots & 0 \\
0 & 0 & 0 & 0 & 0 & a_e^k \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}_{m \times m}
\]  

(9.14)

with \( m = 2k \).

Furthermore, \( T_e \) is modified as

\[
T_e\big|_{(x_1, x_2)} = \begin{bmatrix}
\frac{\partial N_e}{\partial x_1} \\
\frac{\partial N_e}{\partial x_2}
\end{bmatrix}_{(x_1, x_2)} = \begin{bmatrix}
\frac{\partial N_e(\xi, \eta)}{\partial \xi} a \frac{\partial \xi}{\partial x_1} \\
\frac{\partial N_e(\xi, \eta)}{\partial \eta} a \frac{\partial \eta}{\partial x_2}
\end{bmatrix} = \frac{1}{a_e} T_e\big|_{(\xi, \eta)} a
\]  

(9.15)
Similarly, we have
\[
Q_e\big|_{(x_1,x_2)} = \frac{1}{ae} Q_e\big|_{(\xi,\eta)} a \quad (9.16)
\]

Finally, the matrices $H_e$ and $G_e$ are written as†
\[
H_e\big|_{(x_1,x_2)} = \left( \int_{\Gamma_e} Q_e^T N_e d\Gamma \right) \bigg|_{(x_1,x_2)}
= \int_{\Gamma_e} \frac{1}{ae} a^T Q_e^T \big|_{(\xi,\eta)} N_e \big|_{(\xi,\eta)} a d\Gamma \big|_{(\xi,\eta)}
= a H_e\big|_{(\xi,\eta)} a
\quad (9.17)
\]
\[
G_e\big|_{(x_1,x_2)} = \int_{\Gamma_e} Q_e^T \tilde{N}_e d\Gamma \bigg|_{(x_1,x_2)}
= \int_{\Gamma_e} \frac{1}{ae} a^T Q_e^T \big|_{(\xi,\eta)} \tilde{N}_e a d\Gamma \big|_{(\xi,\eta)}
= a G_e\big|_{(\xi,\eta)}
\quad (9.18)
\]
and consequently the inverse of the matrix $H_e$ can be derived
\[
H_e^{-1}\big|_{(x_1,x_2)} = \left[ a H_e\big|_{(\xi,\eta)} a \right]^{-1} = a^{-1} H_e^{-1}\big|_{(\xi,\eta)} a^{-1}
\quad (9.19)
\]
with
\[
a^{-1} = \begin{bmatrix}
a_e^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a_e^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a_e^{-2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a_e^{-2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a_e^{-k} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & a_e^{-k} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a_e^{-k} \\
\end{bmatrix}_{m \times m}
\quad (9.20)
\]
Substitution of the matrices $H_e$ and $G_e$ into the stiffness matrix $K_e$ produces
\[
K_e\big|_{(x_1,x_2)} = \left( G_e^T H_e^{-1} G_e \right) \big|_{(x_1,x_2)} = \left( G_e^T H_e^{-1} G_e \right) \big|_{(\xi,\eta)} = K_e\big|_{(\xi,\eta)}
\quad (9.21)
\]
from which we find that the stiffness matrix does not change in value after dimensionless transformation.

In addition, the coefficient vector appearing in the intra-element potential field can be changed as
\[
c_e\big|_{(x_1,x_2)} = (H_e^{-1} G_e) \big|_{(x_1,x_2)} d_e = a^{-1} (H_e^{-1} G_e) \big|_{(\xi,\eta)} d_e
\quad (9.22)
\]
†Note: the relation
\[
d\Gamma\big|_{(x_1,x_2)} = a_e \sqrt{d \xi^2 + d \eta^2} = a_e d\Gamma\big|_{(\xi,\eta)}
\]
is used for the transformation of matrices $H_e$ and $G_e$. 
9.3.2 Dimensionless transformation in special HT element for plane potential problems

For a special T-element including a circular hole with radius $b$ as displayed in Figure 9.3, dimensionless transformation of Eq. (9.6) leads to

$$
r(|x_1, x_2|) = ae \tilde{r}(|\xi, \eta|), \quad \theta|_{(x_1, x_2)} = \tilde{\theta}|_{(\xi, \eta)}, \quad b|_{(x_1, x_2)} = ae \tilde{b}|_{(\xi, \eta)} \quad (9.23)
$$

Substituting Eq. (9.23) into the potential Trefftz functions defined in Eq. (8.13) leads to the same form as that of Eq. (9.12), but different content. The corresponding matrices $T_e, H_e$ and $G_e$ also have the same forms as those of Eqs. (9.12) - (9.19).

9.3.3 Dimensionless transformation in regular element for plane elastic problems

Consider again the regular HT element shown in Figure 9.2. Introduction of the dimensionless transformation (9.6) to the homogeneous displacement solutions in Eqs. (6.52) - (6.55) and stress solutions in Eqs. (6.57) - (6.60) yields

$$
N_{ik}|_{(x_1, x_2)} = a_e^k N_{ik}|_{(\xi, \eta)}
$$

$$
T_{ik}|_{(x_1, x_2)} = a_e^{k-1} T_{ik}|_{(\xi, \eta)} \quad (9.24)
$$
where \( i = 1, 2, 3, 4, k = 1, 2, 3, \ldots \). In the above derivation, the following relations

\[
z|_{(x_1,x_2)} = a_e z|_{(\xi,\eta)}, \quad \bar{z}|_{(x_1,x_2)} = a_e \bar{z}|_{(\xi,\eta)} \quad (9.25)
\]

have been used. Using the relations (9.24), the interpolation matrix defined in Eq. (6.12) yields the same form as that of Eq. (9.13) except that the matrix \( a \) is now defined by

\[
a = \begin{bmatrix}
a_e & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a_e & 0 & 0 & 0 & 0 & 0 \\
0 & a_e & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a_e & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a_e^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a_e^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a_e^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & a_e^2 \\
\end{bmatrix} \quad (9.26)
\]

for the case of \( m = 7 \).

The corresponding \( H_e, G_e, K_e \) and \( c_e \) in terms of the dimensionless coordinate system \((\xi, \eta)\) can be obtained in a way similar to that described in Section 9.3.1. It is found that these matrices have the same form as those of Eqs. (9.17), (9.18), (9.21) and (9.22).

### 9.3.4 Dimensionless transformation in hole element for plane elastic problems

Consider again the circular hole element shown in Figure 9.3. When the dimensionless coordinates \((\xi, \eta)\) defined in Eq. (9.6) are used, the displacement homogeneous solutions \( \hat{N}_{ik}^e \) and stress solutions \( \hat{T}_{ik}^e \) in Section 8.4 become

\[
\begin{align*}
\hat{N}_{i0}^{e(1)}|_{(x_1,x_2)} &= a_e \hat{N}_{i0}^e|_{(\xi,\eta)} \\
\hat{N}_{i1}^{e(1)}|_{(x_1,x_2)} &= a_e^2 \hat{N}_{i1}^e|_{(\xi,\eta)} \\
\hat{N}_{i2}^{e(1)}|_{(x_1,x_2)} &= a_e^2 \hat{N}_{i2}^e|_{(\xi,\eta)} \\
\hat{N}_{ikk}^{e(1)}|_{(x_1,x_2)} &= a_e^{1+k} \hat{N}_{ikk}^e|_{(\xi,\eta)} \quad (k \geq 2) \\
\hat{N}_{2kk}^{e(1)}|_{(x_1,x_2)} &= a_e^{1+k} \hat{N}_{2kk}^e|_{(\xi,\eta)} \quad (k \geq 2) \\
\hat{N}_{3kk}^{e(1)}|_{(x_1,x_2)} &= a_e^{1-k} \hat{N}_{3kk}^e|_{(\xi,\eta)} \quad (k \geq 2) \\
\hat{N}_{4kk}^{e(1)}|_{(x_1,x_2)} &= a_e^{1-k} \hat{N}_{4kk}^e|_{(\xi,\eta)} \quad (k \geq 2)
\end{align*}
\]
and
\[
\hat{T}_{10}^{(x_1,x_2)} = \hat{T}_{10}^{(\xi,\eta)}
\]
\[
\hat{T}_{11}^{(x_1,x_2)} = a_e \hat{T}_{11}^{(\xi,\eta)}
\]
\[
\hat{T}_{21}^{(x_1,x_2)} = d_e \hat{T}_{11}^{(\xi,\eta)}
\]
\[
\hat{T}_{1k}^{(x_1,x_2)} = a^k_e \hat{T}_{1k}^{(\xi,\eta)} \quad (k \geq 2)
\]
\[
\hat{T}_{2k}^{(x_1,x_2)} = d^k_e \hat{T}_{2k}^{(\xi,\eta)} \quad (k \geq 2)
\]
\[
\hat{T}_{3k}^{(x_1,x_2)} = a^{e-2-k} \hat{T}_{3k}^{(\xi,\eta)} \quad (k \geq 2)
\]
\[
\hat{T}_{4k}^{(x_1,x_2)} = a^{e-2-k} \hat{T}_{4k}^{(\xi,\eta)} \quad (k \geq 2)
\]
(9.28)

Making use of the dimensionless coordinate system \((\xi, \eta)\), the interpolation matrix defined in Eq. (6.12) yields the same form as that of Eq. (9.13) except that the matrix \(a\) is now defined by

\[
a = \begin{bmatrix}
a_e & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a^2_e & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a^2_e & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a^3_e & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a^2_e & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a^3_e & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & a^{-3}_e \\
0 & 0 & 0 & 0 & 0 & 0 & a^{-3}_e
\end{bmatrix}
\]
(9.29)

for the case of \(m = 7\).

The corresponding \(H_e\), \(G_e\), \(K_e\) and \(c_e\) for the case of hole element can again be obtained similarly and it is found that these matrices also have the same form as those of Eqs. (9.17), (9.18), (9.21) and (9.22).

### 9.4 Nodal stress evaluation-smooth techniques

Like conventional FEM, HT-FE analysis generally involves the minimisation of some functional defined in terms of piecewise functions including internal Trefftz interpolation and element boundary frame functions. These functions are generally required to have a certain degree of inter-element continuity depending on terms in the functional. In many engineering problems, the quantities of primary engineering interest involve the function derivatives and in many instances, these derivatives do not possess continuity between elements. For example, in elastic analysis, continuity of displacement fields obtained by HT-FEM is guaranteed within and between elements. However, the stress components evaluated from displacement derivatives are usually discontinuous between elements (see Figure 9.4); they are continuous within each element only. As a result, histogram-type distributions of these discontinuous
fields are usually encountered in practical computation and the analyst is therefore faced with the problem of interpreting quantities with such distributions. In fact, this discontinuity phenomenon is usually not rational from the theoretical point of view in practical engineering, since in the so-called displacement-based methods, for example, FEM, BEM and HT-FEM, the discontinuous stresses between elements are caused by the different measurements of the assumed displacement variation in each element.

![Illustration of discontinuous stress field between elements in HT-FEM](image)

**FIGURE 9.4**
Illustration of discontinuous stress field between elements in HT-FEM

Although in quite a few commercial FE software programs such as ABAQUS and ANSYS some rational and consistent post-process procedures are adopted for the interpretation of discontinuous functions and can produce good smooth results, it is necessary to review some treatments to obtain smooth distribution of certain quantities at element nodes in the domain under consideration for further improvement of HT-FEM.

Consider again the elastic problems discussed in Chapter 6. Unlike in the conventional FEM, the stress distributions (6.14) within each element

\[
\sigma_e = T_e \mathbf{c}_e
\]

exactly satisfy the governing equations according to HT-FE theory and the domain integration is removed in the calculation of stiffness matrix in the HT-FEM. The extrapolation treatment employed widely in conventional FEM [10], using the stress values at Gauss sampling points is not suitable for the HT-FEM. Here we introduce a simpler way to obtain nodal values of stress. It is achieved by averaging the values of stress at the corner nodes. For instance, for the four elements shown in Figure 9.4,
the value of stress at the common node \( p \) is calculated using the following expression

\[
\bar{\sigma}_p = \frac{\sigma_{pi} + \sigma_{pj} + \sigma_{pk} + \sigma_{pl}}{4}
\]  \hspace{1cm} (9.31)

or

\[
\bar{\sigma}_p = \frac{\sigma_{pi} A_i + \sigma_{pj} A_j + \sigma_{pk} A_k + \sigma_{pl} A_l}{A_i + A_j + A_k + A_l}
\]  \hspace{1cm} (9.32)

where \( \bar{\sigma}_p \) represents the smoothed nodal values of stress. \( \sigma_{pi} \), \( \sigma_{pj} \), \( \sigma_{pk} \), and \( \sigma_{pl} \) are the unsmoothed stresses evaluated from Eq. (9.30) in different elements, and \( A_i, A_j, A_k, \) and \( A_l \) denote the areas of elements \( i, j, k, \) and \( l \), respectively.

### 9.5 Generating intra-element points for outputting field results

Unlike in the conventional FEM, all integrals involved in the calculation of a stiffness matrix are defined on the element boundary in the HT-FEM. Therefore, the output of field results at Gaussian sample points used in the conventional FEM is not suitable for the HT-FEM. In the previous chapters, only the internal fields at the centroid of an element are calculated and outputted. To obtain field distributions at more intra-element points for further reference, the following expression can be used to generate internal points:

\[
x_{m1} = \frac{x_{i1}}{2}, \quad x_{m2} = \frac{x_{i2}}{2}
\]  \hspace{1cm} (9.33)

where \((x_{m1}, x_{m2})\) represents the midpoint coordinate of the line with ends at the centroid and a particular element node (see Figure 9.5a) and \((x_{i1}, x_{i2})\) are the nodal coordinates of the node.

For the special circular hole element discussed in Chapter 8, let the origin of the local coordinates be the centre of the circular hole. We can generate extra computing points \((x_{m1}, x_{m2})\) on the circular boundary through the formulation (see Figure 9.5b):

\[
x_{m1} = \frac{x_{i1} b}{\sqrt{x_{i1}^2 + x_{i2}^2}}, \quad x_{m2} = \frac{x_{i2} b}{\sqrt{x_{i1}^2 + x_{i2}^2}}
\]  \hspace{1cm} (9.34)

where \( b \) represents the radius of the circular hole.

### 9.6 Sparse matrix generation and solving procedure

In Chapters 5 and 6, the final global stiffness equation is solved using a Gaussian elimination algorithm which is based on the full stiffness matrix of the finite element equation. However, after a node-by-node assembling procedure the resulting
FIGURE 9.5
Generation of more internal points within a regular T-element and a special circular hole element

The linear system is characterised by a system matrix which is usually large and sparse. “Sparse” here indicates that many elements in the global stiffness matrix are zero and some elements are located near the main diagonal line. To increase computational efficiency and reduce the space requirement for storing the stiffness matrix, the modified banded storage strategy is generally employed (see Figure 9.6). For the sake of convenience, we present here a brief review of this strategy.

As usual, the bandwidth can be evaluated by [11]

$$\text{bandwidth} = (\max \epsilon_D + 1) \times DOF \quad (9.35)$$

where $D_e$ is the maximum difference between any node numbers occurring in a specific element in the HT-FEM, and $DOF$ denotes the number of degrees of freedom per node.

From Eq. (9.35), we can see that to reduce bandwidth we should number systematically and try to ensure a minimum number difference between adjacent nodes. The narrow bandwidth means a small storage requirement, especially for large-scale computation.

Subsequently, the corresponding banded-symmetric solver based on modified Gaussian elimination can be designed to obtain the final nodal displacements. The detailed algorithm can be found in Ref. [11].

Alternatively, the wavefront or frontal method may be used to optimise equation solution time. In the wavefront method, elements rather than nodes are automatically renumbered, and assembly of the stiffness equations alternates with their solution by Gaussian elimination. Thus, it is somewhat more difficult to understand and to program than the classic banded-symmetric method, but it has greater computational efficiency than the latter and is therefore becoming popular in large-scale programs [12, 13].
9.7 An alternative formulation to HT-FEM

It is well known that HT-FEM is based on a hybrid-type method which includes the use of an independent auxiliary inter-element frame field defined on each element boundary and an independent internal field chosen so as to *a priori* satisfy the homogeneous governing differential equations by means of a suitable truncated T-complete functions set of homogeneous solutions. Inter-element continuity is enforced by using a modified variational principle, which is used to construct the standard force-displacement relationship, that is, the stiffness equation, and to establish linkage of frame field and internal fields of the element. The property of non-singular element boundary integrals in HT-FEM enables us to construct arbitrary shaped elements conveniently. Moreover, special T-element can be designed to perform special-purpose analyses. However, the terms of truncated T-complete functions must be carefully selected to achieve the desired results. Further, T-complete functions are difficult to generate for some physical problems. Moreover, due to the inherent properties of T-complete functions used, that is, higher orders of the Euclidean distance variable, a relatively complex coordinate transformation is usually required in HT-FEM to keep the inverse of matrix $H$ stable.

To circumvent this drawback while maintaining the advantages of HT-FEM, a novel hybrid finite formulation based on the fundamental solutions, called HFS-FEM, has been developed by Wang and Qin [14]. In the HFS-FEM, the fundamental solution is used to replace the Trefftz functions in the HT-FEM. The proposed HFS-
FEM can be viewed as a fourth type of FEM, which is significantly different from the other three types including conventional FEM [11, 15], natural FEM [16, 17], and HT-FEM [9]. In the analysis, a linear combination of the fundamental solutions at different source points is used to approximate the field variable within the element. The independent frame field is used to guarantee inter-element continuity and defined in the same way as in HT-FEM along the element boundary. The modified variational principle employed is similar to that in HT-FEM and is used to generate the final stiffness equation and establish linkage between the boundary frame field and internal field in an element. The proposed HFS-FEM inherits all the advantages of the HT-FEM and removes the difficulties encountered in constructing and selecting T-functions.

It can be seen from the discussion above that the major difference between HT-FEM and HFS-FEM lies in the different intra-element trial functions used. T-complete functions are used in HT-FEM, whereas fundamental solutions are used as internal trial functions in HFS-FEM to construct the approximated interpolation field within an element. We take the Laplace equation as an example to demonstrate the basic concept of the proposed HFS-FEM.

![FIGURE 9.7](image)

Intra-element field, frame field in a particular element in HFS-FEM, and the generation of source points
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For a typical polygonal element as shown in Figure 9.7, the following two groups of potential fields are assumed:

- The intra-element field is defined within the element

\[ u_e(x) \approx \sum_{j=1}^{n_s} N_e(x,y_j) c_{ej} = N_e(x) c_e \quad \forall x \in \Omega_e, \ y_j \notin \Omega_e \]  \hspace{1cm} (9.36)

where \( c_e \) is a vector of undetermined coefficients (or virtual source value) and \( n_s \) is the number of virtual sources outside the element under consideration. \( N_e^j(x) = N_e(x,y_j) \) is the fundamental solution at point \( y_j \) satisfying the Laplace operator equation in an infinite domain:

\[ \nabla^2 N_e(x,y) = -\delta(x,y) \quad \forall x,y \in \mathbb{R}^2 \]  \hspace{1cm} (9.37)

where

\[ \delta(x,y) = \begin{cases} 0 & x \neq y \\ 1 & x = y \end{cases} \]  \hspace{1cm} (9.38)

For the two-dimensional case, we have

\[ N_e(x,y) = -\frac{1}{2\pi} \ln r \]  \hspace{1cm} (9.39)

Note that coordinates \( x, y_j \) are defined in the local coordinates system \((x_1,x_2)\) and

\[ r = \|x - y\| \]  \hspace{1cm} (9.40)

Substituting Eq. (9.36) into the standard Laplace equation (5.1), we have

\[ \nabla^2 u(x) = \sum_{j=1}^{n_s} \nabla^2 N_e(x,y_j) c_{ej} = 0 \quad \forall x \in \Omega_e, \ y_j \notin \Omega_e \]  \hspace{1cm} (9.41)

in which the solution property of \( N_e(x,y_j) \) and the point \( y_j \) located outside the element are used.

The virtual source point \( y_j \) can be generated in the same manner as in the method of fundamental solutions (MFS) [18 - 21]

\[ y = x_b + \gamma (x_b - x_e) \]  \hspace{1cm} (9.42)

where \( \gamma \) is a dimensionless coefficient, \( x_b \) and \( x_e \) are the boundary point and the geometrical centroid of an element, respectively. For the element shown in Figure 9.7, we can utilise element nodes as \( x_b \) to generate corresponding sources outside the element. It can be proved that generating source points using all element nodes naturally satisfies the rank requirement of minimal terms (5.45).

- Frame field defined along the element boundary
\[ \tilde{u}_e(x) = \tilde{N}_e(x) \mathbf{d}_e \quad x \in \Gamma_e \] 

(9.43)

is designed to enforce conformity on the field variable \( u \) between neighbouring elements. \( \mathbf{d}_e \) denotes nodal DOF vector of all element nodes and \( \tilde{N}_e \) represents the conventional finite element interpolating functions. This procedure is the same as that used in HT-FEM.

- A modified functional in the same form as Eq. (5.21)

\[
\Psi_{me} = \frac{1}{2} \int_{\Omega_e} (\tilde{q}_1^2 + \tilde{q}_2^2) \tau_e d\Omega - \int_{\Gamma_e} \tilde{q}\tilde{u}_\tau d\Gamma + \int_{\Gamma_{eq}} \tilde{q}\tilde{u}_\tau d\Gamma 
\]

(9.44)

is used to establish the linkage of unknown \( \mathbf{c}_e \) and \( \mathbf{d}_e \) and to obtain the final stiffness equation.

In summary, in this section we have presented the basic concept and procedure of the newly developed HFS-FEM. Further studies on this area are under way.

References


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